Mediterr. J. Math. (2017) 14:38
DOI 10.1007/s00009-017-0881-2
1660-5446/17/020001-13
published online March 2, 2017

Mediterranean Journal of Mathematics

Singular Integral Operator Involving Higher Order Lipschitz Classes

Juan Bory-Reyes, Lianet De la Cruz-Toranzo and Ricardo Abreu-Blaya

Abstract. In this paper, we investigate a singular integral operator with polyanalytic Cauchy kernel. In particular, we will prove that the higher order Lipschitz classes (of order $1+\alpha$) behave invariant under the action of that operator.

Mathematics Subject Classification. 30G35.

Keywords. Singular integral operator, Lipschitz classes, polyanalytic functions.

1. Introduction

Polyanalytic functions have been investigated thoroughly, notably by the Russian school led by Balk [1], and they provide extensions of classical operators from complex analysis [1,2,4,8,10]. These functions represent one of the more natural generalizations of the analytic ones, and are closely related to polyharmonic functions, which have numerous applications in physics and engineering. Any polyharmonic function of order k can be decomposed into a sum of some polyanalytic function of order k and its conjugate.

A complex valued function $f(z) = u(x, y)+iv(x, y)$ is said to be polyanalytic of order k (k-analytic) in a domain $\Omega \subset \mathbb{C}$ if it has partial derivatives (with respect to x and y) up to the order k and in Ω satisfies the iteration of the Cauchy–Riemann condition:

$$\frac{\partial^k f}{\partial \overline{z}^k} = 0, \tag{1}$$

where

$$\partial_{\overline{z}} = \frac{\partial}{\partial \overline{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

A real valued function $u = u(x, y)$ is called polyharmonic of order k on a domain Ω if $u \in C^\infty(\Omega)$ and $\triangle^k u = 0$, where $\triangle$ denotes the ordinary Laplacian and $\triangle^k u = \triangle^{k-1}(\triangle u)$.

N. Thédoresco [15, page 23], (see also [1, page 204]), was probably the first to propose a Cauchy type formula for polyanalytic functions, a formula that expresses the values of a polyanalytic function within some contour Γ in terms of its values and those of its successive derivatives.

Until further notice, we always suppose that Ω is a simple connected bounded domain of $\mathbb{R}^2$ with sufficiently smooth boundary Γ.

Recently, Begehr developed the following higher order Borel–Pompeiu formula for $\mathbb{C}$-valued and C^k-smooth functions, see [3, page 230].

Theorem 1.1. *Let $f \in C^k(\overline{\Omega}) \cap C^{k+1}(\Omega)$ for $k \geq 0$. Then for $z \in \Omega$*

$$f(z) = \sum_{n=0}^{k} \frac{1}{2\pi i} \int_\Gamma \frac{(\overline{z-\zeta})^n}{n!(\zeta - z)} \partial_{\overline{\zeta}}^n f(\zeta) \mathrm{d}\zeta - \frac{1}{\pi} \int_\Omega \frac{(\overline{z-\zeta})^k}{k!(\zeta - z)} \partial_{\overline{\zeta}}^{k+1} f(\zeta) \mathrm{d}\xi \mathrm{d}\eta. \quad (2)$$

When f is polyanalytic of order $k+1$ in Ω, then formula (2) reduces to

$$f(z) = \sum_{n=0}^{k} \frac{1}{2\pi i} \int_\Gamma \frac{(\overline{z-\zeta})^n}{n!(\zeta - z)} \partial_{\overline{\zeta}}^n f(\zeta) \mathrm{d}\zeta. \quad (3)$$

Here, the sum of contour integrals may be thought of as a sort of Cauchy integral operator in the theory of polyanalytic functions.

In this paper, we suggest a very natural function space where the boundary values of such a Cauchy integral are well behaved. More concretely, we introduce a related singular integral operator in this context and prove that the higher order Lipschitz classes [14] behave invariant under its action. This result can be interpreted as a generalization of the classical Plemelj–Privalov theorem [7,9].

Bianalytic functions [the solutions of (1) for $k = 2$] deserve special attention because of their connection with biharmonic functions. The biharmonic equation $\triangle^2 u = 0$ is encountered in plane problems of elasticity and it is also used to describe radar imaging and slow flows of viscous incompressible fluids [5,11,12]. Due to this fact, and for other reasons for which the case $k = 2$ is interesting, we restrict ourselves to that case. In a forthcoming publication, on the basis of approach developed here, we will study the general case.

2. Preliminaries

We start this section by defining the so called higher order Lipschitz classes, which are directly related to a very deep theorem in real analysis due to H. Whitney [16]. We shall follow the notation used in [14] but restricted to $\mathbb{R}^2$.

2.1. Lipschitz Classes

Let $\mathbf{E}$ be a closed subset of $\mathbb{R}^2$, k a non-negative integer and $0 < \alpha \leq 1$. We shall say that a real valued function f, defined in $\mathbf{E}$, belongs to $\mathrm{Lip}(\mathbf{E}, k+\alpha)$ if there exist real valued bounded functions $f^{(j)}$, $0 < |j| \leq k$, defined on $\mathbf{E}$, with $f^{(0)} = f$, and so that

$$R_j(x,y) = f^{(j)}(x) - \sum_{|j+l| \leq k} \frac{f^{(j+l)}(y)}{l!}(x-y)^l, \quad x, y \in \mathbf{E} \quad (4)$$

satisfies

$$|R_j(x,y)| = \mathcal{O}(|x-y|^{k+\alpha-|j|}), \quad x,y \in \mathbf{E}, |j| \le k. \tag{5}$$

In general, an element of $\mathrm{Lip}(\mathbf{E}, k+\alpha)$ should be interpreted as a collection $\{f^{(j)} : \mathbf{E} \mapsto \mathbb{R}, \, |j| \le k\}$. When $k = 0$, the Lipschitz class becomes the usual class $C_b^{0,\alpha}(\mathbf{E})$ of bounded Hölder continuous functions in $\mathbf{E}$.

Remark 2.1. We remark that the function $f^{(0)} = f$ does not necessarily determine the functions $f^{(j)}$ for arbitrary $\mathbf{E}$, but for $\mathbf{E} = \mathbb{R}^2$ the functions $f^{(j)}$ are uniquely determined by $f^{(0)}$ and $\mathrm{Lip}(\mathbb{R}^2, k+\alpha)$ actually consists of continuous and bounded functions f with continuous and bounded partial derivatives $\partial^{(j)} f$ up to the order k. Moreover, for $|j| = k$ the functions $\partial^{(j)} f$ belongs to the space $\mathrm{Lip}(\mathbb{R}^2, \alpha)$.

For completeness of exposition, we recall the multi-index notation

$$\partial^{(j)} := \frac{\partial^{|j|}}{\partial x_1^{j_1} \partial x_2^{j_2}},$$

with $(j) = (j_1, j_2)$.

We shall say that a complex valued function $f = u + iv$ belongs to $\mathrm{Lip}(\mathbf{E}, k+\alpha)$ if both u and v do so. In this context, however, this definition can be reformulated in purely complex terms as

$$R_j(\tau,\zeta) = f^{(j)}(\tau) - \sum_{|j+l| \le k} \frac{f^{(j+l)}(\zeta)}{l!} (\tau-\zeta)^{l_1} (\overline{\tau-\zeta})^{l_2}, \quad \tau,\zeta \in \mathbf{E} \tag{6}$$

$$|R_j(\tau,\zeta)| = \mathcal{O}(|\tau-\zeta|^{k+\alpha-|j|}), \quad \tau,\zeta \in \mathbf{E}, |j| \le k, \tag{7}$$

the functions $f^{(j)}$ being this time complex-valued as well.

Indeed, assume $f = u + iv$ to be in $\mathrm{Lip}(\mathbf{E}, k+\alpha)$. Then, in accordance with the above definition there exist real valued bounded functions $u^{(j)}$, $v^{(j)}$ $0 < |j| \le k$, defined on $\mathbf{E}$ and such that

$$u^{(j)}(x) = \sum_{|j+l| \le k} \frac{u^{(j+l)}(y)}{l!} (x_1 - y_1)^{l_1} (x_2 - y_2)^{l_2} + U_j(x,y) \tag{8}$$

and

$$v^{(j)}(x) = \sum_{|j+l| \le k} \frac{v^{(j+l)}(y)}{l!} (x_1 - y_1)^{l_1} (x_2 - y_2)^{l_2} + V_j(x,y), \tag{9}$$

with

$$|U_j(x,y)| = \mathcal{O}(|x-y|^{k+\alpha-|j|}), |V_j(x,y)| = \mathcal{O}(|x-y|^{k+\alpha-|j|}).$$

A combination of (8) and (9) yields

$$f^{(j)}(x) = \sum_{|j+l| \le k} \frac{f^{(j+l)}(y)}{l!} (x_1 - y_1)^{l_1} (x_2 - y_2)^{l_2} + R_j(x,y), \tag{10}$$

where

$$f^{(j)} = u^{(j)} + iv^{(j)}, \quad R_j(x,y) = U_j(x,y) + iV_j(x,y).$$

It follows from the elementary formulas

$$x_1 - y_1 = \frac{1}{2}((x-y) + (\overline{x-y})), \quad x_2 - y_2 = \frac{1}{2i}((x-y) - (\overline{x-y})),$$

together with the Newton binomial expansion that (10) can be rewritten in the form

$$f^{(j)}(x) = \sum_{|j+l|\leq k} \frac{f^{(j+l)}(y)}{2^{|l|} i^{l_2} l!} \left[\sum_{n=0}^{|l|} a_n^{(l)} (x-y)^{|l|-n} (\overline{x-y})^n \right] + R_j(x,y), \quad (11)$$

where $a_n^{(l)} = \sum_{|s|=n} \binom{l}{s} (-1)^{s_2}, n = \overline{0, |l|}$.

At this stage and after some convenient abuse of notation, to deduce (6) from (11) is a matter of direct computation.

Following [14, page 177], we put

$$P_j(\tau, \zeta) = \sum_{|j+l|\leq k} \frac{f^{(j+l)}(\zeta)}{l!} (\tau - \zeta)^{l_1} (\overline{\tau - \zeta})^{l_2}.$$

Using the Taylor expansion of the polynomial $P_j(\tau, t) - P_j(\tau, \zeta)$ about the point $t \in \mathbb{R}^2$, we easily obtain

$$P_j(\tau, t) - P_j(\tau, \zeta) = \sum_{|j+l|\leq k} \frac{R_{j+l}(t, \zeta)}{l!} (\tau - t)^{l_1} (\overline{\tau - t})^{l_2}. \quad (12)$$

This relation will be needed in Sect. 3.

For methodological reason we here also include the complex version of the celebrated Whitney extension theorem [14, Theorem 4, page 177].

Theorem 2.1. *Let f be a complex valued function in Lip$(\mathbf{E}, k + \alpha)$, $\mathbf{E} \subset \mathbb{R}^2$. Then, there exists a complex valued function $\tilde{f} \in$ Lip$(\mathbb{R}^2, k + \alpha)$ satisfying*

(i) $\tilde{f}|_{\mathbf{E}} = f^{(0)}$, $\partial_z^{l_1} \partial_{\overline{z}}^{l_2} \tilde{f}|_{\mathbf{E}} = f^{(l)}$, $l = (l_1, l_2)$
(ii) $\tilde{f} \in C^\infty(\mathbb{R}^2 \backslash \mathbf{E})$,
(iii) $|\partial_z^{l_1} \partial_{\overline{z}}^{l_2} \tilde{f}(z)| \leqslant c \, \mathrm{dist}(z, \mathbf{E})^{\alpha - 1}$, *for* $|l| = k + 1$ *and* $z \in \mathbb{R}^2 \backslash \mathbf{E}$.

Until the end of this work, c will denote a positive constant, not necessarily the same at different occurrences.

Remark 2.2. In particular, we note that for $f \in \mathrm{Lip}(\Gamma, k + \alpha)$ the Whitney theorem ensures the existence of an extension $\tilde{f}$ such that $\tilde{f} = f$ and $\partial_{\overline{z}}^n \tilde{f} = f^{(0,n)}$ on Γ, for $n = 0, \ldots, k$.

The above remark suggests that the already announced polyanalytic Cauchy operator [the sum of contour integrals in (3)] may be naturally defined for functions $f \in \mathrm{Lip}(\Gamma, k + \alpha)$. These functions (collection of functions) are intrinsically given on Γ, and as we shall see, they here play a similar role as Hölder functions do for the classical analytic Cauchy integral.

2.2. A Singular Integral Operator

In accordance with our last considerations, let us define the polyanalytic Cauchy integral of a function $f \in \mathrm{Lip}(\Gamma, k+\alpha)$ by

$$\mathcal{C}^{(k)} f(z) := \sum_{n=0}^{k} \frac{1}{2\pi i} \int_\Gamma \frac{(\overline{z-\zeta})^n}{n!(\zeta - z)} f^{(0,n)}(\zeta) \mathrm{d}\zeta, \tag{13}$$

where $f^{(0,n)}$ denotes the corresponding function $f^{(j)}$ with $(j) = (0,n)$ associated to $f \in \mathrm{Lip}(\Gamma, k+\alpha)$.

Of course, the function $\mathcal{C}^{(k)} f(z)$ is polyanalytic in $\mathbb{R}^2 \backslash \Gamma$ by definition. A less trivial concern is to know whether it keeps a well boundary behavior when $z \in \Omega$ approaches $t \in \Gamma$. This answer is closely related to the study of the principal value integral (singular integral operator)

$$\mathcal{S}^{(k)} f(t) = \sum_{n=0}^{k} \frac{1}{\pi i} \int_\Gamma \frac{(\overline{t-\zeta})^n}{n!(\zeta - t)} f^{(0,n)}(\zeta) \mathrm{d}\zeta, \quad t \in \Gamma. \tag{14}$$

The first summand in (14) corresponds to the classical singular integral operator (also called Hilbert transform) given by

$$\mathcal{S}^{(0)} f(t) = \frac{1}{\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - t} \mathrm{d}\zeta,$$

which has been intensively studied [7,9]. Probably, the most famous result concerning $\mathcal{S}^{(0)}$ is the well known Plemelj–Privalov theorem about the invariance of the Hölder classes $C^{0,\alpha}(\Gamma)$ under the action of the singular integral operator $\mathcal{S}^{(0)}$. In more concrete terms:

$$\mathcal{S}^{(0)}(C^{0,\alpha}(\Gamma)) \subset C^{0,\alpha}(\Gamma), \quad 0 < \alpha < 1. \tag{15}$$

The following theorem, our main result, states a generalization of (15). Its proof will take essentially the remaining of the paper.

Theorem 2.2. *Let be* Γ *as before and* $0 < \alpha < 1$. *Then we have the inclusion*

$$\mathcal{S}^{(1)}(Lip(\Gamma, 1+\alpha)) \subset Lip(\Gamma, 1+\alpha).$$

We conclude the section with two auxiliary lemmas, which go back as far as [13]. The estimates contained will prove extremely useful in Section 3.

Lemma 2.1. *Let* Γ *be a smooth Jordan curve with diameter* d, *and* $t \in \Gamma$. *Set* $\Gamma_r(t) = \Gamma \cap B(t,r)$, *being* $B(t,r)$ *the closed ball with center* t *and radius* r. *Then, for* $0 \le r \le d$

$$mes(\Gamma_r(t)) = \int_{\Gamma_r(t)} |\mathrm{d}\zeta| \le cr, \quad \int_{\Gamma_r(t)} \frac{|\mathrm{d}\zeta|}{|\zeta - t|^{1-\alpha}} \le c\, r^\alpha,$$

$$r \int_{\Gamma \backslash \Gamma_r(t)} \frac{|\mathrm{d}\zeta|}{|\zeta - t|^{2-\alpha}} \le c\, r^\alpha.$$

Moreover,

$$\left| \int_{\Gamma \backslash \Gamma_r(t)} \frac{\mathrm{d}\zeta}{\zeta - t} \right| \le c,$$

where $c > 0$ is independent of r.

Lemma 2.2. *Let Γ be a smooth Jordan curve and $t, \tau \in \Gamma$. Then*

$$\frac{1}{\pi i}\int_\Gamma \frac{\mathrm{d}\zeta}{(\zeta - t)(\zeta - \tau)} = 0, \quad \frac{1}{\pi i}\int_\Gamma \frac{\mathrm{d}\zeta}{(\zeta - t)^2} = 0.$$

3. Proof of the Main Theorem

Proof. For simplicity of notation, we write $\widehat{f}$ instead of $\mathcal{S}^{(1)}f$.

Set

$$\widehat{f}^{(1,0)}(t) = \frac{1}{\pi i}\int_\Gamma \frac{f(\zeta) + \overline{t - \zeta} f^{(0,1)}(\zeta)}{(\zeta - t)^2}\mathrm{d}\zeta, \quad \widehat{f}^{(0,1)}(t) = \frac{1}{\pi i}\int_\Gamma \frac{f^{(0,1)}(\zeta)}{\zeta - t}\mathrm{d}\zeta$$

and prove that

$$R_{(0,0)}[\widehat{f}](t,\tau) = \widehat{f}(t) - \widehat{f}(\tau) - (t - \tau)\widehat{f}^{(1,0)}(\tau) - (\overline{t - \tau})\widehat{f}^{(0,1)}(\tau)$$
$$R_{(0,1)}[\widehat{f}](t,\tau) = \widehat{f}^{(0,1)}(t) - \widehat{f}^{(0,1)}(\tau)$$
$$R_{(1,0)}[\widehat{f}](t,\tau) = \widehat{f}^{(1,0)}(t) - \widehat{f}^{(1,0)}(\tau)$$

satisfy

$$|R_j[\widehat{f}](t,\tau)| \le c|t - \tau|^{1+\alpha-|j|}$$

for all $|j| \le 1$.

Let us first estimate[1] $|R_0[\widehat{f}](t,\tau)|$.

We have

$$R_0[\widehat{f}] = \frac{1}{\pi i}\int_\Gamma \frac{f(\zeta) + (\overline{t - \zeta})f^{(0,1)}(\zeta)}{\zeta - t}\mathrm{d}\zeta - \frac{1}{\pi i}\int_\Gamma \frac{f(\zeta) + (\overline{\tau - \zeta})f^{(0,1)}(\zeta)}{\zeta - \tau}\mathrm{d}\zeta$$
$$-\frac{t - \tau}{\pi i}\int_\Gamma \frac{f(\zeta) + (\overline{\tau - \zeta})f^{(0,1)}(\zeta)}{(\zeta - \tau)^2}\mathrm{d}\zeta - \frac{\overline{t - \tau}}{\pi i}\int_\Gamma \frac{f^{(0,1)}(\zeta)}{\zeta - \tau}\mathrm{d}\zeta. \quad (16)$$

Making use of the identity $\overline{t - \zeta} = \overline{(t - \tau) + (\tau - \zeta)}$, and by substituting it into the first term on the right-hand side of (16), we obtain

$$R_0[\widehat{f}] = \frac{(t - \tau)^2}{\pi i}\int_\Gamma \frac{f(\zeta) + (\overline{\tau - \zeta})f^{(0,1)}(\zeta)}{(\zeta - t)(\zeta - \tau)^2}\mathrm{d}\zeta$$
$$+\frac{\overline{t - \tau}}{\pi i}\left[\int_\Gamma \frac{f^{(0,1)}(\zeta)}{\zeta - t}\mathrm{d}\zeta - \int_\Gamma \frac{f^{(0,1)}(\zeta)}{\zeta - \tau}\mathrm{d}\zeta\right].$$

[1] We will write it simply $R_j[\widehat{f}]$ when no confusion can arise.

By (6) it follows that $f(\zeta)+(\overline{\tau-\zeta})f^{(0,1)}(\zeta)=f(\tau)-(\tau-\zeta)f^{(1,0)}(\zeta)-R_0[f](\tau,\zeta)$, which implies after substitution

$$\begin{aligned}R_0[\widehat{f}] &= \frac{(t-\tau)^2}{\pi i}\int_\Gamma \frac{f^{(1,0)}(\zeta)}{(\zeta-t)(\zeta-\tau)}\mathrm{d}\zeta - \frac{(t-\tau)^2}{\pi i}\int_\Gamma \frac{R_0[f](\tau,\zeta)}{(\zeta-t)(\zeta-\tau)^2}\mathrm{d}\zeta \\ &\quad + \frac{\overline{t-\tau}}{\pi i}\left[\int_\Gamma \frac{f^{(0,1)}(\zeta)}{\zeta-t}\mathrm{d}\zeta - \int_\Gamma \frac{f^{(0,1)}(\zeta)}{\zeta-\tau}\mathrm{d}\zeta\right] \\ &= \frac{t-\tau}{\pi i}\left[\int_\Gamma \frac{f^{(1,0)}(\zeta)}{\zeta-t}\mathrm{d}\zeta - \int_\Gamma \frac{f^{(1,0)}(\zeta)}{\zeta-\tau}\mathrm{d}\zeta\right] \\ &\quad - \frac{(t-\tau)^2}{\pi i}\int_\Gamma \frac{R_0[f](\tau,\zeta)}{(\zeta-t)(\zeta-\tau)^2}\mathrm{d}\zeta \\ &\quad + \frac{\overline{t-\tau}}{\pi i}\left[\int_\Gamma \frac{f^{(0,1)}(\zeta)}{\zeta-t}\mathrm{d}\zeta - \int_\Gamma \frac{f^{(0,1)}(\zeta)}{\zeta-\tau}\mathrm{d}\zeta\right].\end{aligned}$$

Observe the use of Lemma 2.2 in the above equality.

Since $f\in \mathrm{Lip}(\Gamma,1+\alpha)$, it follows that $f^{(1,0)}, f^{(0,1)}\in \mathrm{Lip}(\Gamma,\alpha)$. From this, by the classic Plemelj–Privalov theorem (15), we conclude that the first and last terms above are both dominated by $c\,|t-\tau|^{1+\alpha}$.

It will thus be sufficient to prove that the second term

$$I(t,\tau)=\frac{(t-\tau)^2}{\pi i}\int_\Gamma \frac{R_0[f](\tau,\zeta)}{(\zeta-t)(\zeta-\tau)^2}\mathrm{d}\zeta$$

is dominated by $c|t-\tau|^{1+\alpha}$.

Let $r=\frac{|t-\tau|}{2}$ and set $\Gamma_1:=\Gamma_r(t)$, $\Gamma_2:=\Gamma_r(\tau)$ and $\Gamma_3:=\Gamma\backslash\Gamma_1\cup\Gamma_2$. Write

$$I_p(t,\tau)=\frac{(t-\tau)^2}{\pi i}\int_{\Gamma_p} \frac{R_0[f](\tau,\zeta)}{(\zeta-t)(\zeta-\tau)^2}\mathrm{d}\zeta$$

for $p=\overline{1,3}$.

The following estimation of $I_2(t,\tau)$ being due to the fact that $|\zeta-t|\geq r$ in Γ_2

$$\begin{aligned}|I_2(t,\tau)| &\leq c|t-\tau|^2\int_{\Gamma_2}\frac{|R_0[f](\tau,\zeta)||\mathrm{d}\zeta|}{|\zeta-t||\zeta-\tau|^2} \leq c|t-\tau|^2\int_{\Gamma_2}\frac{|\tau-\zeta|^{1+\alpha}|\mathrm{d}\zeta|}{|\zeta-t||\zeta-\tau|^2} \\ &\leq c|t-\tau|^2\int_{\Gamma_2}\frac{|\mathrm{d}\zeta|}{|\zeta-t||\zeta-\tau|^{1-\alpha}} \leq c|t-\tau|\int_{\Gamma_2}\frac{|\mathrm{d}\zeta|}{|\zeta-\tau|^{1-\alpha}} \\ &\leq c|t-\tau|^{1+\alpha}. \end{aligned} \tag{17}$$

To examine $I_3(t,\tau)$, we note that if $|\zeta-\tau|\leq|\zeta-t|$, then

$$\frac{1}{|\zeta-t||\zeta-\tau|^{1-\alpha}}\leq\frac{1}{|\zeta-\tau|^{2-\alpha}}.$$

On the other hand, if $|\zeta-t|\leq|\zeta-\tau|$, then

$$\frac{1}{|\zeta-t||\zeta-\tau|^{1-\alpha}}\leq\frac{1}{|\zeta-t|^{2-\alpha}}.$$

From this,

$$\frac{1}{|\zeta - t||\zeta - \tau|^{1-\alpha}} \leq \frac{1}{|\zeta - t|^{2-\alpha}} + \frac{1}{|\zeta - \tau|^{2-\alpha}}.$$

Now, it is clear that $\Gamma_3 \subset \Gamma\backslash\Gamma_1$ and $\Gamma_3 \subset \Gamma\backslash\Gamma_2$, which gives

$$\begin{aligned}|I_3(t,\tau)| &\leq c|t-\tau|^2 \int_{\Gamma_3} \frac{|\mathrm{d}\zeta|}{|\zeta - t||\zeta - \tau|^{1-\alpha}} \\ &\leq c|t-\tau|^2 \left[\int_{\Gamma_3} \frac{|\mathrm{d}\zeta|}{|\zeta - t|^{2-\alpha}} + \int_{\Gamma_3} \frac{|\mathrm{d}\zeta|}{|\zeta - \tau|^{2-\alpha}} \right] \\ &\leq c|t-\tau| \left[|t-\tau| \int_{\Gamma\backslash\Gamma_1} \frac{|\mathrm{d}\zeta|}{|\zeta - t|^{2-\alpha}} + |t-\tau| \int_{\Gamma\backslash\Gamma_2} \frac{|\mathrm{d}\zeta|}{|\zeta - \tau|^{2-\alpha}} \right] \\ &\leq c|t-\tau|^{1+\alpha}. \end{aligned} \tag{18}$$

Observe the use of Lemma 2.1 in (17) and (18).

Finally, let us examine $I_1(t,\tau)$. From (12) it follows that

$$\begin{aligned}I_1(t,\tau) = \frac{(t-\tau)^2}{\pi i} \Bigg[&\int_{\Gamma_1} \frac{R_0[f](\tau,t)\mathrm{d}\zeta}{(\zeta - t)(\zeta - \tau)^2} + \int_{\Gamma_1} \frac{R_0[f](t,\zeta)\mathrm{d}\zeta}{(\zeta - t)(\zeta - \tau)^2} \\ &+ (t-\tau) \int_{\Gamma_1} \frac{R_{(1,0)}[f](t,\zeta)\mathrm{d}\zeta}{(\zeta - t)(\zeta - \tau)^2} + (\overline{t-\tau}) \int_{\Gamma_1} \frac{R_{(0,1)}[f](t,\zeta)\mathrm{d}\zeta}{(\zeta - t)(\zeta - \tau)^2} \Bigg].\end{aligned}$$

The integrals within the bracket signs in the above expression shall be denoted by $I_1^1(t,\tau)$, $I_1^2(t,\tau)$, $I_1^3(t,\tau)$, $I_1^4(t,\tau)$, respectively.

We first examine $I_1^1(t,\tau)$. From the identity

$$\frac{1}{(\zeta - t)(\zeta - \tau)^2} = \frac{1}{(t-\tau)^2}\left(\frac{1}{\zeta - t} - \frac{1}{\zeta - \tau} \right) - \frac{1}{t-\tau}\frac{1}{(\zeta - \tau)^2},$$

we obtain

$$\begin{aligned}|I_1^1(t,\tau)| &\leq \frac{c|R_0[f](\tau,t)|}{|t-\tau|^2} \Big(\Big| \int_{\Gamma_1} \frac{\mathrm{d}\zeta}{\zeta - t} \Big| + \Big| \int_{\Gamma_1} \frac{\mathrm{d}\zeta}{\zeta - \tau} \Big| + |t-\tau| \Big| \int_{\Gamma_1} \frac{\mathrm{d}\zeta}{(\zeta - \tau)^2} \Big| \Big) \\ &\leq \frac{c}{|\tau - t|^{1-\alpha}} \Big(\Big| \int_{\Gamma_1} \frac{\mathrm{d}\zeta}{\zeta - t} \Big| + \int_{\Gamma_1} \frac{|\mathrm{d}\zeta|}{|\zeta - \tau|} + |t-\tau| \int_{\Gamma_1} \frac{|\mathrm{d}\zeta|}{|\zeta - \tau|^2} \Big).\end{aligned}$$

By Lemma 2.1, we have

$$\Big| \int_{\Gamma_1} \frac{\mathrm{d}\zeta}{\zeta - t} \Big| \leq c,$$

$$\int_{\Gamma_1} \frac{|\mathrm{d}\zeta|}{|\zeta - \tau|} \leq \frac{c}{|t-\tau|} \int_{\Gamma_1} |\mathrm{d}\zeta| \leq \frac{c \cdot mes(\Gamma_1)}{|t-\tau|} \leq c,$$

and

$$|t-\tau| \int_{\Gamma_1} \frac{|\mathrm{d}\zeta|}{|\zeta - \tau|^2} \leq \frac{c}{|t-\tau|} \int_{\Gamma_1} |\mathrm{d}\zeta| \leq \frac{c \cdot mes(\Gamma_1)}{|t-\tau|} \leq c.$$

Consequently, we conclude that

$$|I_1^1(t,\tau)| \leq \frac{c}{|t-\tau|^{1-\alpha}}. \tag{19}$$

We now turn to $I_1^2(t,\tau)$. Since $|t-\zeta| \leq \frac{|t-\tau|}{2}$ and $|\zeta-\tau| \geq \frac{|t-\tau|}{2}$ in Γ_1, we have

$$\begin{aligned} |I_1^2(t,\tau)| &\leq \int_{\Gamma_1} \frac{|R_0[f](t,\zeta)||\mathrm{d}\zeta|}{|\zeta-t||\zeta-\tau|^2} \leq c\frac{|t-\tau|^\alpha}{|t-\tau|^2}\int_{\Gamma_1}|\mathrm{d}\zeta| \\ &\leq \frac{c\cdot mes(\Gamma_1)}{|t-\tau|^{2-\alpha}} \leq \frac{c}{|t-\tau|^{1-\alpha}}. \end{aligned} \tag{20}$$

We now apply Lemma 2.1 again to conclude that

$$\begin{aligned} |I_1^3(t,\tau)| &\leq |t-\tau| \int_{\Gamma_1} \frac{|R_{(1,0)}[f](t,\zeta)||\mathrm{d}\zeta|}{|\zeta-t||\zeta-\tau|^2} \\ &\leq \frac{c}{|t-\tau|}\int_{\Gamma_1}\frac{|\mathrm{d}\zeta|}{|\zeta-t|^{1-\alpha}} \leq \frac{c}{|t-\tau|^{1-\alpha}}. \end{aligned} \tag{21}$$

By a similar argument

$$|I_1^4(t,\tau)| \leq \frac{c}{|t-\tau|^{1-\alpha}}. \tag{22}$$

Combining (19)–(22) we can assert that $I_1(t,\tau)$ is bounded by $c|t-\tau|^{1+\alpha}$, hence so is $I(t,\tau)$, by (17) and (18), and finally

$$|R_0[\widehat{f}](t,\tau)| \leq c|t-\tau|^{1+\alpha} \tag{23}$$

Note that, since $f^{(0,1)} \in \mathrm{Lip}(\Gamma,\alpha)$ by hypothesis,

$$|R_{(0,1)}[\widehat{f}](t,\tau)| \leq c|t-\tau|^\alpha$$

holds by the Plemelj–Privalov theorem (15). Then we are reduced to proving

$$|R_{(1,0)}[\widehat{f}](t,\tau)| \leq c|t-t|^\alpha.$$

To get this estimate, we use (6) again to obtain

$$f(\zeta) + \overline{t-\zeta}f^{(0,1)}(\zeta) = f(t) - (t-\zeta)f^{(1,0)}(\zeta) - R_0[f](t,\zeta),$$

which implies after substitution

$$\begin{aligned} R_{(1,0)}[\widehat{f}] = &\frac{1}{\pi i}\int_\Gamma \frac{f^{(1,0)}(\zeta)}{\zeta-t}\mathrm{d}\zeta - \frac{1}{\pi i}\int_\Gamma \frac{f^{(1,0)}(\zeta)}{\zeta-\tau}\mathrm{d}\zeta \\ &+\frac{1}{\pi i}\int_\Gamma \frac{R_0[f](\tau,\zeta)}{(\zeta-\tau)^2}\mathrm{d}\zeta - \frac{1}{\pi i}\int_\Gamma \frac{R_0[f](t,\zeta)}{(\zeta-t)^2}\mathrm{d}\zeta. \end{aligned}$$

Since $f^{(1,0)} \in \mathrm{Lip}(\Gamma,\alpha)$, it follows that

$$\Big|\frac{1}{\pi i}\int_\Gamma \frac{f^{(1,0)}(\zeta)}{\zeta-t}\mathrm{d}\zeta - \frac{1}{\pi i}\int_\Gamma \frac{f^{(1,0)}(\zeta)}{\zeta-\tau}\mathrm{d}\zeta\Big| \leq c|t-\tau|^\alpha.$$

We next turn to estimating

$$J(t,\tau) = \frac{1}{\pi i}\int_\Gamma \frac{R_0[f](\tau,\zeta)}{(\zeta-\tau)^2}\mathrm{d}\zeta - \frac{1}{\pi i}\int_\Gamma \frac{R_0[f](t,\zeta)}{(\zeta-t)^2}\mathrm{d}\zeta.$$

Since the following identities hold

$$\frac{1}{\pi i}\int_\Gamma \frac{R_0[f](\tau,\zeta)}{(\zeta-\tau)^2}\mathrm{d}\zeta = \frac{t-\tau}{\pi i}\int_\Gamma \frac{-R_0[f](\tau,\zeta)}{(\zeta-\tau)^2(\zeta-t)}\mathrm{d}\zeta + \frac{1}{\pi i}\int_\Gamma \frac{R_0[f](\tau,\zeta)}{(\zeta-\tau)(\zeta-t)}\mathrm{d}\zeta \tag{24}$$

$$\frac{1}{\pi i}\int_\Gamma \frac{R_0[f](t,\zeta)}{(\zeta-t)^2}\mathrm{d}\zeta = \frac{t-\tau}{\pi i}\int_\Gamma \frac{R_0[f](t,\zeta)}{(\zeta-t)^2(\zeta-\tau)}\mathrm{d}\zeta + \frac{1}{\pi i}\int_\Gamma \frac{R_0[f](t,\zeta)}{(\zeta-\tau)(\zeta-t)}\mathrm{d}\zeta, \tag{25}$$

we may continue subtracting (25) from (24) to get

$$\begin{aligned} J(t,\tau) = {} & \frac{t-\tau}{\pi i}\int_\Gamma \frac{-R_0[f](\tau,\zeta)}{(\zeta-\tau)^2(\zeta-t)}\mathrm{d}\zeta - \frac{t-\tau}{\pi i}\int_\Gamma \frac{R_0[f](t,\zeta)}{(\zeta-t)^2(\zeta-\tau)}\mathrm{d}\zeta \\ & + \frac{1}{\pi i}\int_\Gamma \frac{R_0[f](\tau,\zeta)-R_0[f](t,\zeta)}{(\zeta-\tau)(\zeta-t)}\mathrm{d}\zeta. \end{aligned} \tag{26}$$

It has been already proved that

$$|I(t,\tau)| = \Big|\frac{(t-\tau)^2}{\pi i}\int_\Gamma \frac{R_0[f](\tau,\zeta)}{(\zeta-t)(\zeta-\tau)^2}\mathrm{d}\zeta\Big| \le c|t-\tau|^{1+\alpha}.$$

Therefore, the first two terms on the right-hand side of (26) are both bounded by $c|t-\tau|^\alpha$.

What is left is to show that the last term,

$$J'(t,\tau) = \frac{1}{\pi i}\int_\Gamma \frac{R_0[f](\tau,\zeta)-R_0[f](t,\zeta)}{(\zeta-\tau)(\zeta-t)}\mathrm{d}\zeta,$$

so is.

From (12) it follows that

$$\begin{aligned} & R_0[f](\tau,\zeta) - R_0[f](t,\zeta) = R_0[f](\tau,t) \\ & \quad + (\tau-t)R_{(1,0)}[f](t,\zeta) + (\overline{\tau-t})R_{(0,1)}[f](t,\zeta). \end{aligned}$$

Substituting into $J'(t,\tau)$ yields

$$J'(t,\tau) = \frac{\tau-t}{\pi i}\int_\Gamma \frac{R_{(1,0)}[f](t,\zeta)}{(\zeta-\tau)(\zeta-t)}\mathrm{d}\zeta + \frac{\overline{\tau-t}}{\pi i}\int_\Gamma \frac{R_{(0,1)}[f](t,\zeta)}{(\zeta-\tau)(\zeta-t)}\mathrm{d}\zeta. \tag{27}$$

Let $J''(t,\tau)$ denote the first term on the right-hand side of (27). Let be Γ_p, $p=\overline{1,3}$, as before and put

$$J''_p(t,\tau) = \frac{\tau-t}{\pi i}\int_{\Gamma_p} \frac{R_{(1,0)}[f](t,\zeta)}{(\zeta-\tau)(\zeta-t)}\mathrm{d}\zeta.$$

It follows that

$$|J''_1(t,\tau)| = \Big|\frac{\tau-t}{\pi i}\int_{\Gamma_1} \frac{R_{(1,0)}[f](t,\zeta)}{(\zeta-\tau)(\zeta-t)}\mathrm{d}\zeta\Big| \le c\int_{\Gamma_1}\frac{|\mathrm{d}\zeta|}{|\zeta-t|^{1-\alpha}} \le c|t-\tau|^\alpha. \tag{28}$$

Since $|t-\tau|^2\int_{\Gamma_3}\frac{|\mathrm{d}\zeta|}{|\zeta-\tau||\zeta-t|^{1-\alpha}} \le c|t-\tau|^{1+\alpha}$ by (18), we conclude that

$$|J_3''(t,\tau)| = \Big|\frac{\tau - t}{\pi i}\int_{\Gamma_3}\frac{R_{(1,0)}[f](t,\zeta)}{(\zeta-\tau)(\zeta-t)}\mathrm{d}\zeta\Big| \le c|t-\tau|\int_{\Gamma_3}\frac{|\mathrm{d}\zeta|}{|\zeta-\tau||\zeta-t|^{1-\alpha}}$$
$$\le c|t-\tau|^{\alpha}. \tag{29}$$

We now turn to $J_2''(t,\tau)$. From (12), we have

$$R_{(1,0)}[f](t,\zeta) = R_{(1,0)}[f](\tau,\zeta) - R_{(1,0)}[f](\tau,t),$$

hence that

$$\Big|\int_{\Gamma_2}\frac{R_{(1,0)}[f](t,\zeta)}{(\zeta-\tau)(\zeta-t)}\mathrm{d}\zeta\Big| \le \Big|\int_{\Gamma_2}\frac{R_{(1,0)}[f](\tau,\zeta)}{(\zeta-\tau)(\zeta-t)}\mathrm{d}\zeta\Big| + \Big|\int_{\Gamma_2}\frac{R_{(1,0)}[f](\tau,t)}{(\zeta-\tau)(\zeta-t)}\mathrm{d}\zeta\Big|.$$

We need to prove that the two terms on the right-hand side above are both bounded by $c|t-\tau|^{\alpha-1}$.

Let us examine the first one

$$\Big|\int_{\Gamma_2}\frac{R_{(1,0)}[f](\tau,\zeta)}{(\zeta-\tau)(\zeta-t)}\mathrm{d}\zeta\Big| \le c\int_{\Gamma_2}\frac{|\mathrm{d}\zeta|}{|\zeta-\tau|^{1-\alpha}|\zeta-t|} \le \frac{c}{|t-\tau|}\int_{\Gamma_2}\frac{|\mathrm{d}\zeta|}{|\zeta-\tau|^{1-\alpha}}$$
$$\le \frac{c}{|t-\tau|^{1-\alpha}}.$$

On the other hand, since

$$\Big|\int_{\Gamma_2}\frac{R_{(1,0)}[f](\tau,t)}{(\zeta-\tau)(\zeta-t)}\mathrm{d}\zeta\Big| = \Big|\frac{R_{(1,0)}[f](\tau,t)}{t-\tau}\Big(\int_{\Gamma_2}\frac{\mathrm{d}\zeta}{\zeta-t} - \int_{\Gamma_2}\frac{\mathrm{d}\zeta}{\zeta-\tau}\Big)\Big|,$$

Lemma 2.1 shows that the modulus of the two integrals in parentheses are both bounded by a non-negative constant c, hence that

$$\Big|\int_{\Gamma_2}\frac{R_{(1,0)}[f](\tau,t)}{(\zeta-\tau)(\zeta-t)}\mathrm{d}\zeta\Big| \le c|t-\tau|^{\alpha-1},$$

and finally that

$$|J_2''(t,\tau)| = \Big|\frac{\tau-t}{\pi i}\int_{\Gamma_2}\frac{R_{(1,0)}[f](\tau,t)}{(\zeta-\tau)(\zeta-t)}\mathrm{d}\zeta\Big| \le c|t-\tau|^{\alpha}.$$

Similar arguments apply to the second term on the right-hand side of (27) and the proof is completed. □

4. Concluding Remarks

We conclude with a couple of remarks

Remark 4.1. Theorem 2.2 may be extended to the case of Carleson Jordan curves (Ahlfors–David). Such a generalization is similar in spirit to that carried out in [13]. In this more general case, it is necessary to introduce a modified singular integral operator given by

$$\mathcal{S}f(t) = \frac{1}{\pi i}\int_{\Gamma}\frac{f(\zeta)-f(t)}{\zeta - t}\mathrm{d}\zeta + f(t),$$

instead of $\mathcal{S}^{(0)}f$.

The estimates developed in the proof of Theorem 2.2 are similarly obtained. The basic geometry ingredient is the fact that Lemma 2.1 still holds for Carleson curves.

Remark 4.2. Future work will explore the hypothesis that our result still hold for the general case $k \in \mathbb{N}$, as well as the idea of extending it to Euclidean space using Clifford analysis [6].

Acknowledgements

J. Bory Reyes was partially supported by Instituto Politécnico Nacional in the framework of SIP programs.

References

[1] Balk, M.B., Zuev, M.F.: On polyanalytic functions. Russ. Math. Surv. **25**(5), 201–223 (1970)

[2] Begehr, H.: Complex Analytic Methods for Partial Differential Equations: An Introductory Text. World Scientific Publishing Co.,Inc., River Edge (1994)

[3] Begehr, H.: Integral representations in complex, hypercomplex and clifford analysis. Integr. Transform. Spec. Funct. **13**(3), 223–241 (2002)

[4] Begehr, H., Jinyuan, D., Yufeng, W.: A Dirichlet problem for polyharmonic functions. Ann. Mat. Pura Appl. (4) **187**(3), 435–457 (2008)

[5] Berdyshev, A.S., Cabada, A., Turmetov, BKh: On solvability of a boundary value problem for a nonhomogeneous biharmonic equation with a boundary operator of a fractional order. Acta Mathematica Scientia **34B**(6), 1695–1706 (2014)

[6] Brackx, F., Delanghe, R., Sommen, F.: Clifford analysis. In: Research Notes in Mathematics, vol. 76. Pitman (Advanced Publishing Program), Boston (1982)

[7] Gakhov, F.D.: Boundary value problems. Nauka, Moscow (1988)

[8] Hayrapetyan, H.M., Hayrapetyan, A.R.: Boundary value problems in weighted spaces of polyanalytic functions in half-plane. J. Contemp. Math. Anal. **47**(1), 1–15 (2012)

[9] Mushelisvili, N.I.: Singular integral equations, Nauka, Moskow (1968) [English transl. of 1st ed., Noodhoff, Groningen (1953); reprint (1972)]

[10] Mazlov, M.Ya.: The Dirichlet problem for polyanalytic functions. Sb. Math. **200**(10), 1473–1493 (2009) [translation from Mat. Sb. **200**(10), 59–80 (2009)]

[11] Andersson, L.E., Elfving, T., Golub, G.H.: Solution of biharmonic equations with application to radar imaging. J. Comput. Appl. Math. **94**(2), 153–180 (1998)

[12] Lai, M.-C., Liu, H.-C.: Fast direct solver for the biharmonic equation on a disk and its application to incompressible flows. Appl. Math. Comput. **164**(3), 679–695 (2005)

[13] Salaev, V.V.: Direct and inverse estimates for a singular Cauchy integral along a closed curve (English). Math. Notes **19**, 221–231 (1976)

[14] Stein, E.M.: Singular Integrals and Differentiability Properties of Functions. Princeton Math. Ser., vol. 30. Princeton Univ. Press, Princeton (1970)

[15] Thédoresco, N.: La dérivée aréolaire et ses applications *à* la physique mathematique. Thèse, Paris (1931)

[16] Whitney, H.: Analytic extensions of differentiable functions defined in closed sets. Trans. Am. Math. Soc. **36**(1), 63–89 (1934)

Juan Bory-Reyes
Instituto Politécnico Nacional, SEPI-ESIME-ZAC
Mexico DF 07738
Mexico
e-mail: juanboryreyes@yahoo.com

Lianet De la Cruz-Toranzo and Ricardo Abreu-Blaya
Facultad de Informática y Matemática
Universidad de Holguín
Holguín 80100
Cuba
e-mail: lcruzt@uho.edu.cu

Ricardo Abreu-Blaya
e-mail: rabreu@uho.edu.cu

Received: October 12, 2016.
Revised: January 26, 2017.
Accepted: February 14, 2017.