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On Perturbed Fractional Differential Inclusions with Nonlocal Multi-point Erdélyi–Kober Fractional Integral Boundary Conditions

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Abstract. In this paper, by using a nonlinear alternative for a sum of compact upper semicontinuous and contractive multivalued operators, we establish sufficient conditions for the existence of solutions for perturbed fractional differential inclusions with nonlocal multi-point Erdélyi–Kober fractional integral boundary conditions. For the applicability of the main result, we include an example.

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1. Introduction

Considerable interest in fractional differential equations has been stimulated due to their numerous applications in many fields of science and engineering. Important phenomena in finance, electromagnetics, acoustics, viscoelasticity, electrochemistry and material sciences are well described by differential equations of fractional order. For examples and recent development of the topic, see [1–5] and the references cited therein.

Differential inclusions, known as generalization of differential equations and inequalities, serve as important and useful mathematical tools in optimal control theory, dynamical systems and stochastic processes, for details, see the text [6]. In fact, the area of initial and boundary value problems involving fractional-order differential equations and inclusions has been extensively investigated in the recent years. The development on the topic includes the existence theory as well as the methods of solution for such problems, for example, see [7–24] and the references cited therein.

It has been noticed that fractional-order boundary value problems supplemented with integral boundary conditions involve either classical, Riemann–Liouville or Hadamard type integrals. Besides these integrals, there is another kind of integral operator, introduced by Erdélyi and Kober [25] in 1940, which is known as Erdélyi–Kober fractional integral operator. These operators are found to be quite useful in solving single, dual and triple integral equations possessing special functions of mathematical physics in their kernels. For applications of the Erdélyi–Kober fractional integrals, we refer the reader to a series of papers and texts [2,25–29].

In this paper, we investigate a boundary value problem of perturbed fractional differential inclusions equipped with nonlocal multi-point Erdélyi–Kober fractional integral boundary conditions given by

$$\begin{cases} D^q x(t) \in F(t, x(t)) + G(t, x(t)), \quad 0 < t < T, \quad 1 < q \leq 2, \\ x(0) = 0, \quad \alpha x(T) = \sum_{i=1}^{m} \beta_i I_{\eta_i}^{\gamma_i, \delta_i} x(\xi_i), \end{cases} \tag{1.1}$$

where D^q is the standard Riemann–Liouville fractional derivative of order q, $I_{\eta_i}^{\gamma_i,\delta_i}$ is the Erdélyi–Kober fractional integral of order $\delta_i > 0$ with $\eta_i > 0$ and $\gamma_i \in \mathbb{R}$, $i = 1, 2, \ldots, m$, $F, G : [0, T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ are multivalued maps, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$, and $\alpha, \beta_i \in \mathbb{R}$, $\xi_i \in (0, T)$, $i = 1, 2, \ldots, m$ are given constants.

The paper is organized as follows. Section 2 contains preliminary concepts related to the proposed study while the main existence result, based on nonlinear alternative for contractive maps, is presented in Sect. 3. For the illustration of the main result, we discuss an example.

We emphasize that the findings for perturbed fractional differential inclusions supplemented with multipoint Erdélyi–Kober fractional integral boundary conditions reported in this paper are new and contribute significantly to the subject of fractional calculus.

2. Preliminaries

The first part of this Section is devoted to some fundamental concepts of fractional calculus, while the second part deals with the background material for multivalued maps related to our problem.

2.1. Basic Material for Fractional Calculus

In this section, we introduce some notations and definitions of fractional calculus and present preliminary results needed in our proofs later [2,5].

Definition 2.1. The Riemann–Liouville fractional derivative of order q of a continuous function $f : (0, \infty) \to \mathbb{R}$ is defined by

$$D^q f(t) = \frac{1}{\Gamma(n-q)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-q-1} f(s) ds, \quad n - 1 < q < n,$$

where $n = [q] + 1, [q]$ denotes the integer part of a real number q. Here Γ is the Gamma function defined by $\Gamma(q) = \int_0^\infty e^{-s} s^{q-1} ds$.

Definition 2.2. The Riemann–Liouville fractional integral of order $q > 0$ of a continuous function $f : (0,\infty) \to \mathbb{R}$ is defined by

$$J^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s)\mathrm{d}s,$$

provided the integral exists.

Definition 2.3. The Erdélyi–Kober fractional integral of order $\delta > 0$ with $\eta > 0$ and $\gamma \in \mathbb{R}$ of a continuous function $f : (0,\infty) \to \mathbb{R}$ is defined by

$$I_\eta^{\gamma,\delta} f(t) = \frac{\eta t^{-\eta(\delta+\gamma)}}{\Gamma(\delta)} \int_0^t \frac{s^{\eta\gamma+\eta-1} f(s)}{(t^\eta - s^\eta)^{1-\delta}} \mathrm{d}s$$

provided the right side is pointwise defined on $\mathbb{R}_+$.

Remark 2.4. For $\eta = 1$ the above operator is reduced to the Kober operator

$$I_\gamma^\delta f(t) = \frac{t^{-(\delta+\gamma)}}{\Gamma(\delta)} \int_0^t \frac{s^\gamma f(s)}{(t-s)^{1-\delta}} \mathrm{d}s, \quad \gamma,\ \delta > 0,$$

that was introduced for the first time by Kober in [30]. For $\gamma = 0$, the Kober operator is reduced to the Riemann–Liouville fractional integral with a power weight:

$$I_0^\delta f(t) = \frac{t^{-\delta}}{\Gamma(\delta)} \int_0^t \frac{f(s)}{(t-s)^{1-\delta}} \mathrm{d}s, \quad \delta > 0.$$

From the definition of the Riemann–Liouville fractional derivative and integral, we can obtain the following lemmas.

Lemma 2.5 (See [2]). *Let $y \in C(0,T) \cap L^1(0,T)$. Then the fractional differential equation $D^q y(t) = 0$ has a solution*

$$y(t) = c_1 t^{q-1} + c_2 t^{q-2} + \cdots + c_n t^{q-n},$$

where $c_i \in \mathbb{R}$, $i = 1,2,\ldots,n$ and $n-1 < q < n$.

Lemma 2.6 (See [2]). *For $y \in C(0,T) \cap L^1(0,T)$, it holds that*

$$J^q D^q y(t) = y(t) + c_1 t^{q-1} + c_2 t^{q-2} + \cdots + c_n t^{q-n},$$

where $c_i \in \mathbb{R}$, $i = 1,2,\ldots,n$ and $n-1 < q < n$.

2.2. Some Auxiliary Lemmas

For easy reference we include the following well known formula as a lemma.

Lemma 2.7. *Let $\delta, \eta > 0$ and $\gamma, q \in \mathbb{R}$. Then we have*

$$I_\eta^{\gamma,\delta} t^q = \frac{t^q \Gamma(\gamma + (q/\eta) + 1)}{\Gamma(\gamma + (q/\eta) + \delta + 1)}. \tag{2.1}$$

Lemma 2.8. *Let $1 < q \leq 2$, $\delta_i, \eta_i > 0$, $\alpha, \gamma_i, \beta_i \in \mathbb{R}$, $\xi_i \in (0,T)$, $i = 1,2,\ldots,m$ and $h \in C([0,T],\mathbb{R})$. Then the linear Riemann–Liouville fractional differential equation subject to the Erdélyi–Kober fractional integral boundary conditions*

$$\begin{cases} D^q x(t) = h(t), \quad t \in (0,T), \\ x(0) = 0, \quad \alpha x(T) = \displaystyle\sum_{i=1}^m \beta_i I_{\eta_i}^{\gamma_i,\delta_i} x(\xi_i), \end{cases} \tag{2.2}$$

is equivalent to the following fractional integral equation

$$x(t) = J^q h(t) - \frac{t^{q-1}}{\Lambda}\left(\alpha J^q h(T) - \sum_{i=1}^{m} \beta_i I_{\eta_i}^{\gamma_i,\delta_i} J^q h(\xi_i)\right), \tag{2.3}$$

where

$$\Lambda := \alpha T^{q-1} - \sum_{i=1}^{m} \frac{\beta_i {\xi_i}^{q-1}\Gamma(\gamma_i + (q-1)/\eta_i + 1)}{\Gamma(\gamma_i + (q-1)/\eta_i + \delta_i + 1)} \neq 0. \tag{2.4}$$

Proof. Using Lemmas 2.5, 2.6, the Eq. (2.2) can be expressed as an equivalent integral equation

$$x(t) = J^q h(t) - c_1 t^{q-1} - c_2 t^{q-2}, \tag{2.5}$$

for $c_1, c_2 \in \mathbb{R}$. The first condition of (2.2) implies that $c_2 = 0$. Taking the Erdélyi–Kober fractional integral of order $\delta_i > 0$ with $\eta_i > 0$ and $\gamma_i \in \mathbb{R}$ for (2.5) and using Lemma 2.7, we have

$$I^{\delta_i}_{\eta_i,\gamma_i} x(t) = I^{\delta_i}_{\eta_i,\gamma_i} J^q h(t) - c_1 \frac{t^{q-1}\Gamma(\gamma + (q-1)/\eta + 1)}{\Gamma(\gamma + (q-1)/\eta + \delta + 1)}.$$

The second condition of (2.2) yields

$$\begin{aligned}\alpha J^q h(T) - c_1 \alpha T^{q-1} &= \sum_{i=1}^{m} \beta_i I_{\eta_i}^{\gamma_i,\delta_i} J^q h(\xi_i) \\ &\quad - c_1 \sum_{i=1}^{m} \frac{\beta_i \xi_i^{q-1}\Gamma(\gamma_i + (q-1)/\eta_i + 1)}{\Gamma(\gamma_i + (q-1)/\eta_i + \delta_i + 1)},\end{aligned}$$

which implies

$$c_1 = \frac{1}{\Lambda}\left(\alpha J^q h(T) - \sum_{i=1}^{m} \beta_i I_{\eta_i}^{\gamma_i,\delta_i} J^q h(\xi_i)\right).$$

Substituting the values of c_1 and c_2 in (2.5), we obtain the desired solution (2.3).

Conversely, it can easily be shown by direct computation that the integral equation (2.3) satisfies the problem (2.2). This completes the proof. □

2.3. Basic Material for Multivalued Maps

Here we outline some basic concepts of multivalued analysis [31,32].

Let $C([0,T],\mathbb{R})$ denote the Banach space of all continuous functions from $[0,T]$ into $\mathbb{R}$ with the norm $\|x\| = \sup\{|x(t)|,\ t \in [0,T]\}$. Also by $L^1([0,T],\mathbb{R})$ we denote the space of functions $x : [0,T] \to \mathbb{R}$ such that $\|x\|_{L^1} = \int_0^T |x(t)|\mathrm{d}t$.

For a normed space $(X, \|\cdot\|)$, let

$$\begin{aligned}\mathcal{P}_{cl}(X) &= \{Y \in \mathcal{P}(X) : Y \text{ is closed}\},\\ \mathcal{P}_b(X) &= \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\},\\ \mathcal{P}_{cp,c}(X) &= \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}.\end{aligned}$$

A multi-valued map $G : X \to \mathcal{P}(X)$:

(i) is *convex (closed) valued* if $G(x)$ is convex (closed) for all $x \in X$.
(ii) is *bounded* on bounded sets if $G(Y) = \cup_{x\in Y} G(x)$ is bounded in X for all $Y \in \mathcal{P}_b(X)$ (i.e. $\sup_{x\in Y}\{\sup\{|y| : y \in G(x)\}\} < \infty$).
(iii) is called *upper semi-continuous (u.s.c.)* on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood $\mathcal{N}_0$ of x_0 such that $G(\mathcal{N}_0) \subseteq N$.
(iv) G is *lower semi-continuous (l.s.c.)* if the set $\{y \in X : G(y) \cap Y \neq \emptyset\}$ is open for any open set Y in X.
(v) is said to be *completely continuous* if $G(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in \mathcal{P}_b(X)$; If the multi-valued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph, i.e., $x_n \to x_*,\ y_n \to y_*,\ y_n \in G(x_n)$ imply $y_* \in G(x_*)$.
(vi) is said to be *measurable* if for every $y \in X$, the function

$$t \longmapsto d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$$

is measurable.
(vii) *has a fixed point* if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by $FixG$.

Definition 2.9. A multivalued map $F : [0, T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if

(i) $t \longmapsto F(t, x)$ is measurable for each $x \in \mathbb{R}$;
(ii) $x \longmapsto F(t, x)$ is upper semicontinuous for almost all $t \in [0, T]$; Further a Carathéodory function F is called L^1-Carathéodory if
(iii) for each $\rho > 0$, there exists $\varphi_\rho \in L^1([0, T], \mathbb{R}^+)$ such that

$$\|F(t, x)\| = \sup\{|v| : v \in F(t, x)\} \leq \varphi_\rho(t)$$

for all $\|x\| \leq \rho$ and for a.e. $t \in [0, T]$.

Define the function $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R} \cup \{\infty\}$ by

$$H_d(A, B) = \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\right\},$$

where $d(A, b) = \inf_{a\in A} d(a; b)$ and $d(a, B) = \inf_{b\in B} d(a; b)$.

Definition 2.10. A multivalued operator $N : X \to \mathcal{P}_{cl}(X)$ is called

(a) γ-Lipschitz if and only if there exists $\gamma > 0$ such that

$$H_d(N(x), N(y)) \leq \gamma d(x, y) \quad \text{for each } x, y \in X;$$

(b) a contraction if and only if it is γ-Lipschitz with $\gamma < 1$.

For each $y \in C([0, T], \mathbb{R})$, define the set of selections of F by

$$S_{F,y} := \{v \in L^1([0, T], \mathbb{R}) : v(t) \in F(t, y(t)) \text{ on } [0, T]\}.$$

We define the graph of G to be the set $Gr(G) = \{(x, y) \in X \times Y, y \in G(x)\}$ and recall a result for closed graphs and upper-semicontinuity.

Lemma 2.11 ([31, Proposition 1.2]). *If $G : X \to \mathcal{P}_{cl}(Y)$ is u.s.c., then $Gr(G)$ is a closed subset of $X \times Y$; i.e., for every sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ and $\{y_n\}_{n\in\mathbb{N}} \subset Y$, if when $n \to \infty$, $x_n \to x_*$, $y_n \to y_*$ and $y_n \in G(x_n)$, then $y_* \in G(x_*)$. Conversely, if G is completely continuous and has a closed graph, then it is upper semi-continuous.*

The following lemma will be used in the sequel.

Lemma 2.12 ([33]). *Let X be a Banach space. Let $F : J \times \mathbb{R} \to \mathcal{P}_{cp,c}(X)$ be an L^1- Carathéodory multivalued map and let Θ be a linear continuous mapping from $L^1(J, X)$ to $C(J, X)$. Then the operator*

$$\Theta \circ S_F : C(J,X) \to \mathcal{P}_{cp,c}(C(J,X)), \quad x \mapsto (\Theta \circ S_F)(x) = \Theta(S_{F,x})$$

is a closed graph operator in $C(J, X) \times C(J, X)$.

To prove our main result in this section, we use the following form of the nonlinear alternative for contractive maps [34, Corollary 3.8].

Theorem 2.13. *Let X be a Banach space, and D a bounded neighborhood of $0 \in X$. Let $Z_1 : X \to \mathcal{P}_{cp,c}(X)$ and $Z_2 : \bar{D} \to \mathcal{P}_{cp,c}(X)$ two multi-valued operators satisfying*

(a) *Z_1 is contraction, and*
(b) *Z_2 is upper semicontinuous and compact.*

Then, if $G = Z_1 + Z_2$, either

(i) *G has a fixed point in $\bar{D}$ or*
(ii) *there is a point $u \in \partial D$ and $\lambda \in (0, 1)$ with $u \in \lambda G(u)$.*

3. Existence Results

Throughout this paper, for convenience, we use the following expressions

$$J^q f(z) = \frac{1}{\Gamma(q)} \int_0^z (z-s)^{q-1} f(s)\mathrm{d}s, \quad z \in \{t, T\},$$

for $t \in [0, T]$ and

$$I_{\eta_i}^{\gamma_i,\delta_i} J^q f(\xi_i) = \frac{\eta_i \xi_i^{-\eta_i(\delta_i+\gamma_i)}}{\Gamma(q)\Gamma(\delta_i)} \int_0^{\xi_i} \int_0^r \frac{r^{\eta_i\gamma_i+\eta_i-1}(r-s)^{q-1}}{(\xi_i^{\eta_i} - r^{\eta_i})^{1-\delta_i}} f(s)\mathrm{d}s\, \mathrm{d}r,$$

where $\xi_i \in (0, T)$ for $i = 1, 2, \ldots, m$.

Let us list the following assumptions:

(H_1) $F : [0, T] \times \mathbb{R} \to \mathcal{P}_{cp,c}(\mathbb{R})$ is L^1-Carathéodory;

(H_2) there exists a continuous nondecreasing function $\Phi : [0, \infty) \to (0, \infty)$ and a function $p \in L^1([0, T], \mathbb{R}^+)$ such that

$$\|F(t,x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t,x)\} \le p(t)\Phi(\|x\|) \quad \text{for each } (t,x) \in [0,T] \times \mathbb{R};$$

(H_3) the multi-valued map $t \to G(t, x)$ is measurable for each $x \in \mathbb{R}$ and integrably bounded, i.e. there exists a function $M \in L^1([0, T], \mathbb{R}^+)$ such that

$$|G(t,x)| := \sup\{|g| : g(t) \in G(t,x)\} \le M(t), \quad \text{for a.e. } t \in [0,T] \quad \text{and } x \in \mathbb{R};$$

(H_4) $G : [0,T] \times \mathbb{R} \to \mathcal{P}_{cp,c}(\mathbb{R})$ and there exists a function $\ell \in L^1([0,T],\mathbb{R})$ such that

$$H_d(G(t,x), G(t,y)) \le \ell(t)|x-y|, \quad t \in [0,T]$$

for all $x, y \in \mathbb{R}$ with

$$J^q\ell(T) + \frac{T^{q-1}}{|\Lambda|}\left(\alpha J^q \ell(T) + \sum_{i=1}^{m} \beta_i I_{\eta_i}^{\gamma_i,\delta_i} J^q \ell(\xi_i)\right) < 1; \tag{3.1}$$

(H_5) there exists a constant $r > 0$ such that

$$\frac{r}{\Phi(r)\Psi_1 + \Psi_2} > 1, \tag{3.2}$$

where

$$\Psi_1 = J^q p(T) + \frac{|\alpha| T^{q-1}}{|\Lambda|} J^q p(T) + \frac{T^{q-1}}{|\Lambda|} \sum_{i=1}^{m} |\beta_i| I_{\eta_i}^{\gamma_i,\delta_i} J^q p(\xi_i)$$

and

$$\Psi_2 = J^q M(T) + \frac{|\alpha| T^{q-1}}{|\Lambda|} J^q M(T) + \frac{T^{q-1}}{|\Lambda|} \sum_{i=1}^{m} |\beta_i| I_{\eta_i}^{\gamma_i,\delta_i} J^q M(\xi_i).$$

We transform the problem (1.1) into a fixed point problem. Consider the operator

$$\mathcal{N}(x) = \left\{ \begin{array}{l} h \in C([0,T],\mathbb{R}) : \\ h(t) = \left\{ \begin{array}{l} J^q[f(t) + g(t)] \\ -\dfrac{t^{q-1}}{\Lambda}\Bigg(\alpha J^q[f(T) + g(T)] \\ -\displaystyle\sum_{i=1}^{m} \beta_i I_{\eta_i}^{\gamma_i,\delta_i} J^q[f(\xi_i) + g(\xi_i)]\Bigg) \end{array} \right. \end{array} \right\} \tag{3.3}$$

for $f \in S_{F,x}$ and $g \in S_{G,x}$.

Define the operators $\mathcal{F}, \mathcal{G} : C([0,T],\mathbb{R}) \to \mathcal{P}(C([0,T],\mathbb{R}))$ by

$$\mathcal{F}(x) = \left\{ \begin{array}{l} h \in C([0,T],\mathbb{R}) : \\ h(t) = \left\{ \begin{array}{l} J^q f(t) \\ -\dfrac{t^{q-1}}{\Lambda}\left(\alpha J^q f(T) - \displaystyle\sum_{i=1}^{m} \beta_i I_{\eta_i}^{\gamma_i,\delta_i} J^q f(\xi_i)\right) \end{array} \right. \end{array} \right\} \tag{3.4}$$

for $f \in S_{F,x}$, and

$$\mathcal{G}(x) = \left\{ \begin{array}{l} z \in C([0,T],\mathbb{R}) : \\ z(t) = \left\{ \begin{array}{l} J^q g(t) \\ -\dfrac{t^{q-1}}{\Lambda}\left(\alpha J^q g(T) - \displaystyle\sum_{i=1}^{m} \beta_i I_{\eta_i}^{\gamma_i,\delta_i} J^q g(\xi_i)\right) \end{array} \right. \end{array} \right\} \tag{3.5}$$

for $g \in S_{G,x}$. Observe that $\mathcal{N} = \mathcal{F} + \mathcal{G}$. We shall show that the operators $\mathcal{F}$ and $\mathcal{G}$ satisfy all the conditions of Theorem 2.13 on $[0,T]$.

Lemma 3.1. *The operators $\mathcal{F}$ and $\mathcal{G}$, defined by (3.4) and (3.5) respectively, are compact and convex-valued.*

Proof. First, we show that the operators $\mathcal{F}$ and $\mathcal{G}$ define the multivalued operators $\mathcal{F}, \mathcal{G} : B_r \to \mathcal{P}_{cp,c}(C([0,T],\mathbb{R}))$ where $B_r = \{x \in C([0,T],\mathbb{R}) : \|x\| \le r\}$ is a bounded set in $C([0,T],\mathbb{R})$. We shall prove that $\mathcal{F}$ is compact-valued on B_r. Note that the operator $\mathcal{F}$ is equivalent to the composition $\mathcal{L} \circ S_F$, where $\mathcal{L}$ is the continuous linear operator on $L^1([0,T],\mathbb{R})$ into $C([0,T],\mathbb{R})$, defined by

$$\mathcal{L}(v)(t) = J^q v(t) + \frac{t^{q-1}}{\Lambda}\Big\{\alpha I_\eta^{\gamma,\delta} J^q v(\xi) - J^q v(T)\Big\}.$$

Suppose that $x \in B_r$ is arbitrary and let $\{v_n\}$ be a sequence in $S_{F,x}$. Then, by definition of $S_{F,x}$, we have $v_n(t) \in F(t, x(t))$ for almost all $t \in [0,T]$. Since $F(t, x(t))$ is compact for all $t \in J$, there is a convergent subsequence of $\{v_n(t)\}$ (we denote it by $\{v_n(t)\}$ again) that converges in measure to some $v(t) \in S_{F,x}$ for almost all $t \in J$. On the other hand, $\mathcal{L}$ is continuous, so $\mathcal{L}(v_n)(t) \to \mathcal{L}(v)(t)$ pointwise on $[0,T]$.

In order to show that the convergence is uniform, we have to show that $\{\mathcal{L}(v_n)\}$ is an equi-continuous sequence. Let $t_1, t_2 \in [0,T]$ with $t_1 < t_2$. Then, we have

$$\begin{aligned}
&|\mathcal{L}(v_n)(t_2) - \mathcal{L}(v_n)(t_1)| \\
&\quad \le |J^q v_n(t_2) - J^q v_n(t_1)| + \frac{|t_2^{q-1} - t_1^{q-1}|}{|\Lambda|} J^q |v_n(T)| \\
&\qquad + \frac{|\alpha||t_2^{q-1} - t_1^{q-1}|}{|\Lambda|} |I_\eta^{\gamma,\delta} J^q |v_n(\xi)| \\
&\quad \le \frac{\psi(r)}{\Gamma(q)} \left| \int_0^{t_1} [(t_2 - s)^{q-1} - (t_1 - s)^{q-1}] p(s) \mathrm{d}s + \int_{t_1}^{t_2} (t_2 - s)^{q-1} p(s) \mathrm{d}s \right| \\
&\qquad + \frac{|t_2^{q-1} - t_1^{q-1}|}{|\Lambda|} \big(J^q |v_n(T)| + |\alpha||I_\eta^{\gamma,\delta} J^q |v_n(\xi)|\big).
\end{aligned}$$

We see that the right hand of the above inequality tends to zero as $t_2 \to t_1$. Thus, the sequence $\{\mathcal{L}(v_n)\}$ is equi-continuous and by using the Arzelá–Ascoli theorem, we get that there is a uniformly convergent subsequence. So, there is a subsequence of $\{v_n\}$ (we denote it again by $\{v_n\}$) such that $\mathcal{L}(v_n) \to \mathcal{L}(v)$. Note that, $\mathcal{L}(v) \in \mathcal{L}(S_{F,x})$. Hence, $\mathcal{F}(x) = \mathcal{L}(S_{F,x})$ is compact for all $x \in B_r$. So $\mathcal{F}(x)$ is compact.

Now, we show that $\mathcal{F}(x)$ is convex for all $x \in C([0,T],\mathbb{R})$. Let $h_1, h_2 \in \mathcal{F}(x)$. We select $f_1, f_2 \in S_{F,x}$ such that

$$h_i(t) = J^q f_i(t) + \frac{t^{q-1}}{\Lambda}\big\{\alpha I_\eta^{\gamma,\delta} J^q f_i(\xi) - J^q f_i(T)\big\}, \quad i = 1, 2$$

for almost all $t \in [0,T]$. Let $0 \le \lambda \le 1$. Then, we have

$$\begin{aligned}
&[\lambda h_1 + (1-\lambda) h_2](t) = J^q[\lambda f_1(t) + (1-\lambda) f_2(t)] \\
&\quad + \frac{t^{q-1}}{\Lambda}\big\{\alpha I_\eta^{\gamma,\delta} J^q[\lambda f_1(\xi) + (1-\lambda) f_2(\xi)] - J^q[\lambda f_1(T) + (1-\lambda) f_2(T)]\big\}.
\end{aligned}$$

Since F has convex values, so $S_{F,u}$ is convex and $\lambda f_1(s)+(1-\lambda)f_2(s)\in S_{F,x}$. Thus

$$\lambda h_1+(1-\lambda)h_2\in\mathcal{F}(x).$$

Consequently, $\mathcal{F}$ is convex-valued. Similarly, $\mathcal{G}$ is compact and convex-valued. □

Lemma 3.2. *Assume that* (H_3) *and* (H_4) *hold. Then the operator* $\mathcal{G}$ *defined by* (3.5) *is a contraction.*

Proof. Let $x,\bar{x}\in C^2([0,T],\mathbb{R})$ and $h_1\in\mathcal{G}(x)$. Then there exists $g_1(t)\in G(t,x(t))$ such that, for each $t\in[0,T]$,

$$h_1(t)=J^q g_1(t)-\frac{t^{q-1}}{\Lambda}\left(\alpha J^q g_1(T)-\sum_{i=1}^m\beta_i I_{\eta_i}^{\gamma_i,\delta_i}J^q g_1(\xi_i)\right).$$

By (H_4), we have

$$H_d(F(t,x),F(t,\bar{x}))\le\ell(t)|x(t)-\bar{x}(t)|.$$

So, there exists $w\in G(t,\bar{x}(t))$ such that

$$|v_1(t)-w(t)|\le\ell(t)|x(t)-\bar{x}(t)|,\quad t\in[0,T].$$

Define $U:[0,T]\to\mathcal{P}(\mathbb{R})$ by

$$U(t)=\{w\in\mathbb{R}:|v_1(t)-w|\le\ell(t)|x(t)-\bar{x}(t)|\}.$$

Since the multivalued operator $U(t)\cap G(t,\bar{x}(t))$ is measurable (Proposition III.4 [35]), there exists a function $v_2(t)$ which is a measurable selection for U. So $v_2(t)\in G(t,\bar{x}(t))$ and for each $t\in[0,T]$, we have $|v_1(t)-v_2(t)|\le\ell(t)|x(t)-\bar{x}(t)|$.

For each $t\in[0,T]$, let us define

$$h_2(t)=J^q g_2(t)-\frac{t^{q-1}}{\Lambda}\left(\alpha J^q g_2(T)-\sum_{i=1}^m\beta_i I_{\eta_i}^{\gamma_i,\delta_i}J^q g_2(\xi_i)\right).$$

Thus,

$$\begin{aligned}&|h_1(t)-h_2(t)|\\&\quad\le J^q|g_1(t)-g_2(t)|\\&\qquad+\frac{t^{q-1}}{|\Lambda|}\left(\alpha J^q|g_1(T)-g_2(T)|+\sum_{i=1}^m\beta_i I_{\eta_i}^{\gamma_i,\delta_i}J^q|g_1(\xi_i)-g_2(\xi_i)|\right)\\&\quad\le\left\{J^q\ell(T)+\frac{T^{q-1}}{|\Lambda|}\left(\alpha J^q\ell(T)+\sum_{i=1}^m\beta_i I_{\eta_i}^{\gamma_i,\delta_i}J^q\ell(\xi_i)\right)\right\}\|x-\bar{x}\|.\end{aligned}$$

Hence,

$$\|h_1-h_2\|\le\left\{J^q\ell(T)+\frac{T^{q-1}}{|\Lambda|}\left(\alpha J^q\ell(T)+\sum_{i=1}^m\beta_i I_{\eta_i}^{\gamma_i,\delta_i}J^q\ell(\xi_i)\right)\right\}\|x-\bar{x}\|.$$

Analogously, interchanging the roles of x and $\overline{x}$, we obtain

$$H_d(\mathcal{F}(x),\mathcal{F}(\bar{x}))\le\delta\|x-\bar{x}\|,$$

where

$$\delta = J^q\ell(T) + \frac{T^{q-1}}{|\Lambda|}\left(\alpha J^q\ell(T) + \sum_{i=1}^{m}\beta_i I_{\eta_i}^{\gamma_i,\delta_i} J^q\ell(\xi_i)\right).$$

So $\mathcal{G}$ is a contraction, since $\delta < 1$ by (3.1). This completes the proof. □

Lemma 3.3. *Assume that* (H_1) *and* (H_2) *hold. Then the operator* $\mathcal{F}$ *defined by* (3.4) *is upper semicontinuous.*

Proof. For the sake of convenience, we break the proof into several steps.
Step 1. $\mathcal{F}$ *maps bounded sets (balls) into bounded sets in* $C([0,T],\mathbb{R})$.
For a positive number ρ, let $B_\rho = \{x \in C([0,T],\mathbb{R}) : \|x\| \le \rho\}$ be a bounded ball in $C([0,T],\mathbb{R})$. Then, for each $h \in \mathcal{F}(x), x \in B_\rho$, there exists $f \in S_{F,x}$ such that

$$h(t) = J^q f(t) - \frac{t^{q-1}}{\Lambda}\left(\alpha J^q f(T) - \sum_{i=1}^{m}\beta_i I_{\eta_i}^{\gamma_i,\delta_i} J^q f(\xi_i)\right).$$

Then we have

$$\begin{aligned}|h(x)| &\le J^q|f(T)| + \frac{|\alpha|T^{q-1}}{|\Lambda|}J^q|f(T)| + \frac{T^{q-1}}{|\Lambda|}\sum_{i=1}^{m}|\beta_i| I_{\eta_i}^{\gamma_i,\delta_i} J^q|f(\xi)|\\ &\le \Phi(\|x\|)J^q p(T) + \Phi(\|x\|)\frac{|\alpha|T^{q-1}}{|\Lambda|}J^q p(T)\\ &\quad +\Phi(\|x\|)\frac{T^{q-1}}{|\Lambda|}\sum_{i=1}^{m}|\beta_i| I_{\eta_i}^{\gamma_i,\delta_i} J^q p(\xi_i),\end{aligned}$$

and consequently,

$$\|h\| \le \Phi(r)\left\{J^q p(T) + \frac{|\alpha|T^{q-1}}{|\Lambda|}J^q p(T) + \frac{T^{q-1}}{|\Lambda|}\sum_{i=1}^{m}|\beta_i| I_{\eta_i}^{\gamma_i,\delta_i} J^q p(\xi_i)\right\}.$$

Step 2. $\mathcal{F}$ *maps bounded sets into equicontinuous sets of* $C([0,T],\mathbb{R})$.
Let $\tau_1, \tau_2 \in [0,T]$ with $\tau_1 < \tau_2$ and $x \in B_\rho$. For each $h \in \mathcal{F}(x)$, we obtain

$$\begin{aligned}&|h(\tau_2) - h(\tau_1)|\\ &\le |J^q v(\tau_2) - J^q v(\tau_1)| + \frac{|\alpha||\tau_2^{q-1} - \tau_1^{q-1}|}{|\Lambda|}J^q|v(T)|\\ &\quad + \frac{|\tau_2^{q-1} - \tau_1^{q-1}|}{|\Lambda|}\sum_{i=1}^{m}|\beta_i| I_{\eta_i}^{\gamma_i,\delta_i} J^q|v(\xi)|\\ &\le \frac{\Phi(r)}{\Gamma(q)}\left|\int_0^{\tau_1}[(\tau_2 - s)^{q-1} - (\tau_1 - s)^{q-1}]p(s)ds + \int_{\tau_1}^{\tau_2}(\tau_2 - s)^{q-1}p(s)ds\right|\\ &\quad + \frac{|\tau_2^{q-1} - \tau_1^{q-1}|\Phi(r)}{|\Lambda|}\left(|\alpha|J^q p(T) + \sum_{i=1}^{m}|\beta_i| I_{\eta_i}^{\gamma_i,\delta_i} J^q(\xi_i)\right).\end{aligned}$$

Obviously the right hand side of the above inequality tends to zero independently of $x \in B_\rho$ as $\tau_2 - \tau_1 \to 0$. As $\mathcal{F}$ satisfies the above three assumptions, therefore it follows by the Arzelá–Ascoli theorem that $\mathcal{F} : C([0,T],\mathbb{R}) \to \mathcal{P}(C([0,T],\mathbb{R}))$ is completely continuous.

Since $\mathcal{F}$ is completely continuous, in order to prove that it is u.s.c., it is enough to prove that it has a closed graph. Thus, in our next step, we show that

Step 3. $\mathcal{F}$has a closed graph.

Let $x_n \to x_*, h_n \in \mathcal{F}(x_n)$ and $h_n \to h_*$. Then we need to show that $h_* \in \mathcal{F}(x_*)$. Associated with $h_n \in \mathcal{F}(x_n)$, there exists $v_n \in S_{F,x_n}$ such that for each $t \in [0,T]$,

$$h_n(t) = J^q v_n(t) - \frac{t^{q-1}}{\Lambda}\left(\alpha J^q v_n(T) - \sum_{i=1}^{m} \beta_i I_{\eta_i}^{\gamma_i,\delta_i} J^q v_n(\xi_i)\right).$$

Thus it suffices to show that there exists $v_* \in S_{F,x_*}$ such that for each $t \in [0,T]$,

$$h_*(t) = J^q v_*(t) - \frac{t^{q-1}}{\Lambda}\left(\alpha J^q v_*(T) - \sum_{i=1}^{m} \beta_i I_{\eta_i}^{\gamma_i,\delta_i} J^q v_*(\xi_i)\right).$$

Let us consider the linear operator $\Theta : L^1([0,T],\mathbb{R}) \to C([0,T],\mathbb{R})$ given by

$$f \mapsto \Theta(v)(t) = J^q v(t) - \frac{t^{q-1}}{\Lambda}\left(\alpha J^q v(T) - \sum_{i=1}^{m} \beta_i I_{\eta_i}^{\gamma_i,\delta_i} J^q v(\xi_i)\right).$$

Observe that

$$\|h_n(t) - h_*(t)\| = \left\| J^q(v_n(t) - v_*(t)) - \frac{t^{q-1}}{\Lambda}\left(\alpha J^q(v_n(T) - v_*(T)) - \sum_{i=1}^{m} \beta_i I_{\eta_i}^{\gamma_i,\delta_i} J^q(v_n(\xi_i) - v_*(\xi_i))\right)\right\| \to 0,$$

as $n \to \infty$.

Thus, it follows by Lemma 2.12 that $\Theta \circ S_F$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,x_n})$. Since $x_n \to x_*$, therefore, we have

$$h_*(t) = J^q v_*(t) - \frac{t^{q-1}}{\Lambda}\left(\alpha J^q v_*(T) - \sum_{i=1}^{m} \beta_i I_{\eta_i}^{\gamma_i,\delta_i} J^q v_*(\xi_i)\right),$$

for some $v_* \in S_{F,x_*}$. Thus the operator $\mathcal{F}$ is upper semicontinuous. □

Theorem 3.4. *Assume that (H_1)–(H_5) are satisfied. Then the boundary value problem (1.1) has at least one solution on $[0,T]$.*

Proof. Define an open ball $B_r = \{x \in C([0,T],\mathbb{R}) : \|x\| \le r\}$, where r satisfies the inequality (3.6) given in condition (H_5). As a consequence of Lemmas 3.1, 3.2, 3.3 the operators $\mathcal{F}$ and $\mathcal{G}$ satisfy all the conditions of Theorem 2.13 and hence its conclusion implies either condition (i) or condition (ii) holds.

We show that the conclusion (ii) is not possible. If $x \in \lambda\mathcal{F}(x) + \lambda\mathcal{G}(x)$ for $\lambda \in (0,1)$, then there exist $f \in S_{F,x}$ and $g \in S_{G,x}$ such that

$$x(t) = \lambda J^q f(t) - \lambda \frac{t^{q-1}}{\Lambda}\left(\alpha J^q f(T) - \sum_{i=1}^{m} \beta_i I_{\eta_i}^{\gamma_i,\delta_i} J^q f(\xi_i)\right)$$
$$+ \lambda J^q g(t) - \lambda \frac{t^{q-1}}{\Lambda}\left(\alpha J^q g(T) - \sum_{i=1}^{m} \beta_i I_{\eta_i}^{\gamma_i,\delta_i} J^q g(\xi_i)\right).$$

In view of $(H_2), (H_3)$ we obtain

$$|x(t)| \le \Phi(\|x\|)\left\{ J^q p(T) + \frac{|\alpha| T^{q-1}}{|\Lambda|} J^q p(T) + \frac{T^{q-1}}{|\Lambda|} \sum_{i=1}^{m} |\beta_i| I_{\eta_i}^{\gamma_i,\delta_i} J^q p(\xi_i) \right\}$$
$$+ J^q M(T) + \frac{|\alpha| T^{q-1}}{|\Lambda|} J^q M(T) + \frac{T^{q-1}}{|\Lambda|} \sum_{i=1}^{m} |\beta_i| I_{\eta_i}^{\gamma_i,\delta_i} J^q M(\xi_i),$$

or

$$\|x\| \le \Phi(\|x\|)\left\{ J^q p(T) + \frac{|\alpha| T^{q-1}}{|\Lambda|} J^q p(T) + \frac{T^{q-1}}{|\Lambda|} \sum_{i=1}^{m} |\beta_i| I_{\eta_i}^{\gamma_i,\delta_i} J^q p(\xi_i) \right\}$$
$$+ J^q M(T) + \frac{|\alpha| T^{q-1}}{|\Lambda|} J^q M(T) + \frac{T^{q-1}}{|\Lambda|} \sum_{i=1}^{m} |\beta_i| I_{\eta_i}^{\gamma_i,\delta_i} J^q M(\xi_i). \quad (3.6)$$

If condition (ii) of Theorem 2.13 holds, then there exists $\lambda \in (0,1)$ and $x \in \partial B_r$ with $x = \lambda\mathcal{F}(x) + \lambda\mathcal{G}(x)$. Then, x is a solution of (1.1) with $\|x\| = r$. Now, by the inequality (3.6), we get

$$\frac{r}{\Phi(r)\Psi_1 + \Psi_2} \le 1,$$

which contradicts (3.2). Hence, $\mathcal{N}$ has a fixed point in $[0,T]$ by Theorem 2.13, and consequently the problem (1.1) has a solution. This completes the proof. □

3.1. Example

In this section, we will illustrate our main result with the help of an example. Let us consider the following boundary value problem for Riemann–Liouville fractional differential inclusions with Erdélyi–Kober fractional integral boundary conditions

$$\begin{cases} D^{3/2}x(t) \in F(t,x(t)) + G(t,x(t)), \quad t \in (0,1), \\ x(0) = 0, \\ \alpha x(1) = \frac{3}{2} I_{\frac{1}{5}}^{\frac{1}{3},\frac{1}{7}} x\left(\frac{1}{4}\right) + I_{\frac{2}{5}}^{\frac{2}{3},\frac{4}{7}} x\left(\frac{1}{2}\right) + 2 I_{\frac{3}{5}}^{\frac{3}{4},\frac{6}{7}} x\left(\frac{3}{4}\right). \end{cases} \quad (3.7)$$

Here $q = 3/2$, $m = 3$, $T = 1$, $\beta_1 = 3/2$, $\beta_2 = 1$, $\beta_3 = 2$, $\eta_1 = 1/5$, $\eta_2 = 2/5$, $\eta_3 = 3/5$, $\gamma_1 = 1/3$, $\gamma_2 = 2/3$, $\gamma_3 = 3/4$, $\delta_1 = 1/7$, $\delta_2 = 4/7$, $\delta_3 = 6/7$, $\xi_1 = 1/4$, $\xi_2 = 1/2$, $\xi_3 = 3/4$, the multivalued maps $F, G : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ are given by

$$x \to F(t,x) = \left[\frac{t\cos^2 x}{(16+t^2)(1+\cos^2 x)}, \frac{x^2 \sin t}{(1+x^2)(1+t^4)} + \frac{1}{2} \right], \quad (3.8)$$

$$x \to G(t,x) = \left[\frac{2}{\pi\sqrt{t+9}}\tan^{-1} x, \frac{t|x|}{1+|x|}+1\right] \tag{3.9}$$

and α will be fixed later.

For $f \in F$, we have

$$|f| \leq \max\left(\frac{t\cos^2 x}{(16+t^2)(1+\cos^2 x)}, \frac{x^2 \sin t}{(1+x^2)(1+t^4)} + \frac{1}{2}\right) \leq \frac{3}{2}, \quad x \in \mathbb{R}.$$

According to the condition (H_2), let us fix $p(t) = 3/2, \Phi(\|x\|) = 1$. In view of the assumptions (H_3) and (H_4), we have $M(t) = 1+t, \ell(t) = t$. Using the given data, we find that the condition (3.1) holds for any $\alpha > 3.479637$. In our analysis, we take $\alpha = 4$. With this choice, $\Lambda \approx 2.185595$, $\Psi_1 \approx 3.532777$, $\Psi_2 \approx 3.260379$. Hence by the condition (3.2) given by (H_5), there exists $r > 6.793156$. Thus all the conditions of Theorem 3.4 are satisfied. In consequence, there exists a solution for problem (3.7) on $[0,1]$.

Remark 3.5. The existence result obtained in this paper corresponds to the Dirichlet boundary value problem of perturbed fractional differential inclusions if we take $\alpha = 1$ and $\beta_i = 0$, $i = 1, \ldots, m$ in the statement of (1.1).

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