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Trigonometric B-Spline Collocation Method for Solving PHI-Four and Allen–Cahn Equations

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Abstract. In this paper, we develop a numerical solution based on non-polynomial B-spline (trigonometric B-spline) collocation method for solving time-dependent equations involving PHI-Four and Allen–Cahn equations. A three-time-level implicit algorithm has been derived. This algorithm combines the trigonometric B-spline interpolant and the θ-weighted scheme for space and time discretization, respectively. Convergence analysis is discussed and the accuracy of the presented method is $O(\tau^2 + h^2)$. Applying von Neumann stability analysis, the proposed technique is shown to be unconditionally stable. Three test problems are demonstrated to reveal that our method is reliable, efficient and very encouraging.

Mathematics Subject Classification. 65M70, 65M12.

Keywords. Time-dependent problems, PHI-Four equation, Allen–Cahn equation, trigonometric B-spline, collocation method, convergence analysis, von Neumann stability.

1. Introduction

Partial differential equations (PDEs) are a great tool to model many applications that appear in science and engineering. PDEs are essential in various fields such as plasma physics, fluid dynamics and quantum field theory [1]. Consider the following time-dependent equation:

$$au_t + bu_{tt} + cu_{xx} + du = \psi(x, t, u, u^q, u_x), \quad \bar{a} \le x \le \bar{b} \text{ and } t \ge 0, \quad (1.1)$$

subject to initial conditions

$$u(x, 0) = f_1(x), \quad u_t(x, 0) = f_2(x), \quad (1.2)$$

and boundary conditions as follows:

$$u(a, t) = g_1(t), \quad u(b, t) = g_2(t), \qquad t \ge t_0, \quad (1.3)$$

where $u = u(x,t)$ represents the wave displacement at position x and time t and a, b, c, q, and d are known constants and $\psi(x, t, u, u^q, u_x)$ is the force term.

Changing the values of the constants in Eq. (1.1) gives various types of equations. For example, if we set the constants as $a = 0, b = 1, c = -1, d = -1$ and $\psi(x, t, u, u^q, u_x) = -u^3$, Eq. (1.1) becomes PHI-Four equation. This equation plays an important role in mathematical physics. Triki and Wazwaz [1], have been addressed the exact bright and dark soliton solutions of the form:

$$u(x,t) = \pm \tanh\left(\frac{(x - vt)}{\sqrt{2(1 - v^2)}}\right), \tag{1.4}$$

where v is the wave speed. Also, in [1], the authors applied sine–cosine ansatz method to investigate the solution of the following two types of PHI-Four equation:

$$(u^{\pm n})_{tt} - c(u^{\pm n})_{xx} - u^m + u^{\pm n} = 0. \tag{1.5}$$

Also, Triki and Wazwaz [2], considered a new variety of soliton solutions of the generalized Fitzhugh–Nagumo equation by specific solitary wave ansatz and the tanh method.

Also, if we set the constants as $a = 1, b = 0, c = -1, d = -1$ and $\psi(x, t, u^q, u_x) = -u^3$. Equation (1.1) becomes Allen–Cahn equation. Allen–Cahn equation is used to model many applications such as mathematical biology, quantum mechanics, plasma physics and image processing [3–5].

The solution of these types of NPDEs are significant in various applications. Bahrawy et al. [6] presented a Jacobi–Gauss–Lobatto collocation method for solving the PHI-Four equation. Ehsani et al. [7] solved the PHI-Four equation using the homotopy perturbation method. Wazwaz and Triki [8] developed non-topological and topological soliton solutions for the PHI-Four equation via solitary wave ansatz method. Also, Alofi [9] proposed the generalized tanh method to find the solution of Drinfeld–Sokolov system and PHI-Four equation. Soliman and Abdo [10] established new solutions for RLW, PHI-Four and Boussinesq equations. Also, Sassaman and Biswas [11] investigated the solution of PHI-Four and Klein Gordon equations using the soliton perturbation theory. Lastly, Najafi [12] discussed the soliton solution of this equation using the He's variational method for solving PHI-Four equation.

Hariharan [3], Hariharan and Kannan [4] adapted a wavelet-based method for solving the Newell–Whitehead and Allen–Cahn equations, respectively. Beneš et al. [5] developed an algorithm of pattern recovery based on the solution of the Allen–Cahn equation. Ishtiaq et al. [13] demonstrated some numerical methods for Allen–Cahn equation using different time stepping and space discretization methods with non-periodic boundary conditions.

Ordinary B-spline collocation method has been used for solving PDEs by many authors as in [14–20]. Mohammadi [21], used another type of spline named exponential B-spline to present a numerical algorithm for the time-dependent generalized regularized long wave equation. Also, Mohammadi [22], developed an exponential B-spline collocation method for solving convection–diffusion equation.

In this paper, we aim to illustrate an alternative type of nonpolynomial B-spline function named trigonometric B-spline for solving time-dependent problems. Based on nonpolynomial B-spline function a three-time-level implicit algorithm has been derived. This algorithm uses the nonpolynomial B-spline interpolant and the θ-weighted scheme for the spatial and temporal discretization, respectively, to solve the time-dependent problem (1.1)–(1.3). We discussed the convergence analysis in details and the stability analysis of the proposed method are presented via von Neumann technique. The main advantage of this approach is easy to implement for both linear and nonlinear problems.

This paper is organized as follows: Sect. 2 is devoted to temporal discretization via finite difference technique to discretize the time-dependent problem (1.1). In Sect. 3 we described the cubic trigonometric B-spline and implemented our method to solve the problem (1.1)–(1.3). Convergence analysis of the presented scheme is discussed in Sect. 4. While the stability analysis using von Neumann approach is shown to be unconditionally stable in Sect. 5. In Sect. 6 we presented three numerical examples to validate the efficiency and convergence of our method. Finally, in Sect. 7 we conclude our results.

2. Temporal Discretization

Consider a uniform mesh Δ with the grid points π_{ij} to discretize the region $\Omega = [\bar{a}, \bar{b}] \times [0, T]$. Each π_{ij} is the vertices of the grid points (x_i, t_j) where $x_i = a + ih, i = 0, 1, 2, \ldots, N$ and $t_j = jk, j = 0, 1, 2, \ldots, M, Mk = T$. The quantities h and k are the mesh size in the space and time directions, respectively. First we discretize the problem in, time variable using the forward and central difference formulas and apply θ-weighted scheme to the space derivatives to Eq. (1.1) where $(0 \le \theta \le 1)$ then it can be written as:

$$\begin{aligned}
&\frac{a}{k}(u(x,t_{n+1}) - u(x,t_n)) + \frac{b}{k^2}(u(x,t_{n+1}) - 2u(x,t_n) + u(x,t_{n-1})) \\
&\quad = \theta[-cu''(x,t_{n+1}) - du(x,t_{n+1}) \\
&\qquad + \psi(x,t_{n+1},u(x,t_{n+1}),u^q(x,t_{n+1}),u'(x,t_{n+1}))] \\
&\qquad + (1-\theta)[-cu''(x,t_n) - du(s,t_n) \\
&\qquad + \psi(x,t_n,u(x,t_n),u^q(x,t_n),u'(x,t_n))],
\end{aligned} \tag{2.1}$$

where the subscription $n-1, n$ and $n+1$ denotes the adjacent time levels. Rearrange the above equation, we get

$$\begin{aligned}
&\alpha u''(x,t_{n+1}) + \beta u(x,t_{n+1}) + \gamma\psi(x,t_{n+1},\ u(x,t_{n+1}),u^q(x,t_{n+1}),u'(x,t_{n+1})) \\
&\quad = \varphi(x,t_n,\ u(x,t_n),u^q(x,t_n),u'(x,t_n)),
\end{aligned} \tag{2.2}$$

where $\alpha = \theta c, \beta = \frac{a}{k} + \frac{b}{k^2} + d\theta, \gamma = -\theta$ and

$$\begin{aligned}&\varphi(x,t_n,\ u(x,t_n),u^q(x,t_n),u'(x,t_n))\\&\quad=\left(\frac{a}{k}+\frac{2b}{k^2}-d(1-\theta)\right)u(x,t_n)-(1-\theta)cu''(x,t_n)\\&\qquad+(1-\theta)\psi(x,t_n,\ u(x,t_n),u^q(x,t_n),u'(x,t_n))-\frac{b}{k^2}u(x,t_{n-1}).\end{aligned}\tag{2.3}$$

With the boundary conditions:

$$u(a,t_{n+1})=g_1(t_{n+1}),\quad u(b,t_{n+1})=g_2(t_{n+1}).\tag{2.4}$$

The space derivatives are approximated by trigonometric B-spline which are presented in the next section.

3. Trigonometric B-Spline Collocation Method

We use the cubic trigonometric B-spline function to solve the boundary value problems given by (2.2)–(2.4) at each time level. Let $\bar{\Delta}\equiv\{\bar{a}=x_0<x_1,\ldots,x_{N-1},x_N=\bar{b}\}$ be a uniform partition of the solution domain $\bar{a}\le x\le\bar{b}$ by the knots x_j with $h=x_{j+1}-x_j=\frac{\bar{b}-\bar{a}}{N}, j=0,1,\ldots,N$.

Let $\mathrm{TS}(\bar{\Delta})$ be the space of trigonometric spline functions over the partition $\bar{\Delta}$. We can define the trigonometric B-spline functions $\{\mathrm{TBS}_j(x)\}$, for $j=-1,0,1,\ldots,N+1$. for $\mathrm{TS}(\bar{\Delta})$ after including two more points on each side of the partition $\bar{\Delta}$. Thus, the cubic trigonometric B-spline function is defined as in [13,23,27,28].

$$\mathrm{TBS}_j(x)=\frac{1}{\mu(h)}\begin{cases}\sin^3(\frac{x-x_{j-2}}{2}), & x\in[x_{j-2},x_{j-1}]\\ \sin(\frac{x-x_{j-2}}{2})[\sin(\frac{x-x_{j-2}}{2})\sin(\frac{x_j-x}{2})\\ \quad+\sin(\frac{x_{j+1}-x}{2})\sin(\frac{x-x_{j-1}}{2})]\\ \quad+\sin(\frac{x-x_{j-2}}{2})\sin^2(\frac{x_{j+1}-x}{2}), & x\in[x_{j-1},x_j]\\ \sin(\frac{x_{j+2}-x}{2}),[\sin(\frac{x-x_{j-1}}{2})\sin(\frac{x_{j+1}-x}{2})\\ \quad+\sin(\frac{x_{j+2}-x}{2})\sin(\frac{x-x_j}{2})]\\ \quad+\sin(\frac{x-x_{j-2}}{2})\sin^2(\frac{x_{j+1}-x}{2}), & x\in[x_j,x_{j+1}]\\ \sin^3(\frac{x_{j+2}-x}{2}), & x\in[x_{j+1},x_{j+2}]\\ 0, & \text{otherwise}\end{cases}$$

where

$$\mu(h)=\sin(\bar{h})\sin(h)\sin(3\bar{h}),\quad \bar{h}=\frac{h}{2}.$$

Properties of trigonometric B-spline basis are given below, see [23]

1. $\mathrm{TBS}_j(x)$ is nonnegative,
2. $\mathrm{TBS}_j(x)\ge 0$ when $x\in[x_{j-2},x_{j+2}]$ and zero otherwise,
3. $\sum_j \mathrm{TBS}_j(x)=1$.

Then the coefficients of TBS_j and its derivatives are given in the following table.

Table 1. Coefficients of TBS_j and its derivatives, see [24,25]

x	x_{j-2}	x_{j-1}	x_j	x_{j+1}	x_{j+2}
TBS_j	0	γ_1	σ	γ_1	0
TBS'_j	0	λ	0	$-\lambda$	0
$\mathrm{TBS}^{(2)}_j$	0	δ	σ_1	δ	0

Where

$$\gamma_1 = \sin^2(\bar{h})\csc(h)\csc(3\bar{h}), \quad \sigma = \frac{2}{1+2\cos(h)}, \quad \lambda = \frac{3}{4}\csc(3\bar{h}),$$
$$\delta = \frac{3(1+3\cos(h))\csc^2(\bar{h})}{16(2\cos(\bar{h})+\cos(3\bar{h}))}, \quad \sigma_1 = \frac{-3\cot^2(\bar{h})}{2(1+2\cos(h))}.$$

We consider that $U(x,t)$ be the approximation to the exact solution $u(x,t)$ of problems (2.2)–(2.4), thus we have

$$U(x,t_{n+1}) = \sum_{j=-1}^{N+1} c_j(t_{n+1})\mathrm{TBS}_j(x), \tag{3.1}$$

where c_j are unknown coefficients and $\mathrm{TBS}_j(x)$ are trigonometric B-spline functions. The nodal values U_j, U'_j and $U^{(2)}_j$ at the knots x_j are derived from expression (3.1) and Table 1 in the following form

$$U_j = \gamma_1 c_{j+1} + \sigma c_j + \gamma_1 c_{j-1}, \tag{3.2}$$
$$U'_j = -\lambda c_{j+1} + \lambda c_{j-1}, \tag{3.3}$$
$$U^{(2)}_j = \delta c_{j+1} + \sigma_1 c_j + \delta c_{j-1}. \tag{3.4}$$

Substituting Eqs. (3.2)–(3.4) into Eq. (2.2) yields the following equation

$$\alpha \sum_{j=-1}^{n+1} c_j(t_{n+1})\mathrm{TBS}''_j(x_j) + \beta \sum_{j=-1}^{n+1} c_j(t_{n+1})\mathrm{TBS}_j(x_j)$$
$$+\gamma\psi\left(x_j, t_{n+1}, \sum_{j=-1}^{n+1} c_j(t_{n+1})\mathrm{TBS}_j(x_j), \left(\sum_{j=-1}^{n+1} c_j(t_{n+1})\mathrm{TBS}_j(x_j)\right)^q,\right.$$
$$\left.\sum_{j=-1}^{n+1} c_j(t_{n+1})\mathrm{TBS}'_j(x_j)\right) = \varphi(x_j). \tag{3.5}$$

Simplifying the above relation leads to the following system of equations

$$\alpha_1 c_{j+1} + \beta_1 c_j + \alpha_1 c_{j-1} + \gamma\psi_j = \varphi_j, \quad j = 0,1,2,\ldots,N, \tag{3.6}$$

where $\alpha_1 = \alpha\delta + \beta\gamma_1, \beta_1 = \alpha\sigma_1 + \beta\sigma$ and $\psi_j = \psi(x_j, \gamma_1 c_{j+1} + \sigma c_j + \gamma_1 c_{j-1}, \lambda c_{j+1} - \lambda c_{j-1}, (\gamma_1 c_{j+1} + \sigma c_j + \gamma_1 c_{j-1})^q)$.

To obtain a unique solution of the last system, two additional constraints are required. These constrains are obtained from the boundary conditions given by Eq. (2.4).

Using the first boundary condition, we get

$$u(a,t_{n+1}) = g_1(t_{n+1}) = \gamma_1 c_1 + \sigma c_0 + \gamma_1 c_{-1}.$$

Eliminating the constant c_{-1} from the above equation and Eq. (3.6) for $j = 0$, we obtain

$$\alpha^* c_0 + \gamma_1\gamma\psi_0 = \gamma_1\varphi_0 - \alpha_1 g_1(t_{n+1}), \quad \text{for } j = 0, \tag{3.7}$$

where $\alpha^* = \gamma_1\beta_1 - \alpha_1\sigma$.

Similarly, for the second end condition, we have

$$\alpha^* c_N + \gamma_1\gamma\psi_N = \gamma_1\varphi_N - \alpha_1 g_2(t_{n+1}), \quad \text{for } j = N. \tag{3.8}$$

Equations (3.6)–(3.8) can be written in the following matrix form as:

$$\boldsymbol{MC} + \boldsymbol{\psi} = \boldsymbol{\varphi}, \tag{3.9}$$

where

$$\boldsymbol{M} = \begin{bmatrix} \alpha^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha_1 & \beta_1 & \alpha_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_1 & \beta_1 & \alpha_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & 0 & 0 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \alpha_1 & \beta_1 & \alpha_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha^* \end{bmatrix},$$

$$\boldsymbol{\psi} = (\gamma_1\gamma\psi_0, \gamma\psi_1, \gamma\psi_2, \ldots, \gamma_1\gamma\psi_N)^{\mathrm{T}},$$
$$\boldsymbol{\varphi} = (\gamma_1\varphi_0 - \alpha_1 g_1(t_{n+1}), \varphi_1, \varphi_2, \ldots, \gamma_1\varphi_N - \alpha_1 g_2(t_{n+1}))^{\mathrm{T}},$$
$$\boldsymbol{C} = (c_0, c_1, c_2, \ldots, c_N)^{\mathrm{T}}.$$

4. Convergence Analysis

Let $\bar{u}(x)$ be the exact solution of the boundary value problems (3.6)–(3.8), and also $U(x) = \sum_{j=-1}^{n+1} c_j \mathrm{TBS}_j(x)$ be the trigonometric B-spline collocation approximation to $\bar{u}(x)$. Due to the round of errors, we assume that $\bar{U}(x) = \sum_{j=-1}^{n+1} \bar{c}_j \mathrm{TBS}_j(x)$ be the computed B-spline approximation to $\bar{u}(x)$, where $\bar{c} = (\bar{c}_0, \bar{c}_1, \ldots, \bar{c_n})^{\mathrm{T}}$. To estimate $\|\bar{u}(x) - U(x)\|$, we must estimate the errors $\|\bar{u}(x) - \bar{U}(x)\|$ and $\|\bar{U}(x) - U(x)\|$, respectively. Following (3.9) for $\bar{U}(x)$, we get

$$\boldsymbol{M\bar{C}} + \boldsymbol{\bar{\psi}} = \boldsymbol{\bar{\varphi}}, \tag{4.1}$$

where $\boldsymbol{\bar{\psi}} = (\gamma_1\gamma\bar{\psi}_0, \gamma\bar{\psi}_1, \gamma\bar{\psi}_2, \ldots, \gamma_1\gamma\bar{\psi}_N)^{\mathrm{T}}$,

$$\boldsymbol{\bar{\varphi}} = (\gamma_1\bar{\varphi}_0 - \alpha_1 g_1(t_{n+1}), \bar{\varphi}_1, \bar{\varphi}_2, \ldots, \gamma_1\bar{\varphi}_N - \alpha_1 g_2(t_{n+1}))^{\mathrm{T}},$$
$$\boldsymbol{\bar{C}} = (\bar{c}_0, \bar{c}_1, \bar{c}_2, \ldots, \bar{c}_N)^{\mathrm{T}}.$$

Using the systems (3.9) and (4.1), and following [26], we obtain

$$\boldsymbol{M}(\boldsymbol{C} - \boldsymbol{\bar{C}}) + (\boldsymbol{\psi} - \boldsymbol{\bar{\psi}}) = \boldsymbol{\varphi} - \boldsymbol{\bar{\varphi}}, \tag{4.2}$$

Before we proceed, we need the following theorem.

Theorem 1. *Suppose that* $y(x) \in C^4[\bar{a}, \bar{b}]$ *and* $\bar{\Delta} \equiv \{\bar{a} = x_0 < x_1 < \cdots < x_{N-1} < x_N = \bar{b}\}$ *be a uniform partition of* $[\bar{a}, \bar{b}]$ *with a step size* h*. If* $z(x)$ *be the unique trigonometric B-spline approximation for* $y(x)$ *at the knots* $x_0, x_1, \ldots, x_{N-1}, x_N$*, then*

$$\begin{aligned} &\|z(x) - y(x)\| \leq O(h^3), \\ &\|z^{(k)}(x) - y^{(k)}(x)\| \leq O(h^2), \quad k = 1, 2, \\ &\|z^{(k)}(x) - y^{(k)}(x)\| \leq O(h), \quad k = 3. \end{aligned}$$

Proof. See [27].

Recall to Eq. (2.2) and Theorem 1, we get a bound on $\|\boldsymbol{\varphi} - \bar{\boldsymbol{\varphi}}\|$ as:

$$\begin{aligned} |\varphi_j - \bar{\varphi}_j| &= |\alpha u_j'' + \beta u_j + \gamma \psi_j - \alpha \bar{u}_j'' - \beta \bar{u}_j - \gamma \bar{\psi}_j| \\ &\leq |\alpha||u_j'' - \bar{u}_j''| + |\beta||u_j - \bar{u}_j| + |\gamma||\psi_j - \bar{\psi}_j|. \end{aligned}$$

Then using Theorem 1, we have $\|\boldsymbol{\varphi} - \bar{\boldsymbol{\varphi}}\| \leq |\alpha|O(h^2) + |\beta|O(h^3) + L(O(h^3) + O(h^2))$, where $\|\psi'(z)\| < L$, see [27,28].

$$\|\boldsymbol{\varphi} - \bar{\boldsymbol{\varphi}}\| \leq K h^2, \quad K = |\alpha| + L + (|\beta| + L)O(h). \tag{4.3}$$

Now for the term $(\boldsymbol{\psi} - \bar{\boldsymbol{\psi}})$ and apply the mean value theorem, we get

$$\boldsymbol{\psi} - \bar{\boldsymbol{\psi}} = (\psi_u(\propto_1)\boldsymbol{\omega}_1 + \psi_{u_x}(\propto_2)\boldsymbol{\omega}_2)(\boldsymbol{C} - \bar{\boldsymbol{C}}), \tag{4.4}$$

where $\propto_1$ and $\propto_2$ are in $[\bar{a}, \bar{b}]$ and ω_1 and ω_2 are given matrices which have the following form:

$$\boldsymbol{\omega}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \gamma_1 & \sigma & \gamma_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \gamma_1 & \sigma & \gamma_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & 0 & 0 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \gamma_1 & \sigma & \gamma_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\boldsymbol{\omega}_2 = \begin{bmatrix} \frac{-\lambda\sigma}{\gamma_1} & -2\lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda & 0 & -\lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & -\lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & 0 & 0 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & -\lambda \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\lambda & \frac{-\lambda\sigma}{\gamma_1} \end{bmatrix}$$

Using Eq. (4.4) into Eq. (4.2), we get

$$\bar{\boldsymbol{M}}(\boldsymbol{C} - \bar{\boldsymbol{C}}) = \boldsymbol{\varphi} - \bar{\boldsymbol{\varphi}}, \tag{4.5}$$

where $\bar{\boldsymbol{M}} = \boldsymbol{M} + \psi_u(\propto_1)\boldsymbol{\omega}_1 + \psi_{u_x}(\propto_2)\boldsymbol{\omega}_2$.

Then from Eq. (4.5), we have

$$\|\boldsymbol{C} - \bar{\boldsymbol{C}}\| \leq \|\bar{M}^{-1}\|\|\boldsymbol{\varphi} - \bar{\boldsymbol{\varphi}}\| \leq K h^2 \|\bar{M}^{-1}\|.$$

Using the properties of trigonometric B-spline basis, see [23,26,28], we note that trigonometric B-spline basis is defined only in the interval $[x_{j-2}, x_{j+2}]$ and outside of this interval it is zero. Therefore, $\mathrm{TBS}_j(x)$ is having non-vanishing values at the mesh points $[x_{j-2}, x_{j+2}]$ and at other mesh points the value of $\mathrm{TBS}_j(x)$ is zero. It is clear that from the definition of trigonometric B-spline basis, the derivatives of $\mathrm{TBS}_j(x)$ up to second order also have the same nature at the mesh points as in the case of $\mathrm{TBS}_j(x)$. Using these facts, we can say that the matrix $\|\bar{M}\|$ is a tridiagonal band matrix with nonzero entries and dominant principal diagonal elements. Hence, the matrix is nonsingular and $\bar{M}^{-1}$ is bounded. Then, we get

$$\|\boldsymbol{C} - \bar{\boldsymbol{C}}\| \leq K_1 h^2, \quad K_1 = K\|\bar{M}^{-1}\|. \tag{4.6}$$

□

Lemma 2. *The B-spline $TBS_{-1}, TBS_0, \ldots TBS_{N+1}$ defined in Eq.* (3.1), *satisfies the inequality*

$$\sum_{j=-1}^{n+1} |TBS_j(x)| \leq 6.$$

Proof. We know that $|\sum_{j=-1}^{n+1} \mathrm{TBS}_j(x)| \leq \sum_{j=-1}^{n+1} |\mathrm{TBS}_j(x)|$.

At any node x_j, we have

$$\begin{aligned}\sum_{j=-1}^{n+1} |\mathrm{TBS}_j(x)| &= |\mathrm{TBS}_{j-1}(x)| + |\mathrm{TBS}_j(x)| + |\mathrm{TBS}_{j+1}(x)| \\ &= |\gamma_1| + |\sigma| + |\gamma_1| \leq 4.\end{aligned}$$

Also, we have at any point in each subinterval $x_{j-1} \leq x \leq x_j$

$$\sum_{j=-1}^{N+1} |\mathrm{TBS}_j(x)| = |\mathrm{TBS}_{j-2}(x)| + |\mathrm{TBS}_{j-1}(x)| + |\mathrm{TBS}_j(x)| + |\mathrm{TBS}_{j+1}(x)| \leq 6$$

Then, we have

$$\sum_{j=-1}^{N+1} |\mathrm{TBS}_j(x)| = |\mathrm{TBS}_{j-2}(x)| + |\mathrm{TBS}_{j-1}(x)| + |\mathrm{TBS}_j(x)| \leq 6$$

Hence, this proves the lemma.

From $U(x) - \bar{U}(x) = \sum_{j=-1}^{n+1}(c_j - \bar{c}_j)\mathrm{TBS}_j(x)$, [28] and Lemma 2, we get that

$$\|\bar{U}(x) - U(x)\| \leq \bar{K}h^2, \quad \bar{K} = 6K_1. \tag{4.7}$$

□

Theorem 3. *Let $\bar{u}(x)$ be the exact solution of* (3.6)–(3.8) *and let $U(x)$ be the trigonometric B-spline collocation approximation, then*

$$\|\bar{u}(x) - U(x)\| = O(h^2).$$

Proof. From Theorem 1 and Eq. (4.7), we get that

$$\begin{aligned}\|\bar{u}(x) - U(x)\| &= \|\bar{u}(x) - U(x) + \bar{U}(x) - \bar{U}(x)\| \leq \|\bar{u}(x) - \bar{U}(x)\| \\ &\quad + \|\bar{U}(x) - U(x)\| \\ &\leq O(h^3) + \bar{K}h^2 \leq O(h^2).\end{aligned}$$

Suppose that $u(x,t)$ be the exact solution to (1.1)–(1.3) and $U(x,t)$ be the approximation to this solution, then we have $\|u(x,t_{n+1}) - U(x,t_{n+1})\| \leq O(\tau^2 + h^2)$. □

5. Stability Analysis

The stability of the proposed method is investigated by von Neumann method. The form of the proposed scheme takes the following form by letting $\psi(x,t,u,u^q,u_x) = 0$ for simplicity

$$\begin{aligned}(\alpha\delta + \beta\gamma_1)c_{j+1}^{n+1} &+ (\alpha\sigma_1 + \beta\sigma)c_j^{n+1} + (\alpha\delta + \beta\gamma_1)c_{j-1}^{n+1} + (\alpha_1\delta + \beta_1\gamma_1)c_{j+1}^{n} \\ &+ (\alpha_1\sigma_1 + \beta_1\sigma)c_j^{n} + (\alpha_1\delta + \beta_1\gamma_1)c_{j-1}^{n} + \frac{b\gamma_1}{k^2}c_{j-1}^{n-1} + \frac{b\sigma_1}{k^2}c_j^{n-1} \\ &+ \frac{b\gamma_1}{k^2}c_{j+1}^{n-1} = 0. \end{aligned} \tag{5.1}$$

where $\alpha_1 = c(1-\theta)$ and $\beta_1 = d(1-\theta) - \frac{a}{k} - \frac{b}{k^2}$.

We discuss two cases, the stability analysis for both PHI-Four and Allen–Cahn equations.

5.1. Stability Analysis of PHI-Four Equation

For the stability analysis of PHI-Four equation, we let $a = 0, b = 1, c = -1, d = -1$, then substitute by the values of γ_1, δ, σ and σ_1 in Eq. (5.1) and after simplification, one obtain

$$\begin{aligned}(r_1 + k^2\theta r_2)c_{j+1}^{n+1} &+ (r_3 - k^2\theta r_4)c_j^{n+1} + (r_1 + k^2\theta r_2)c_{j-1}^{n+1} \\ &+ (-2r_1 + k^2(1-\theta)r_2)c_{j+1}^{n} + (-2r_3 - k^2(1-\theta)r_4)c_j^{n} \\ &+ (-2r_1 - k^2(1-\theta)r_2)c_{j-1}^{n} + r_1c_{j-1}^{n-1} + r_3c_j^{n-1} + r_1c_{j+1}^{n-1} = 0, \end{aligned} \tag{5.2}$$

where

$$\begin{aligned} r_1 &= \sin^2(\bar{h})(32\cos(\bar{h}) + 16\cos(3\bar{h}))(1 + 2\cos(\bar{h})), \\ r_2 &= \sin(h)\sin(3\bar{h})(3 + 9\cos(\bar{h}))(1 + 2\cos(\bar{h})), \\ r_3 &= \sin(h)\sin(3\bar{h})(64\cos(\bar{h}) + 32\cos(3\bar{h})), \\ r_4 &= -\cos^2(\bar{h})\sin(h)\sin(3\bar{h})\mathrm{cosec}^2(\bar{h})(48\cos(\bar{h}) + 24\cos(3\bar{h})). \end{aligned}$$

Put $c_j^n = \xi^n e^{i\epsilon jh}, i = \sqrt{-1}$ in Eq. (5.2) and simplifying, we get

$$\xi^2\left[\mu_1\cos(\epsilon h) + \mu_2\right] + \xi\left[\mu_3\cos(\epsilon h) + \mu_4\right] + \left[r_1\cos(\epsilon h) + r_3\right] = 0. \tag{5.3}$$

where

$$\mu_1 = (r_1 + k^2\theta r_2), \quad \mu_2 = (r_3 - k^2\theta r_4), \quad \mu_3 = (-2r_1 + k^2(1-\theta)r_2),$$
$$\mu_4 = (-2r_3 - k^2(1-\theta)r_4).$$

Then Eq. (5.3) becomes

$$(P + \theta Q)\xi^2 + (-2P + (1-\theta)Q)\xi + P = 0. \tag{5.4}$$

where

$$P = r_3 + r_1\cos(\epsilon h), \quad Q = k^2(-r_4 + r_2\cos(\epsilon h)).$$

The necessary and sufficient condition for Eq. (5.4) to be stable ($|\xi| \le 1$), then we get:

$$\xi = \frac{-(-2P + (1-\theta)Q) \pm \sqrt{(-2P + (1-\theta)Q)^2 - 4P(P+\theta Q)}}{2(P+\theta Q)}.$$

After simplification, we get that

$$|\xi| = \sqrt{\frac{(2P - (1-\theta)Q)^2}{4(P+\theta Q)^2} + \frac{4PQ - ((1-\theta)Q)^2}{4(P+\theta Q)^2}},$$

Then we have

$$|\xi| = \sqrt{\frac{P^2 + P\theta Q}{P^2 + 2P\theta Q + (\theta Q)^2}} \le 1.$$

It is evidence that the scheme is unconditionally stable.

5.2. Stability Analysis of Allen–Cahn Equation

For the stability analysis of Allen–Cahn equation, we let $a = 1, b = 0, c = -1$ and $d = -1$, then substitute by the values of γ_1, δ, σ and σ_1 in Eq. (5.1), after simplification and following the procedure given in Sect. 5.1, one obtain

$$(P + \theta Q)\xi + (-P + (1-\theta)Q) = 0. \tag{5.5}$$

Rewrite Eq. (5.5) in the following form:

$$\xi = \frac{(-P + (1-\theta)Q) + iz}{(P+\theta Q) + iz},$$

where z is an arbitrary constant.

Then we have

$$|\xi| = \sqrt{\frac{(-P + (1-\theta)Q)^2 + z^2}{(P+\theta Q)^2 + z^2}} \le 1.$$

This means that the scheme is unconditionally stable.

6. Numerical Experiments and Discussion

To illustrate the performance and the accuracy of the presented method, three test problems are given in this section, one for PHI-Four equation and the other two test problems are for Allen–Cahn equation. We compute E_j norm described by the following relation

$$E_j = |u_j^{\text{exact}} - U_j^{\text{approximate}}|.$$

Also, all numerical computations were made using MATLAB R2009a.

Test Problem (1) Consider Eq. (1.1) with the initial and boundary conditions given in Eqs. (1.2)–(1.3) with the values of the constants as $a = 0, b = 1, c = -1, d = -1$ and $\psi(x, t, u^q, u_x) = -u^3$ which gives the PHI-Four equation as:

$$u_{tt} - u_{xx} - u + u^3 = 0, \tag{6.1}$$

with the initial and boundary conditions as follows:

Table 2. Absolute error for test Problem 1 at $\theta = \frac{1}{2}$, $0 < x < 1$ and $0 \leq \mathrm{t} \leq 0.01$

x/t	0.002	0.004	0.006	0.008	0.01
0	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00
0.1	1.28E−10	6.75E−10	1.63E−09	2.98E−09	4.75E−09
0.2	9.38E−10	2.70E−09	5.24E−09	8.55E−09	1.26E−08
0.3	6.94E−10	1.08E−10	1.73E−09	4.83E−09	9.22E−09
0.4	5.61E−09	1.28E−08	2.14E−08	3.15E−08	4.29E−08
0.5	1.31E−08	2.39E−08	3.23E−08	3.82E−08	4.15E−08
0.6	4.43E−08	9.06E−08	1.38E−07	1.87E−07	2.37E−07
0.7	1.18E−07	2.32E−07	3.41E−07	4.45E−07	5.43E−07
0.8	3.00E−07	6.01E−07	9.02E−07	1.20E−06	1.50E−06
0.9	5.53E−07	1.10E−06	1.64E−06	2.17E−06	2.69E−06
1	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00

$$\begin{aligned} u(x,0) &= 0, \quad u_t(x,0) = x, \quad u(0,t) = 0, \\ u(1,t) &= t + \frac{t^3}{6} - \frac{t^5}{20} + \frac{t^5}{120} - \frac{t^7}{140} - \frac{t^7}{840} - \frac{t^{11}}{23760} + \frac{t^{13}}{37440} - \frac{t^{15}}{168000} \\ &\quad + \frac{t^{17}}{2176000}, \end{aligned}$$

with the analytical exact solution as given in [7] as follows:

$$\begin{aligned} u(x,t) = xt + \frac{xt^3}{6} - \frac{x^3t^5}{20} + \frac{xt^5}{120} - \frac{xt^7}{140} - \frac{x^3t^7}{840} - \frac{x^3t^{11}}{23760} + \frac{x^5t^{13}}{37440} \\ - \frac{x^7t^{15}}{168000} + \frac{x^9t^{17}}{2176000}. \end{aligned} \tag{6.2}$$

Test Problem (2) Consider Eq. (1.1) with the initial and boundary conditions given in Eqs. (1.2) and (1.3) with the values of the constants as $a = 1, b = 0, c = -1, d = -1$ and $\psi(x,t,u^q,u_x) = -u^3$ which gives the Allan–Cahn equation as

$$u_t - u_{xx} - u + u^3 = 0, \tag{6.3}$$

with the initial condition

$$u(x,0) = -0.5 + 0.5\tanh(0.3536\,x),$$

and the boundary conditions

$$u(0,t) = -0.5+0.5\tanh(-0.75\,t), \quad u(1,t) = -0.5 + 0.5\tanh(0.3536 - 0.75t),$$

and the exact solution as follows from [3]

$$u(x,t) = -0.5 + 0.5\tanh(0.3536\,x - 0.75\,t) \tag{6.4}$$

Test Problem (3) Consider Eq. (1.1) with the initial and boundary conditions given in Eqs. (1.2) and (1.3) with the values of the constants as $a = 1, b = 0,$

Table 3. Absolute error for test problem 1 at $\theta = \frac{1}{2}$, $0 < x < 1$ and $0 \leq t \leq 0.1$

x/t	0.02	0.04	0.06	0.08	0.1
0	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00
0.1	9.33E−08	5.04E−07	1.45E−06	2.73E−06	4.29E−06
0.2	1.65E−08	1.20E−06	2.64E−06	5.05E−06	8.69E−06
0.3	6.84E−07	1.05E−06	4.59E−06	9.09E−06	1.35E−05
0.4	1.74E−06	4.35E−06	4.89E−06	6.86E−06	1.38E−05
0.5	8.61E−06	5.87E−06	6.52E−06	2.45E−05	3.75E−05
0.6	3.38E−05	3.44E−05	2.11E−05	8.31E−06	2.89E−05
0.7	0.000142	0.000102	6.91E−05	3.24E−05	0.000158
0.8	0.000585	0.000322	0.000352	0.000224	1.45E−05
0.9	0.00242	0.000703	0.000976	0.001062	0.000986
1	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00

Table 4. Comparison between the exact and approximate solutions for test Problem 1 at $\theta = \frac{1}{2}$, $0 < x < 1$ and $0 \leq t \leq 0.1$

x	t	Exact solution	Approximate solution
0	0	0	0
0.1	0.01	0.00100001	0.00100000
0.2	0.02	0.00400026	0.00400028
0.3	0.03	0.00900135	0.00900139
0.4	0.04	0.01600426	0.01600861
0.5	0.05	0.02501041	0.02500943
0.6	0.06	0.03602159	0.03604266
0.7	0.07	0.04903999	0.04901543
0.8	0.08	0.06406820	0.06429252
0.9	0.09	0.08110917	0.08006743
1	0.1	0.10016624	0.10016524

$c = -1, d = -1$ and $\psi(x, t, u^q, u_x) = -u^3$ which gives the Allan Cahn equation as

$$u_t - u_{xx} - u + u^3 = 0, \tag{6.5}$$

with the initial condition

$$u(x, 0) = \left(1 + \mathrm{e}^{-(\frac{\sqrt{2}}{2})x}\right)^{-1},$$

and the boundary conditions

$$u(0, t) = \left(1 + \mathrm{e}^{-\frac{3}{2}t}\right)^{-1}, \quad u(1, t) = \left(1 + \mathrm{e}^{-\left(\frac{\sqrt{2}}{2}\right)\left[1+\frac{3\sqrt{2}}{2}t\right]}\right)^{-1},$$

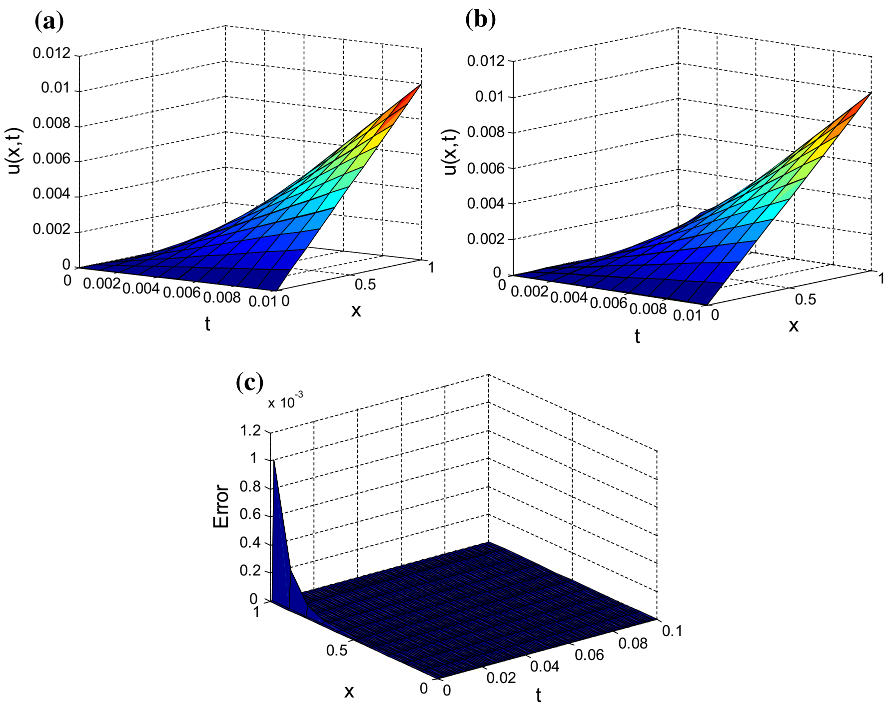

Figure 1. Space–time graph of Problem 1 at $\theta = \frac{1}{2}$, $0 < x < 1$ and $0 \leq t \leq 0.1$. **a** Approximate solution, **b** Exact solution, **c** Error

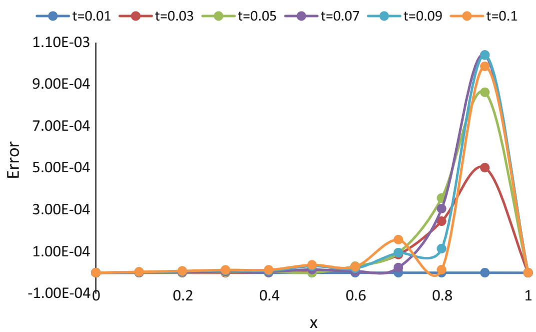

Figure 2. Maximum error for Problem 1 for different time levels at $\theta = \frac{1}{2}$, $0 < x < 1$ and $0 \leq t \leq 0.1$

and the exact solution as follows from [3]

$$u(x,t) = \left(1 + \mathrm{e}^{-\left(\frac{\sqrt{2}}{2}\right)\left[x + \frac{3\sqrt{2}}{2}t\right]}\right)^{-1}. \tag{6.6}$$

Table 5. Absolute error for Problem 2 at $\theta = \frac{1}{2}$, $0 < x < 1$ and $0 \leq t \leq 0.01$

x/t	0.001	0.003	0.005	0.007	0.009	0.01
0.0	0.000E+00	0.000E+00	1.110E−16	1.110E−16	0.000E+00	0.000E+00
0.1	2.448E−04	6.443E−04	9.687E−04	1.246E−03	1.492E−03	1.606E−03
0.2	2.000E−04	6.128E−04	1.016E−03	1.399E−03	1.761E−03	1.935E−03
0.3	1.797E−04	5.411E−04	9.082E−04	1.277E−03	1.644E−03	1.826E−03
0.4	1.594E−04	4.806E−04	8.052E−04	1.134E−03	1.465E−03	1.632E−03
0.5	1.410E−04	4.252E−04	7.124E−04	1.003E−03	1.296E−03	1.444E−03
0.6	1.242E−04	3.747E−04	6.280E−04	8.841E−04	1.143E−03	1.273E−03
0.7	1.091E−04	3.290E−04	5.520E−04	7.770E−04	1.003E−03	1.116E−03
0.8	9.502E−05	2.896E−04	4.824E−04	6.705E−04	8.536E−04	9.434E−04
0.9	8.886E−05	2.412E−04	3.725E−04	4.910E−04	6.009E−04	6.534E−04
1.0	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00

Table 6. Comparison between the exact and approximate solutions for test Problem 2 at $\theta = \frac{1}{2}$, $0 < x < 1$ and $0 \leq t \leq 0.01$

x	t	Exact solution	Approximate solution
0	0	−0.5	−0.5
0.1	0.001	−0.48294667	−0.48270191
0.2	0.002	−0.46584916	−0.46544517
0.3	0.003	−0.44881129	−0.44827076
0.4	0.004	−0.43186126	−0.43121885
0.5	0.005	−0.41504090	−0.41432848
0.6	0.006	−0.39839276	−0.39763717
0.7	0.007	−0.38195995	−0.38118064
0.8	0.008	−0.36571227	−0.36499250
0.9	0.009	−0.34910129	−0.34910400
1	0.01	−0.33054967	−0.33354385

Tables 2 and 3 show the absolute error for PHI-Four equation and a comparison between the exact and the approximate solutions is also reported in Table 4. Figure 1 shows the space–time graph of the exact, the approximate solutions and the error for PHI-Four equation. The error at different time levels is depicted in Fig. 2. While, Tables 5 and 6 show the absolute error for Allen–Cahn equation for Problems 2 and 3. Also, Table 6 reports the exact and the approximate solutions for Problem 2. Figures 3 and 5 show the space–time graph of the exact, the approximate solutions and the error for Problems 2 and 3, and finally, Figs. 4 and 6 the error at different time levels. We can conclude from these tables and figures that our numerical results are in good agreement with the exact solution for the given time-dependent problems.

Table 7. Absolute error for test problem 3 at $\theta = \frac{1}{2}$, $0 < x < 1$ and $0 \leq t \leq 0.01$

x/t	0.001	0.003	0.005	0.007	0.009	0.01
0.0	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00
0.1	2.509E−04	7.070E−04	1.127E−03	1.528E−03	1.917E−03	2.108E−03
0.2	3.351E−04	1.022E−03	1.715E−03	2.408E−03	3.100E−03	3.446E−03
0.3	4.389E−04	1.328E−03	2.233E−03	3.153E−03	4.085E−03	4.555E−03
0.4	5.456E−04	1.649E−03	2.768E−03	3.905E−03	5.057E−03	5.639E−03
0.5	6.542E−04	1.975E−03	3.314E−03	4.669E−03	6.042E−03	6.735E−03
0.6	7.626E−04	2.301E−03	3.857E−03	5.434E−03	7.028E−03	7.830E−03
0.7	8.697E−04	2.620E−03	4.399E−03	6.188E−03	7.968E−03	8.850E−03
0.8	9.653E−04	2.956E−03	4.902E−03	6.755E−03	8.510E−03	9.353E−03
0.9	1.161E−03	3.064E−03	4.617E−03	5.951E−03	7.136E−03	7.685E−03
1.0	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00

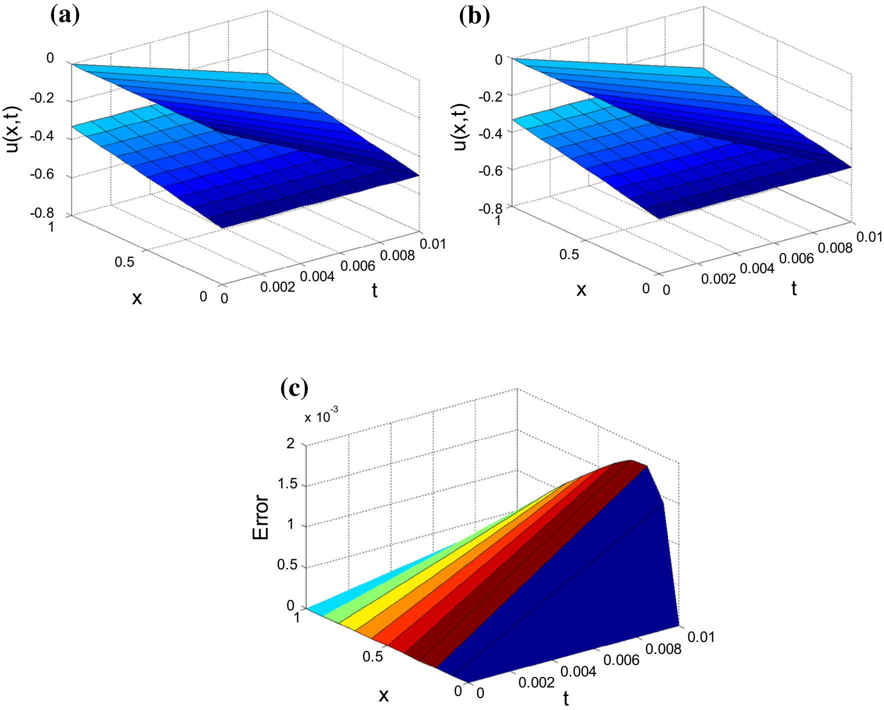

Figure 3. Space–time graph of Problem 2 at $\theta = \frac{1}{2}$, $0 < x < 1$ and $0 \leq t \leq 0.01$. **a** Approximate solution, **b** Exact solution, **c** Error

7. Conclusion

Nonpolynomial B-spline collocation method is implemented for the solution of time-dependent problems involving the PHI-Four and Allen–Cahn

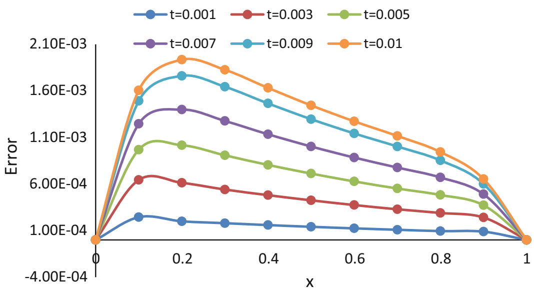

Figure 4. Maximum error for Problem 2 for different time levels at $\theta = \frac{1}{2}$, $0 < x < 1$ and $0 \leq t \leq 0.01$

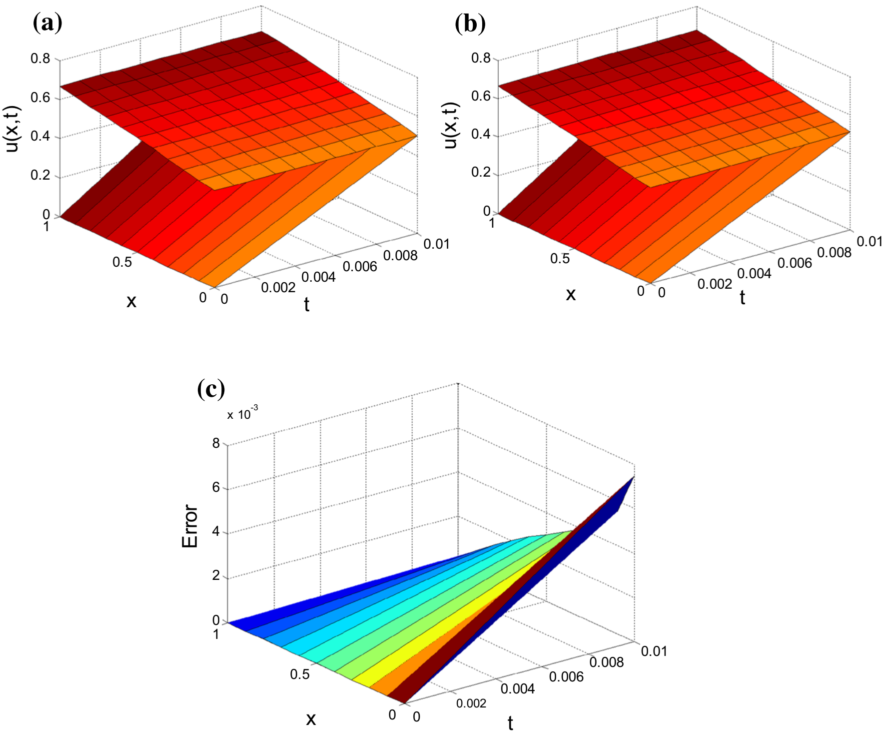

Figure 5. Space–time graph of Problem 3 at $\theta = \frac{1}{2}$, $0 < x < 1$ and $0 \leq t \leq 0.01$. **a** Approximate solution, **b** Exact solution, **c** Error

equations. The numerical solution is obtained using three-time-level implicit scheme based on a trigonometric cubic B-spline interpolant for spatial discretization and the θ-weighted scheme for temporal discretization. A detailed convergence analysis of the proposed method is shown. Also, von Neumann approach is applied to show that our scheme is unconditionally stable. It is evident from the three problems that the approximate solution is very close

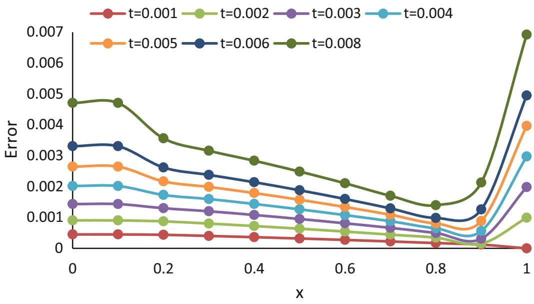

Figure 6. Maximum error for Problem 3 for different time levels at $\theta = \frac{1}{2}$, $0 < x < 1$ and $0 \leq t \leq 0.01$

to the exact solution and the results are very encouraging. The suggested scheme is easy, simple and good alternative to some other techniques when dealing with the numerical solution of time-dependent problems.

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