Mediterr. J. Math. (2017) 14:106
DOI 10.1007/s00009-017-0908-8
1660-5446/17/030001-13
published online April 10, 2017

Mediterranean Journal of Mathematics

On the Density Condition of a Multiresolution Analysis in Lebesgue Spaces

A. San Antolín

Abstract. We are interested in the problem of when the density condition in a multiresolution analysis defined in $L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, holds. Indeed, if $2 \leq p < \infty$, we obtain sufficient conditions on the generators of a multiresolution analysis in order to the density condition is satisfied. We emphasis on the requirement of the Fourier transform in a neighborhood of the origin. This involves the notion of density point. When $1 \leq p \leq 2$, the obtained condition is necessary. Moreover, we study the same problem when a multiresolution analysis is defined in the subspace of $L^\infty(\mathbb{R}^n)$ of the set of all continuous functions vanishing at infinite.

Mathematics Subject Classification. 42B10, 46F12, 43A15, 42A65.

Keywords. Fourier transform, Lebesgue spaces, multiresolution analysis, point of density.

1. Introduction

A multiresolution analysis is a general method for constructing orthonormal wavelets, Riesz basis of wavelets, and wavelet frames. Moreover, it plays a main role in approximation of functions spaces by dilated of shift-invariant subspaces. Here, we are interested in the problem of when the density condition in a multiresolution analysis defined in $L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, holds.

1.1. Notation and Basic Definitions

The sets of strictly positive integers, integers, and real numbers will be denoted by $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{R}$, respectively. Let $n \in \mathbb{N}$, if we write $L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, we mean the usual Lebesgue space, and $C_0(\mathbb{R}^n)$ will be the subspace of $L^\infty(\mathbb{R}^n)$ consisting of all continuous functions vanishing at infinity. For $1 < p < \infty$, the conjugate number q will be the real number, such that $\frac{1}{p} + \frac{1}{q} = 1$, and if $p = 1$, the conjugate will be $q = \infty$. We will denote $B_r(\mathbf{y}) = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{y}| < r\}$, and we will write B_r if $\mathbf{y}$ is the origin. If $A : \mathbb{R}^n \to \mathbb{R}^n$ is a linear map, A^* will mean the adjoint of A. With some

The author was partially supported by MEC/MICINN Grant #MTM2011-27998 (Spain) and by Generalitat Valenciana Grant GV/2015/035.

abuse in the notation, if we write A, we also mean the corresponding matrix respect to the canonical basis. Moreover $d_A = |\det A|$. For a Lebesgue measurable set $E \subset \mathbb{R}^n$, $E^c = \mathbb{R}^n \setminus E$ and the Lebesgue measure of E in $\mathbb{R}^n$ will be denoted by $|E|_n$. If $\mathbf{x} \in \mathbb{R}^n$, then $\mathbf{x} + E = \{\mathbf{x} + \mathbf{y} : \quad \text{for} \quad \mathbf{y} \in E\}$. We will denote $A(E) = \{ \mathbf{x} \in \mathbb{R}^n : \mathbf{x} = A(\mathbf{t}) \quad \text{for} \quad \mathbf{t} \in E \}$ and the volume of E changes under A according to $|AE|_n = d_A |E|_n$. The characteristic function of a set $E \subset \mathbb{R}^n$ will be denoted by χ_E, i.e., $\chi_E(\mathbf{x}) = 1$ if $\mathbf{x} \in E$, and $\chi_E(\mathbf{x}) = 0$ otherwise.

A linear map A is called expansive if all (complex) eigenvalues of A have absolute value greater than 1. It is said that a diagonalizable linear map A is isotropic if all (complex) eigenvalues of A have the same absolute value. If A is invertible, we consider the operator D_A on $L^p(\mathbb{R}^n)$, $1 \le p \le \infty$, defined by $D_A f(\mathbf{t}) = f(A\mathbf{t})$. The translation of a function $f \in L^p(\mathbb{R}^n)$ by $\mathbf{b} \in \mathbb{R}^n$ will be denoted by $\tau_{\mathbf{b}} f(\mathbf{t}) = f(\mathbf{t} - \mathbf{b})$. For a subspace S of $L^p(\mathbb{R}^n)$, $\mathbf{b} \in \mathbb{R}^n$, and $j \in \mathbb{Z}$,

$$D_A^j S = \{D_A^j f : f \in S\}, \quad \text{and} \quad \tau_{\mathbf{b}} S = \{\tau_{\mathbf{b}} f : f \in S\}.$$

Given $f : \mathbb{R}^n \to \mathbb{C}$ a Lebesgue measurable function, $\text{supp}(f) = \{\mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}) \ne 0\}$. Sets are defined modulo a null set. Some equations are understood except a null measurable set in $\mathbb{R}^n$.

Definition 1. Let $\mathbf{x} \in \mathbb{R}^n$, we will say that $\mathbf{x}$ is a point of density for a set $E \subset \mathbb{R}^d$, $|E|_n > 0$, if

$$\lim_{r \to 0} \frac{|E \cap B_r(\mathbf{x})|_n}{|B_r(\mathbf{x})|_n} = 1.$$

The following generalization of the classical notions of point of density was introduced in [4].

Definition 2. Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be an expansive linear map. Let $E \subseteq \mathbb{R}^n$ be a set of positive measure. We say that $\mathbf{x} \in \mathbb{R}^n$ is an A-density point for E if for any positive number r, one has

$$\lim_{j \to \infty} \frac{|E \cap [(A^{-j} B_r) + \mathbf{x}]|_n}{|A^{-j} B_r|_n} = 1\,.$$

Observe that when A is isotropic, the notion of A-density point coincides with the classical notion of density point.

Moreover, we need the following definition of [4].

Definition 3. Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be an expansive linear map. A measurable function $f : \mathbb{R}^n \to \mathbb{C}$ is said to be A-*locally nonzero* at a point $\mathbf{z} \in \mathbb{R}^n$ if for any $\varepsilon > 0$ and $r > 0$ there exists $j \in \mathbb{N}$, such that

$$|\{ \mathbf{x} \in (A^{-j} B_r + \mathbf{z}) : f(\mathbf{x}) = 0 \}|_n < \varepsilon |A^{-j} B_r|_n.$$

We adopt the convention that the Fourier transform of a function $f \in L^1(\mathbb{R}^n)$ is defined by

$$\widehat{f}(\mathbf{t}) = \int_{\mathbb{R}^n} f(\mathbf{x}) \mathrm{e}^{-2\pi i \mathbf{x} \cdot \mathbf{t}} \mathrm{d}\mathbf{x}.$$

The Fourier transform can be extended to the space of *tempered distributions* in the usual way. In particular, the Fourier transform is well defined on the spaces $L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, in the sense of tempered distributions. By the Hausdorff–Young inequality, if $f \in L^p(\mathbb{R}^n)$ with $1 \leq p \leq 2$, the Fourier transform of f can be considered as a function in $L^q(\mathbb{R}^n)$. Indeed, one has that

$$\|\widehat{f}\|_{L^q(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)}.$$

For more details on the Fourier transform defined on tempered distributions, one can read, e.g., [5].

For $2 \leq p < \infty$ and its conjugate number q, we consider the following space of functions:

$$\mathcal{I}L^p(\mathbb{R}^n) = \{f \in L^p(\mathbb{R}^n) \ : \ \widehat{f} \in L^q(\mathbb{R}^n)\}.$$

We also need the following:

$$\mathcal{I}C_0(\mathbb{R}^n) = \{f \in C_0(\mathbb{R}^n) \ : \ \widehat{f} \in L^r(\mathbb{R}^n) \text{ for some } 1 \leq r \leq 2\}.$$

1.2. Multiresolution Analysis and Historical Results on the Density Condition

Otherwise will be mentioned and to shorten the notation, if we write A, we mean a dilation given by a fixed expansive linear map $A : \mathbb{R}^n \to \mathbb{R}^n$, such that $A(\mathbb{Z}^n) \subset \mathbb{Z}^n$.

A multiresolution analysis was introduced in [15] (see also [16]) with the dyadic dilation in $L^2(\mathbb{R}^n)$, $n \geq 1$. A multiresolution analysis associated with A was considered, for instance, in [6,14,21,22]. Afterwards, the notion of generalized multiresolution analysis was introduced in [1]. Generalizations to the $L^p(\mathbb{R}^n)$ context appeared in [9,10,23]. Furthermore, multiresolution analyses on $L^p(\Omega)$ where Ω is a compact set were studied in [12].

By a multiresolution analysis defined on $L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, and associated with a dilation A (A-MRA), we will mean a sequence of closed subspaces $\{V_j : j \in \mathbb{Z}\} \subset L^p(\mathbb{R}^n)$, that satisfies the following conditions:

(i) $\forall j \in \mathbb{Z}, \quad V_j \subset V_{j+1}$.
(ii) $\forall j \in \mathbb{Z}, \quad f \in V_j \Leftrightarrow D_A f \in V_{j+1}$.
(iii) $\cap_{j\in\mathbb{Z}} V_j = \{\mathbf{0}\}$.
(iv) $\overline{\cup_{j\in\mathbb{Z}} V_j} = L^p(\mathbb{R}^n)$.
(v) There exists a countable set of function $\Phi = \{\phi_1, \phi_2, \ldots\} \subset V_0$, such that

$$V_0 = \overline{\text{span}}\{\ \tau_{\mathbf{k}}\phi \ : \ \phi \in \Phi, \mathbf{k} \in \mathbb{Z}^n\ \}$$

where the closure is in $L^p(\mathbb{R}^n)$.

In this work, we study the problem of when $W = \cup_{j\in\mathbb{Z}} V_j$ is dense in $L^p(\mathbb{R}^n)$ if the subspaces in $\{V_j \ : \ j \in \mathbb{Z}\}$ satisfy the conditions (i), (ii), and (v) in the definition of an A-MRA. This problem was extensively studied in the literature. Let us focus on those known results that use conditions of different nature on the generators of the subspace V_0. The first necessary and sufficient conditions on $\phi \in L^2(\mathbb{R})$, a generator of a core subspace V_0, to have that $\cup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$ were proved by Madych [14] and by de Boor et al. [2] independently. The result proved by Madych is the following.

Theorem A. *Let $\phi \in L^2(\mathbb{R}^n)$, such that $\{\tau_{\mathbf{k}}\phi \ : \ \mathbf{k} \in \mathbb{Z}^n\}$ is an orthonormal system. Suppose $\{V_j \ : \ j \in \mathbb{Z}\}$ is a sequence of closed subspaces of $L^2(\mathbb{R}^n)$ which enjoys the properties* (ii) *and* (v) *with $\Phi = \{\phi\}$. If $P_j f$ denotes the orthogonal projection of f onto V_j, then the following conditions are equivalent:*

(a) *For all f in $L^2(\mathbb{R}^n)$*

$$\lim_{j\to\infty} \| f - P_j f \| = 0.$$

(b) *The function ϕ in* (v) *satisfies*

$$\lim_{j\to\infty} \frac{1}{|(A^*)^{-j}Q|} \int_{(A^*)^{-j}Q} |\widehat{\phi}(\mathbf{t})|^2 \mathrm{d}\mathbf{t} = 0$$

for every cube Q of finite diameter in $\mathbb{R}^n$.

The result by de Boor, DeVore, and Ron is the following.

Theorem B. *Let $\phi \in L^2(\mathbb{R}^n)$ be such that if V_0 is the $L^2(\mathbb{R}^n)$-closure of the finite linear combinations of the multi-integer translates of ϕ and let, for any $j \in \mathbb{Z}$,*

$$V_j = \{f(2^j\mathbf{x}) \ : \ f \in V_0\}.$$

If the condition (i) *is satisfied for the sequence of subspaces V_j, $j \in \mathbb{Z}$, then the condition* (iv) *holds if and only if*

$$\cup_{j\in\mathbb{Z}} \ (2^j \ supp \ \widehat{\phi}) = \mathbb{R}^n \quad \textit{(modulo a null set)},$$

where

$$supp \ \widehat{\phi} = \{\mathbf{t} \in \mathbb{R}^n \ : \ \widehat{\phi}(\mathbf{t}) \neq 0\}.$$

Afterwards, Hernández et al. [7] (see also [8]) proved the following result.

Theorem C. *Let $\phi \in L^2(\mathbb{R})$, such that $\{\tau_k\phi \ : \ \mathbf{k} \in \mathbb{Z}\}$ is an orthonormal system. Let $V_0 = \overline{span}\{\tau_k \ : \ k \in \mathbb{Z}\}$. Suppose $\{V_j \ : \ j \in \mathbb{Z}\}$ is a sequence of closed subspaces of $L^2(\mathbb{R})$ satisfying the properties* (i) *and* (ii) *with de dyadic dilation. Then, the condition* (iv) *holds if and only if $\lim_{j\to\infty} |\widehat{\phi}(2^{-j}\xi)| = 1$, a.e.*

A generalization of this result to the context of an A-MRA when the core subspace V_0 is generated by the shift of several scaling functions was formulated by Calogero [3].

More necessary and sufficient conditions were proved by Lorentz et al. [13].

Theorem D. *Let $\phi \in L^2(\mathbb{R})$, such that $\{\tau_k\phi \ : \ k \in \mathbb{Z}\}$ is an orthonormal system. Let V_j, $j \in \mathbb{Z}$, be a sequence of closed subspaces of $L^2(\mathbb{R})$ satisfying* (i), (ii), and (v) *with $\Phi = \{\phi\}$ and the dyadic dilation. Then, the condition* (iv) *is equivalent to the following conditions:*

(a) $\lim_{j\to\infty} |\widehat{\phi}(2^{-j}y)|$ *exists and is positive for a.e. $y \in \mathbb{R}$;*

(b) *The set* $\{y \in \mathbb{R} \ : \mid \widehat{\phi}(y) \mid > 0\}$ *is dyadically absorbing, i.e., for a.e.* $y \in \mathbb{R}$*, there exists a positive integer* j_0*, which may depend on* y*, such that if* $j \geq j_0$*, then* $\mid \widehat{\phi}(2^{-j}y) \mid > 0$*.*

(c) $\lim_{j\to\infty} 2^j \phi * \widetilde{\phi}(2^j y)$ *exists in the distributional sense and is a nonzero multiple of the Dirac distribution at the origin. Here,* $\widetilde{\phi}(y) = \overline{\phi(-y)}$ *and* $*$ *denotes the usual convolution.*

In [4], a necessary and sufficient condition to have the density condition in a multiresolution analysis is given in terms of the classical notion of density point and approximate continuity.

Theorem E. *Let* $\phi \in L^2(\mathbb{R}^n)$*, such that* $\{\tau_{\mathbf{k}}\phi \ : \ \mathbf{k} \in \mathbb{Z}^n\}$ *is an orthonormal system. Let* V_j *be a sequence of closed subspaces in* $L^2(\mathbb{R}^n)$ *satisfying the conditions* (i), (ii)*, and* (v) *with* $\Phi = \{\phi\}$*. Then, the following conditions are equivalent:*

(a) $\overline{\cup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}^n)$.

(b) $\widehat{\phi}$ *is* A^**−locally nonzero at the origin.*

(c) *the origin is a point of* A^**−approximate continuity of the function* $|\widehat{\phi}|$ *if we set* $|\widehat{\phi}(\mathbf{0})| = 1$*.*

When Φ is a finite number of functions, a generalization of these last results was proved by Saliani [17]. If it is assumed that a core subspace V_0 is generated by the shifts of a single function, the above result was generalized in [11]. The paper by Soto-Bajo [20] deals with all these mentioned conditions in the context of A-MRA's defined in reducing subspaces and where the core subspace V_0 may be generated by the shift of a non-finite number of functions. In [18], if V_0 is generated by the translations by integers of a function $\phi \in L^2(\mathbb{R}^n)$ and the subspaces V_j are not necessarily nested, the author proved necessary and sufficient conditions on ϕ to have that $\cup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R}^n)$.

In the $L^p(\mathbb{R}^n)$ context, under some decay conditions of a single generator ϕ of V_0 and if $\widehat{\phi}(0) \neq 0$, Jia and Micchelli [10] proved that $\cup_{j\in\mathbb{Z}} V_j$ is dense in $L^p(\mathbb{R}^n)$, $1 \leq p < \infty$.

Theorem F. *If* ϕ *is a function defined on* $\mathbb{R}^n$*, such that*

$$\int_{[0,1)^n} \left| \left(\sum_{\mathbf{k}\in\mathbb{Z}^n} |\phi(\mathbf{x}-\mathbf{k})| \right) \right|^p \mathrm{d}\mathbf{x} < \infty, \quad (1 \leq p < \infty),$$

and $\sum_{\mathbf{k}\in\mathbb{Z}^n} \phi(\mathbf{x}-\mathbf{k}) = 1$*, then for any* $f \in L^p(\mathbb{R}^n)$*,*

$$\left\| f - \sum_{\mathbf{k}\in\mathbb{Z}^n} a_h(\mathbf{k}) \phi(h^{-1} \cdot - \mathbf{k}) \right\|_p \to 0 \quad as \quad h \to 0^+,$$

where

$$a_h(\mathbf{k}) = a_h(f, \mathbf{k}) := h^{-n} \int_{h\mathbf{k}+[0,h)^n} f(\mathbf{x}) \mathrm{d}\mathbf{x} = \int_{[0,h)^n} f(h(\mathbf{x}+\mathbf{k})) \mathrm{d}\mathbf{x}.$$

Zhao [23] obtained an improvement of the above result, because the decay assumptions are weaker, namely, ϕ is in $L^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$. From a different point of view, Zhao also proved the following.

Theorem G. *Let $2 \le p < \infty$ and $1/h$ an integer ≥ 2. Assume that $\varphi \in L^p(\mathbb{R}^n)$, such that $\varphi = \widehat{\psi}$ for some $\psi \in L^q(\mathbb{R}^n)$ and*

$$\mathbb{R}^n \setminus \cup_{j\in\mathbb{Z}} \left((1/h)^j \ supp\ \widehat{\phi}\right) \tag{1}$$

is a null set. Let $\{V_j\}_{j\in\mathbb{Z}}$ be a nested sequence of closed subspaces in $L^p(\mathbb{R}^n)$ satisfying (ii). Then, the span of $\cup_{j\in\mathbb{Z}} V_j$ is dense in $L^p(\mathbb{R}^n)$ if the closure of span of $\cup_{j\in\mathbb{Z}} V_j$ contains φ.

When $1 \le p \le 2$, the condition (1) is necessary is also proved. Finally, the density of $\cup_{j\in\mathbb{Z}} V_j$ in the space L^∞ is studied. A generalization of Zhao's results was obtained by Jia [9], because he worked with finitely generated shift-invariant subspaces and a general dilation matrix. Note that Jia also emphasis in the case when the generators functions of V_0 are compactly supported functions.

We study conditions on the Fourier transform of a single generator of a subspace V_0 in an multiresolution analysis with the dyadic dilation to have $W = \cup_{j\in\mathbb{Z}} V_j$ is dense in $L^p(\mathbb{R}^n)$, $1 \le p \le \infty$. Indeed, when $2 \le p < \infty$, we obtain sufficient conditions in Theorem 1 below. When $p = \infty$, we consider the subspace $C_0(\mathbb{R}^n)$ and our sufficient conditions are written in Theorem 2 below. We focus on the requirement of the Fourier transform of the generators in a neighborhood of the origin and this is given in terms of *density point*. In Proposition 1 below, we prove that this condition is necessary when $1 \le p \le 2$. Although the results we prove here do not appear in the literature in the above context, we prove our results when an A-MRA with a countable number of generators is considered. In this context, we need the notion of A-density point. The condition depends of the dilation A. When the dilation is given by a diagonal matrix with equal entries in the diagonal and greater than 2, the condition (1) also depends of the dilation, but our condition is independent of such a dilation. This is why Corollary 1, 2, and 4 have been written explicitly.

The remainder of this work is the following. In Sect. 2, we write our main results and their proofs can be found in Sect. 3.

2. Main Results

We collect the main results of this manuscript in this section.

When $2 \le p < \infty$, we prove the following sufficient conditions.

Theorem 1. *Let $2 \le p < \infty$, let $\Phi = \{\phi_1, \phi_2, \dots\} \subset \mathcal{I}L^p(\mathbb{R}^n)$, and let $V_j = \overline{span}\{D_A^j \tau_{\mathbf{k}} \phi \ : \ \mathbf{k} \in \mathbb{Z}^n,\ \phi \in \Phi\}$, $j \in \mathbb{Z}$, where the closure is in $L^p(\mathbb{R}^n)$. Assume that $V_j \subset V_{j+1}$. If $\chi_{\cup_{a=1}^{\infty} \operatorname{supp} \widehat{\phi_a}}$ is A^*-locally nonzero at the origin, then $\cup_{j\in\mathbb{Z}} V_j$ is dense in $L^p(\mathbb{R}^n)$.*

Having into account that the definition of point of density and point of A-density are equivalent when A is an isotropic expansive linear map, we have the following.

Corollary 1. *Let $2 \le p < \infty$, let $A : \mathbb{R}^n \to \mathbb{R}^n$ be an expansive isotropic linear map, such that $A(\mathbb{Z}^n) \subset \mathbb{Z}^n$, and let $\Phi = \{\phi_1, \phi_2, \dots\} \subset \mathcal{I}L^p(\mathbb{R}^n)$. Let $V_j = \overline{span}\{D_A^j \tau_{\mathbf{k}} \phi \ : \ \mathbf{k} \in \mathbb{Z}^n, \ \phi \in \Phi\}$, $j \in \mathbb{Z}$, where the closure is in $L^p(\mathbb{R}^n)$. Assume that $V_j \subset V_{j+1}$. If the origin is a point of density for $\cup_{a=1}^{\infty} \operatorname{supp} \widehat{\phi_a}$, then $\cup_{j\in\mathbb{Z}} V_j$ is dense in $L^p(\mathbb{R}^n)$.*

We have the following necessary condition when $1 \le p \le 2$.

Proposition 1. *Let $1 \le p \le 2$, let $\Phi = \{\phi_1, \phi_2, \dots\} \subset L^p(\mathbb{R}^n)$ and let $V_j = \overline{span}\{D_A^j \tau_{\mathbf{k}} \phi \ : \ \mathbf{k} \in \mathbb{Z}^n, \ \phi \in \Phi\}$, $j \in \mathbb{Z}$, where the closure is in $L^p(\mathbb{R}^n)$. Assume that $V_j \subset V_{j+1}$. If $\cup_{j\in\mathbb{Z}} V_j$ is dense in $L^p(\mathbb{R}^n)$, then the origin is a point of A^*-density for $\cup_{a=1}^{\infty} \operatorname{supp} \widehat{\phi_a}$.*

We have the following.

Corollary 2. *Let $1 \le p \le 2$, let $A : \mathbb{R}^n \to \mathbb{R}^n$ be an expansive isotropic linear map, such that $A(\mathbb{Z}^n) \subset \mathbb{Z}^n$ and let $\Phi = \{\phi_1, \phi_2, \dots\} \subset L^p(\mathbb{R}^n)$. Let $V_j = \overline{span}\{D_A^j \tau_{\mathbf{k}} \phi \ : \ \mathbf{k} \in \mathbb{Z}^n, \ \phi \in \Phi\}$, $j \in \mathbb{Z}$, where the closure is in $L^p(\mathbb{R}^n)$. Assume that $V_j \subset V_{j+1}$. If $\cup_{j\in\mathbb{Z}} V_j$ is dense in $L^p(\mathbb{R}^n)$, then the origin is a point of density for $\cup_{a=1}^{\infty} \operatorname{supp} \widehat{\phi_a}$.*

A straight consequence of Theorem 1 and Proposition 1 is the following version of Theorem 1.6 in [20].

Corollary 3. *Let $\Phi := \{\phi_1, \phi_2, \dots\} \subset L^2(\mathbb{R}^n)$ and let $V_j = \overline{span}\{D_A^j \tau_{\mathbf{k}} \phi : \mathbf{k} \in \mathbb{Z}^n, \ \phi \in \Phi\}$, $j \in \mathbb{Z}$, where the closure is in $L^2(\mathbb{R}^n)$. Assume that $V_j \subset V_{j+1}$. The following assertions are equivalent:*

(a) *$\cup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R}^n)$.*
(b) *The origin is a point of A^*-density for $\cup_{a=1}^{\infty} \operatorname{supp} \widehat{\phi_a}$.*
(c) *The function $\chi_{\cup_{a=1}^{\infty} \operatorname{supp} \widehat{\phi_a}}$ is A^*-locally nonzero at the origin.*

Instead of an MRA in $L^\infty(\mathbb{R}^n)$, we consider a multiresolution analysis defined in $C_0(\mathbb{R}^n)$. We prove the following.

Theorem 2. *Let $\Phi := \{\phi_1, \phi_2, \dots\} \subset \mathcal{I}C_0(\mathbb{R}^n)$ and let $V_j = \overline{span}\{D_A^j \tau_{\mathbf{k}} \phi : \mathbf{k} \in \mathbb{Z}^n, \ \phi \in \Phi\}$, $j \in \mathbb{Z}$, where the closure is in $C_0(\mathbb{R}^n)$. Assume that $V_j \subset V_{j+1}$. If $\chi_{\cup_{a=1}^{\infty} \operatorname{supp} \widehat{\phi_a}}$ is A^*-locally nonzero at the origin, then $\cup_{j\in\mathbb{Z}} V_j$ is dense in $C_0(\mathbb{R}^n)$.*

We have the following corollary.

Corollary 4. *Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be an expansive isotropic linear map, such that $A(\mathbb{Z}^n) \subset \mathbb{Z}^n$ and let $\Phi := \{\phi_1, \phi_2, \dots\} \subset \mathcal{I}C_0(\mathbb{R}^n)$. Let $V_j = \overline{span}\{D_A^j \tau_{\mathbf{k}} \phi \ : \ \mathbf{k} \in \mathbb{Z}^n, \ \phi \in \Phi\}$, $j \in \mathbb{Z}$, where the closure is in $C_0(\mathbb{R}^n)$. Assume that $V_j \subset V_{j+1}$. If the origin is a point of density for $\cup_{a=1}^{\infty} \operatorname{supp} \widehat{\phi_a}$, then $\cup_{j\in\mathbb{Z}} V_j$ is dense in $C_0(\mathbb{R}^n)$.*

3. Proofs of the Main Results

We need the following previous lemmas. The next lemma was proved in [4].

Lemma A. *The set* $P = \cup_{k=1}^{\infty} A^{-k}(\mathbb{Z}^n)$ *is dense in* $\mathbb{R}^n$.

The following is a version of Theorem 2.1 in [23].

Lemma B. *Let* $1 \leq p < \infty$, $j \in \mathbb{Z}$, *let* $\Phi := \{\phi_1, \phi_2, \dots\} \subset L^p(\mathbb{R}^n)$, *and let* $V_j = \overline{span}\{D_A^j \tau_{\mathbf{k}} \phi \ :\ \mathbf{k} \in \mathbb{Z}^n,\ \phi \in \Phi\}$, *where the closure is in* $L^p(\mathbb{R}^n)$. *Assume that* $V_j \subset V_{j+1}$. *Then,* $\overline{\cup_{j \in \mathbb{Z}} V_j}$, *where the closure is in* $L^p(\mathbb{R}^n)$, *is invariant under a translation by any* $\mathbf{b} \in \mathbb{R}^n$. *The result remains true if we replace* $L^p(\mathbb{R}^n)$ *by* $C_0(\mathbb{R}^n)$.

Proof. First, we consider that V_j is in $L^p(\mathbb{R}^n)$, $1 \leq p < \infty$. We show that $W := \overline{\cup_{j \in \mathbb{Z}} V_j}$ is invariant under translations by vectors $\mathbf{m} \in P$, where P is defined in Lemma A. Let $\mathbf{m} \in P$, then $\mathbf{m} \in P_\ell$ for some $\ell \in \mathbb{N}$. For any $f \in W$ and $\forall \varepsilon > 0$, there exist $j_0 \in \mathbb{N}$ and $h \in V_{j_0}$, such that $\|f - h\|_{L^p(\mathbb{R}^n)} < \varepsilon$. By (i), when $j \geq j_0$, we have $h \in V_j$, and therefore, $h(\mathbf{x}) = \sum_{a=1}^{\infty} \sum_{\mathbf{k} \in \mathbb{Z}^n} c_{\mathbf{k}}^{(j)} \phi_a(A^j \mathbf{x} - \mathbf{k})$ with convergence in $L^p(\mathbb{R}^n)$. Hence,

$$\tau_{\mathbf{m}} h(\mathbf{x}) = h(\mathbf{x} - \mathbf{m}) = \sum_{a=1}^{\infty} \sum_{\mathbf{k} \in \mathbb{Z}^n} c_{\mathbf{k}}^{(j)} \phi_a(A^j \mathbf{x} - A^j \mathbf{m} - \mathbf{k}).$$

If $j > \max\{\ \ell, j_0\ \}$, then $A^j \mathbf{m} \in \mathbb{Z}^n$. Consequently, $\tau_{\mathbf{m}} h \in V_j$, and therefore, $\tau_{\mathbf{m}} f \in W$. Finally, the density of P in $\mathbb{R}^n$, the closedness of the subspace W, and the continuity of the operator $\tau_{\mathbf{b}}$ in $L^p(\mathbb{R}^n)$ yield the invariance of W under any translation.

When the subspaces V_j are in $C_0(\mathbb{R}^n)$, a proof of the lemma can be done in a similar way. $\square$

We also need the following.

Lemma C. *Let* $2 \leq p < \infty$, *let* $u \in \mathcal{I}L^p(\mathbb{R}^n)$ *and let* $v \in L^q(\mathbb{R}^n)$ *where* $p^{-1} + q^{-1} = 1$. *Then*

$$\int_{\mathbb{R}^n} u(\mathbf{x}) v(\mathbf{x})\, \mathrm{d}\mathbf{x} = \int_{\mathbb{R}^n} \widehat{u}(\mathbf{t}) \widehat{v}(\mathbf{t})\, \mathrm{d}\mathbf{t}.$$

Proof. First, the hypothesis $u \in \mathcal{I}L^p(\mathbb{R}^n)$ and Hausdorff–Young inequality imply that there exists $w \in L^q(\mathbb{R}^n)$, such that $\widehat{u} = \widehat{\widehat{w}} = w(-\cdot) \in L^q(\mathbb{R}^n)$. Moreover, we have $\widehat{v} \in L^p(\mathbb{R}^n)$.

On the other hand, denote by $\mathcal{S}$, the class of Schwartz functions in $\mathbb{R}^n$. Since $\mathcal{S}$ is dense in $L^r(\mathbb{R}^n)$ for any $1 \leq r < \infty$, there exist $\{u_n\}_{m=1}^{\infty}, \{v_n\}_{m=1}^{\infty} \subset \mathcal{S}$, such that

$$\|v - v_m\|_{L^q(\mathbb{R}^n)} \to 0 \quad \text{and} \quad \|\widehat{u} - \widehat{u_m}\|_{L^q(\mathbb{R}^n)} \to 0, \quad \text{as } m \to \infty.$$

By Hausdorff–Young inequality, we have

$$\|\widehat{v} - \widehat{v_m}\|_{L^p(\mathbb{R}^n)} \leq \|v - v_m\|_{L^q(\mathbb{R}^n)} \to 0 \quad \text{as } m \to \infty,$$

and

$$\begin{aligned}\|u - u_m\|_{L^p(\mathbb{R}^n)} &= \|u(-\cdot) - u_m(-\cdot)\|_{L^p(\mathbb{R}^n)} = \|\widehat{\widehat{u}} - \widehat{\widehat{u_m}}\|_{L^p(\mathbb{R}^n)} \\ &\leq \|\widehat{u} - \widehat{u_m}\|_{L^q(\mathbb{R}^n)} \to 0,\end{aligned}$$

as $m \to \infty$. Furthermore, by Parseval's formula for functions in $\mathcal{S}$ and the Hölder inequality, we have

$$\begin{aligned}&\left|\int_{\mathbb{R}^n} u(\mathbf{x})v(\mathbf{x}) - \widehat{u}(\mathbf{x})\widehat{v}(\mathbf{x})\,\mathrm{d}\mathbf{x}\right| \\ &\quad \leq \|v\|_{L^q(\mathbb{R}^n)}\|u - u_m\|_{L^p(\mathbb{R}^n)} + \|u_m\|_{L^p(\mathbb{R}^n)}\|v - v_m\|_{L^q(\mathbb{R}^n)} \\ &\qquad + \|\widehat{v_m}\|_{L^p(\mathbb{R}^n)}\|\widehat{u} - \widehat{u_m}\|_{L^q(\mathbb{R}^n)} + \|\widehat{u}\|_{L^q(\mathbb{R}^n)}\|\widehat{v} - \widehat{v_m}\|_{L^p(\mathbb{R}^n)} \to 0,\end{aligned}$$

as $m \to \infty$, that is what we wanted to prove. □

We are ready to prove Theorem 1.

Proof of Theorem 1. Assume that $W := \overline{\cup_{j\in\mathbb{Z}} V_j}$ is not $L^p(\mathbb{R}^n)$. Then, by Hahn–Banach theorem, there exists a nonzero function $g \in L^q(\mathbb{R}^n)$, such that

$$\int_{\mathbb{R}^n} g(-\mathbf{t})f(\mathbf{t})\mathrm{d}\mathbf{t} = 0, \quad \forall f \in W.$$

Since W is translation invariant,

$$(g * f)(\mathbf{x}) = \int_{\mathbb{R}^n} g(\mathbf{t})f(\mathbf{x} - \mathbf{t})\mathrm{d}\mathbf{t} = 0, \quad \forall \mathbf{x} \in \mathbb{R}^n, \quad \forall f \in W.$$

In particular

$$\left(g * D_A^j \phi_a\right)(\mathbf{x}) = 0, \quad \forall j \in \mathbb{Z}, \quad a = 1, 2, \ldots \tag{2}$$

By Lemma C and (2), we have

$$\int_{\mathbb{R}^n} \mathrm{e}^{2\pi i \mathbf{t}\cdot\mathbf{x}} \widehat{g}(\mathbf{t})\widehat{D_A^j \phi_a}(\mathbf{t})\,\mathrm{d}\mathbf{t} = \left(g * D_A^j \phi_a\right)(\mathbf{x}) = 0, \quad \forall j \in \mathbb{Z}, \quad a = 1, 2, \ldots$$

Thus, having in mind that $\widehat{g}\widehat{D_A^j \phi_a} \in L^1(\mathbb{R}^n)$, we have $\widehat{g}(\mathbf{t})D_{A^*}^{-j}\widehat{\phi_a}(\mathbf{t}) = 0$ a.e. $\forall j \in \mathbb{Z}$, $a = 1, 2, \ldots$

According to our hypothesis, for any positive integer N and $r > 1$, there exists $k \in \mathbb{N}$, such that

$$\left|\left\{\mathbf{t} \in (A^*)^{-k}B_r : \chi_{\cup_{a=1}^{\infty} \operatorname{supp} \widehat{\phi_a}}(\mathbf{t}) = 0\right\}\right|_n < \frac{|(A^*)^{-k}B_r|_n}{N}.$$

Then

$$|\{\, \mathbf{t} \in (A^*)^{-k}B_r : \widehat{g}((A^*)^j\mathbf{t}) \neq 0 \,\}|_n < \frac{|(A^*)^{-k}B_r|_n}{N}$$

and, therefore, taking $j = k$, we obtain

$$|\{\, \mathbf{y} \in B_r : \widehat{g}(\mathbf{y}) \neq 0 \,\}|_n < \frac{|B_r|_n}{N}.$$

Letting $N \to \infty$, we obtain

$$|\{\, \mathbf{y} \in B_r : \widehat{g}(\mathbf{y}) \neq 0 \,\}|_n = 0.$$

Hence, $\widehat{g}(\mathbf{t}) = 0$ a.e. and $g(\mathbf{x}) = 0$ a.e. follows. This is a contradiction with the assumption of W is not dense in $L^p(\mathbb{R}^n)$. This finishes the proof. $\square$

For the proof of Proposition 1, we need the following lemmas.

Lemma D. *Let $1 \leq p \leq 2$, $j \in \mathbb{Z}$, let $\Phi = \{\phi_1, \phi_2, \dots\} \subset L^p(\mathbb{R}^n)$ and let $V_j = \overline{span}\{D_A^j \tau_{\mathbf{k}} \phi \ : \ \mathbf{k} \in \mathbb{Z}^n, \ \phi \in \Phi\}$, where the closure is in $L^p(\mathbb{R}^n)$. Assume that $V_j \subset V_{j+1}$. Then*

$$\cup_{a=1}^{\infty} \operatorname{supp} \widehat{\phi_a} \subset A^* \left(\cup_{a=1}^{\infty} \operatorname{supp} \widehat{\phi_a} \right).$$

Proof. For $a \in \{1, 2, \dots,\}$, we have $\phi_a(A^{-1}\mathbf{x}) \in V_{-1} \subset V_0$. By definition of V_0, we can write

$$\phi_a(A^{-1}\mathbf{x}) = \sum_{\mathbf{k} \in \mathbb{Z}^n} \sum_{\ell=1}^{\infty} \alpha_{\mathbf{k}}^{a,\ell} \phi_\ell(\mathbf{x} - \mathbf{k}),$$

where the convergence is in $L^p(\mathbb{R}^n)$ and $\alpha_{\mathbf{k}}^{a,\ell} \in \mathbb{C}$. Taking the Fourier transform and according to the Hausdorff–Young inequality, we obtain

$$\widehat{\phi}_a(A^*\mathbf{t}) = \sum_{\mathbf{k} \in \mathbb{Z}^n} \sum_{\ell=1}^{\infty} \alpha_{\mathbf{k}}^{a,\ell} \mathrm{e}^{-2\pi i \mathbf{k} \cdot \mathbf{t}} \widehat{\phi}_\ell(\mathbf{t}) \qquad \text{a.e. on } \mathbb{R}^n.$$

Hence

$$\operatorname{supp}(\widehat{\phi_a}) \subset A^* \left(\cup_{\ell=1}^{\infty} \operatorname{supp} \widehat{\phi}_\ell \right),$$

and the conclusion follows. $\square$

The proof of the following lemma is similar to that of Proposition 1 in [19].

Lemma E. *Let $M : \mathbb{R}^n \to \mathbb{R}^n$ be an expansive linear map and let $E \subset \mathbb{R}^n$, $|E|_n > 0$, be a measurable set, such that*

$$\lim_{j \longrightarrow \infty} \chi_E(M^{-j}\mathbf{x}) = 1 \quad \textit{a.e. on } \mathbb{R}^n.$$

Then, the origin is a point of M-density for E.

Proof of Proposition 1. Assume that the origin is not a point of A^*-density for $\cup_{a=1}^{\infty} \operatorname{supp} \widehat{\phi_a}$. By Lemma E with $M = A^*$, there exists a non null measurable set $I \subset B_1$, such that for any $\mathbf{x} \in I$, a sequence $\{j_k\}_{k=1}^{\infty} \subset \mathbb{N}$, $j_k < j_{k+1}$, with the property that $(A^*)^{-j_k}\mathbf{x} \notin \cup_{a=1}^{\infty} \operatorname{supp} \widehat{\phi_a}$ may be taken. Therefore, by Lemma D, we obtain that

$$\cup_{j \in \mathbb{N}} (A^*)^{-j} I \subset \left(\cup_{a=1}^{\infty} \operatorname{supp} \widehat{\phi_a} \right)^c. \tag{3}$$

Consider the linear functional $\Lambda_I : L^p(\mathbb{R}^n) \longrightarrow \mathbb{C}$ defined by

$$\langle \Lambda_I, f \rangle := \langle \chi_I, \widehat{f} \rangle = \int_I \widehat{f}(\mathbf{t})\, \mathrm{d}\mathbf{t}.$$

Since $\langle \Lambda_I, \mathrm{e}^{-2\pi|\cdot|^2}\rangle > 0$, the operator Λ_I is nontrivial. Furthermore, Λ_I is continuous, because

$$|\langle \Lambda_I, f\rangle| = |\langle \chi_I, \widehat{f}\rangle| \le \|\chi_I \widehat{f}\|_{L^1(\mathbb{R}^n)} \le \|\chi_I\|_{L^p(\mathbb{R}^n)} \|\widehat{f}\|_{L^q(\mathbb{R}^n)} \\ \le \|\chi_I\|_{L^p(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)},$$

where the last inequality follows by the Hausdorff–Young inequality.

In particular, for any $a \in \{1, 2, \dots, \}$, $j \in \mathbb{N}$, and $\mathbf{k} \in \mathbb{Z}^n$,

$$\langle \Lambda_I, D_A^j \tau_{\mathbf{k}} \phi_a\rangle = d_A^{-j} \int_I \mathrm{e}^{-2\pi i \mathbf{k}\cdot (A^*)^{-j}\mathbf{t}} \widehat{\phi_a}((A^*)^{-j}\mathbf{t})\,\mathrm{d}\mathbf{t} \\ = \int_{(A^*)^{-j}I} \mathrm{e}^{-2\pi i \mathbf{k}\cdot\mathbf{s}} \widehat{\phi_a}(\mathbf{s})\,\mathrm{d}\mathbf{s} = 0$$

where the last equality follows by (3). Hence, Λ_I is a null operator on $\cup_{j\in\mathbb{N}} V_j$. Therefore, $\cup_{j\in\mathbb{N}} V_j$ is not dense in $L^p(\mathbb{R}^n)$. Finally, bearing in mind that $V_j \subset V_{j+1}$, the result follows. □

Now, we prove Theorem 2.

Proof of Theorem 2. We proceed by contradiction. Assume that $\cup_{j\in\mathbb{Z}} V_j$ is not dense in $C_0(\mathbb{R}^n)$, then by Riesz representation theorem (see example [5, p. 216]), there exists a non trivial Radon measure μ on $\mathbb{R}^n$, such that $|\mu|(\mathbb{R}^n) < \infty$ and

$$\int_{\mathbb{R}^n} f(-\mathbf{t})\mathrm{d}\mu(\mathbf{t}) = 0, \quad \forall f \in W.$$

By Lemma B, we know that $\cup_{j\in\mathbb{Z}} V_j$ is translation invariant, then for $\mathbf{x} \in \mathbb{R}^n$ and $f \in W$, we have

$$(f * \mu)(\mathbf{x}) = \int_{\mathbb{R}^n} f(\mathbf{x} - \mathbf{t})\mathrm{d}\mu(\mathbf{t}) = 0.$$

In particular,

$$(D_A^j \phi_a * \mu)(\mathbf{x}) = \int_{\mathbb{R}^n} D_A^j \phi_a(\mathbf{x} - \mathbf{t})\mathrm{d}\mu(\mathbf{t}) = 0, \quad \forall j \in \mathbb{Z}, \quad a = 1, 2, \dots$$

For any $g \in L^1(\mathbb{R}^n)$ and $a = 1, 2, \dots,$ since $\|D_A^j \phi_a\|_\infty = \|\phi_a\|_{L^\infty(\mathbb{R}^n)}$,

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |g(\mathbf{x} - \mathbf{y}) D_A^j \phi_a(\mathbf{y} - \mathbf{t})| d|\mu|(\mathbf{t})\mathrm{d}\,\mathbf{y} \\ \le \|g\|_{L^1(\mathbb{R}^n)} \|\phi_a\|_{L^\infty(\mathbb{R}^n)} |\mu|(\mathbb{R}^n) < \infty.$$

Now, let g be any compactly supported continuous function and $j \in \mathbb{Z}$. Applying Fubini's theorem (see [5]), we obtain

$$D_A^j \phi_a * (g * \mu) = g * (D_A^j \phi_a * \mu) = 0. \tag{4}$$

Let $g_\mu = g * \mu$. It is well known (see [5]) that $g_\mu \in L^s(\mathbb{R}^n)$ for all $1 \le s \le \infty$. By hypothesis, $\widehat{\phi_a} \in L^r(\mathbb{R}^n)$ for some $1 \le r \le 2$ and there exists $\varphi_a \in L^r(\mathbb{R}^n)$, such that $\widehat{\varphi_a} = \phi_a$. In particular, we have seen that $g_\mu \in L^r(\mathbb{R}^n)$. According to the Hausdorff–Young theorem $\widehat{g_\mu} \in L^{r'}(\mathbb{R}^n)$, where $r^{-1} + r'^{-1} = 1$ when $1 < r \le 2$, and $r' = \infty$ when $r = 1$. By Hölder inequality, $D_A^j \widehat{\phi_a} \widehat{g_\mu} \in L^1(\mathbb{R}^n)$.

Thus, by (4) and following similar ideas as in the proof of Lemma C, we obtain

$$\int_{\mathbb{R}^n} \mathrm{e}^{2\pi i \mathbf{t}\cdot\mathbf{x}} \widehat{g_\mu}(\mathbf{t}) D^j_{A^*} \widehat{\phi_a}(\mathbf{t})\,\mathrm{d}\mathbf{t} = 0, \quad \forall j \in \mathbb{Z}, \quad a = 1, 2, \ldots \tag{5}$$

In an analogous way as the last part of the proof of Theorem 1, we have that $g_\mu(\mathbf{x}) = 0$ for a.e. $\mathbf{x} \in \mathbb{R}^n$. This conclusion is valid for an arbitrary compactly supported continuous function, then μ is the null measure. It contradicts that $\cup_{j\in\mathbb{Z}} V_j$ is not dense in $C_0(\mathbb{R}^n)$. □

Acknowledgements

We would like to thank the anonymous referee for his useful comments and suggestions, which provided an improvement in the overall presentation and clarity of this manuscript.

References

[1] Baggett, L.W., Medina, H.A., Merril, K.D.: Generalized multi-resolution analyses and a construction procedure for all wavelet sets in $\mathbb{R}^n$. J. Fourier Anal. Appl. **5**, 563–573 (1999)

[2] de Boor, C., DeVore, R., Ron, A.: On the construction of multivariate (pre)wavelets. Constr. Approx. **9**, 123–166 (1993)

[3] Calogero, A.: Wavelets on general lattices, associated with general expanding maps of $\mathbb{R}^n$. Electron. Res. Announc. Am. Math. Soc. **5**, 1–10 (1999)**(electronic)**

[4] Cifuentes, P., Kazarian, K.S., San Antolín, A.: Characterization of scaling functions in a multiresolution analysis. Proc. Am. Math. Soc. **133**(4), 1013–1023 (2005)

[5] Folland, G.B.: Real Analysis. Wiley, New York (1984)

[6] Gröchening, K., Madych, W.R.: Multiresolution analysis, Haar bases and self-similar tillings of R^n. IEEE Trans Inf Theory **38**(2), 556–568 (1992)

[7] Hernández, E., Wang, X., Weiss, G.: Characterization of wavelets, scaling functions and wavelets associated with multiresolution analyses, Function spaces, interpolation spaces, and related topics (Haifa, 1995), pp. 51–87, Israel Math. Conf. Proc., vol. 13. Bar-Ilan Univ., Ramat Gan (1999)

[8] Hernández, E., Weiss, G.: A First Course on Wavelets. CRC Press, Boca Raton (1996)

[9] Jia, R.Q.: Multiresolution of L^p spaces. J. Math. Anal. Appl. **184**(3), 620–639 (1994)

[10] Jia, R.Q., Micchelli, C.A.: Using the Refinement Equations for the Construction of Pre-wavelets. II. Powers of Two. Curves and Surfaces, pp. 209–246, Academic Press, Boston (1991)

[11] Kazarian, K.S., San Antolín, A.: Characterization of Scaling Functions in a Frame Multiresolution Analysis in H^2_G, Topics in Classical Analysis and Applications in Honor of Daniel Waterman, pp. 118–140. World Sci. Publ., Hackensack (2008)

[12] Li, B., Wang, H.: Multiresolution analysis of L^p space on compact set. J. Math. Anal. Appl. **317**(1), 245–256 (2006)

[13] Lorentz, R.A., Madych, W.R., Sahakian, A.: Translation and dilation invariant subspaces of $L^2(\mathbb{R})$ and multiresolution analyses. Appl. Comput. Harmon. Anal. **5**(4), 375–388 (1998)

[14] Madych, W.R.: Some elementary properties of multiresolution analyses for $L^2(R^d)$. In: Chui, Ch. (ed.) Wavelets—A Tutorial in Theory and Applications. Academic Press, pp. 259–294 (1992)

[15] Mallat, S.: Multiresolution approximations and wavelet orthonormal bases for $L^2(R)$. Trans. Am. Math. Soc. **315**, 69–87 (1989)

[16] Meyer, Y.: Ondelettes et op*érateurs. I.* Hermann, Paris (1990) [English Translation: Wavelets and operators, Cambridge University Press (1992)]

[17] Saliani, S.: On stable refinable function vectors with arbitrary support. J. Approx. Theory **154**(2), 105–125 (2008)

[18] San Antolín, A.: On the density order of the principal shift-invariant subspaces of $L^2(\mathbb{R}^d)$. J. Approx. Theory **164**(8), 1007–1025 (2012)

[19] San Antolín, A.: Characterization of low pass filters in a multiresolution analysis. Stud. Math. **190**(2), 99–116 (2009)

[20] Soto-Bajo, M.: Closure of dilates of shift-invariant subspaces. Central Eur. J. Math. **11**(10), 1785–1799 (2013)

[21] Strichartz, R.: Construction of Orthonormal Wavelets, Wavelets: Mathematics and Applications, pp. 23–50. Stud. Adv. Math. CRC, Boca Raton (1994)

[22] Woytaszczyk, P.: A Mathematical Introduction to Wavelets. London Mathematical Society, Student Texts, vol. 37 (1997)

[23] Zhao, K.: Density of dilates of a shift-invariant subspace. J. Math. Anal. Appl. **184**(3), 517–532 (1994)

A. San Antolín
Departamento de Matemáticas
Universidad de Alicante
03080 Alicante
Spain
e-mail: `angel.sanantolin@ua.es`

Received: June 4, 2016.
Revised: February 21, 2017.
Accepted: March 24, 2017.