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On Certain Functional Equations on Standard Operator Algebras

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Abstract. In this paper, functional equations related to derivations on semiprime rings and standard operator algebras are investigated. We prove the following result which is related to a classical result of Chernoff. Let X be a real or complex Banach space, let $\mathcal{L}(X)$ be the algebra of all bounded linear operators of X into itself and let $\mathcal{A}(X) \subset \mathcal{L}(X)$ be a standard operator algebra. Suppose there exist linear mappings $D, G : \mathcal{A}(X) \to \mathcal{L}(X)$ satisfying the relations $D(A^{2n+1}) = D(A^{2n})A + A^{2n}G(A)$ and $G(A^{2n+1}) = G(A^{2n})A + A^{2n}D(A)$ for all $A \in \mathcal{A}(X)$. Then there exists $B \in \mathcal{L}(X)$ such that $D(A) = G(A) = [A, B]$ for all $A \in \mathcal{A}(X)$.

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1. Introduction

Throughout the paper, R will denote an associative ring with center $Z(R)$. As usual we write $[x, y]$ for $xy - yx$. Given an integer $n \geq 2$, a ring R is said to be n-torsion free, if for $x \in R$, $nx = 0$ implies $x = 0$. Recall that a ring R is prime, if for $a, b \in R$, $aRb = (0)$ implies $a = 0$ or $b = 0$ and is semiprime in case $aRa = (0)$ implies $a = 0$. We denote by Q_s the symmetric Martindale ring of quotients. For explanation of Q_s, we refer the reader to [1]. Let A be an algebra over the real or complex field and let B be a subalgebra of A. A linear mapping $D : B \to A$ is called a linear derivation if $D(xy) = D(x)y + xD(y)$ for all $x, y \in B$. In case we have a ring R, an additive mapping $D : R \to R$ is called a derivation if $D(xy) = D(x)y + xD(y)$ for all $x, y \in R$ and is called a Jordan derivation in case $D(x^2) = D(x)x + xD(x)$ is fulfilled for all $x \in R$. A derivation D is inner in case there exists $a \in R$ that $D(x) = [x, a]$ for all $x \in R$. Every derivation is a Jordan derivation, but the converse is in general not true. A classical result of Herstein [7] asserts that any Jordan derivation on a 2-torsion free prime ring is a derivation. A brief proof of Herstein's result can be found in [2]. Cusack [6] generalized Herstein's result to 2-torsion free semiprime rings (see also [3] for an alternative proof). An additive mapping

$D : R \to R$, where R is an arbitrary ring, is called a Jordan triple derivation in case $D(xyx) = D(x)yx + xD(y)x + xyD(x)$ for all $x, y \in R$.

Let X be a real or complex Banach space and let $\mathcal{L}(X)$ and $\mathcal{F}(X)$ denote the algebra of all bounded linear operators on X and the ideal of all finite rank operators in $\mathcal{L}(X)$, respectively. An algebra $\mathcal{A}(X) \subset \mathcal{L}(X)$ is said to be a standard if $\mathcal{F}(X) \subset \mathcal{A}(X)$. Let us point out that any standard operator algebra is prime, which is a consequence of a Hahn–Banach theorem.

Brešar [4] has proved the following result.

Theorem A. *Let R be a 2-torsion free semiprime ring and let $D : R \to R$ be an additive mapping satisfying the relation*

$$D(xyx) = D(x)yx + xD(y)x + xyD(x) \tag{1}$$

for all $x, y \in R$. Then D is a derivation.

One can easily prove that any Jordan derivation D on an arbitrary 2-torsion free ring R satisfies the relation (1), which means that Theorem A generalizes Cusack's generalization of Herstein's theorem we have mentioned above.

Motivated by Theorem A, Vukman [15] recently proved the following result.

Theorem B. *Let R be a 2-torsion free semiprime ring and let $D : R \to R$ be an additive mapping. Suppose that either of the relations*

$$\begin{aligned} D(xyx) &= D(xy)x + xyD(x), \\ D(xyx) &= D(x)yx + xD(yx) \end{aligned} \tag{2}$$

for all $x, y \in R$. Then D is a derivation.

The substitution $y = x^{2n-1}$ in relations (2) gives

$$\begin{aligned} D(x^{2n+1}) &= D(x^{2n})x + x^{2n}D(x), \\ D(x^{2n+1}) &= D(x)x^{2n} + xD(x^{2n}). \end{aligned} \tag{3}$$

Recently, Širovnik [9] obtained the following result, which is related to functional Eq. (3) in case $n = 1$.

Theorem C. *Let X be a real or complex Banach space and let $\mathcal{A}(X)$ be a standard operator algebra on X. Suppose that there exist linear mappings $D, G : \mathcal{A}(X) \to \mathcal{L}(X)$ satisfying either the relations*

$$\begin{aligned} D(A^3) &= D(A^2)A + A^2G(A), \\ G(A^3) &= G(A^2)A + A^2D(A) \end{aligned}$$

or the relations

$$\begin{aligned} D(A^3) &= D(A)A^2 + AG(A^2), \\ D(A^3) &= D(A)A^2 + AG(A^2) \end{aligned}$$

for all $A \in \mathcal{A}(X)$. In both cases there exists $B \in \mathcal{L}(X)$, such that $D(A) = G(A) = [A, B]$, which means that D and G are linear derivations.

It is our aim in this paper to prove the following result related to functional Eq. (3). This result, which generalizes Theorem C, is motivated by Theorem B and Theorem C.

Theorem 1. *Let X be a real or complex Banach space and let $\mathcal{A}(X)$ be a standard operator algebra on X. Suppose that there exist linear mappings $D, G : \mathcal{A}(X) \to \mathcal{L}(X)$ satisfying either the relations*

$$D(A^{2n+1}) = D(A^{2n})A + A^{2n}G(A),$$
$$G(A^{2n+1}) = G(A^{2n})A + A^{2n}D(A)$$

or the relations

$$D(A^{2n+1}) = D(A)A^{2n} + AG(A^{2n}),$$
$$D(A^{2n+1}) = D(A)A^{2n} + AG(A^{2n})$$

for all $A \in \mathcal{A}(X)$ and some integer $n \geq 1$. In both cases there exists $B \in \mathcal{L}(X)$, such that $D(A) = G(A) = [A, B]$ for all $A \in \mathcal{A}(X)$, which means that D and G are linear derivations.

The main result of the paper is related to the result below first proved by Chernoff [5] (see also [8, 10, 12–14]).

Theorem D. *Let X be a real or complex Banach space, let $\mathcal{A}(X)$ be a standard operator algebra on X and let $D : \mathcal{A}(X) \to \mathcal{L}(X)$ be a linear derivation. Then D is of the form $D(A) = AB - BA$ for all $A \in \mathcal{A}(X)$ and some $B \in \mathcal{L}(X)$.*

To develop the proof of Theorem 1 we use Herstein's theorem, Theorem D, Lemma 2 and methods which are similar to those used in [12–14].

Lemma 2 [11, Lemma 3]. *Let R be a semiprime ring and let $f : R \to R$ be an additive mapping. If either $f(x)x = 0$ or $xf(x) = 0$ holds for all $x \in R$, then $f = 0$.*

2. Main Result

We begin our discussion with the first result.

Theorem 3. *Let X be a real or complex Banach space, let $\mathcal{A}(X)$ be a standard operator algebra on X. Suppose that there exists a linear mapping $D : \mathcal{A}(X) \to \mathcal{L}(X)$ satisfying either the relation*

$$D(A^{2n+1}) = D(A^{2n})A + A^{2n}D(A)$$

or the relation

$$D(A^{2n+1}) = D(A)A^{2n} + AD(A^{2n})$$

for all $A \in \mathcal{A}(X)$ and some fixed integer $n \geq 1$. Then there exists an operator $B \in \mathcal{L}(X)$, such that $D(A) = [A, B]$ for all $A \in \mathcal{A}(X)$, which means that D is linear derivation.

Proof. In the case when the second relation holds true, then the proof runs similarly, and therefore, it will be omitted. We have

$$D(A^{2n+1}) = D(A^{2n})A + A^{2n}D(A) \tag{4}$$

for all $A \in \mathcal{A}(X)$. First we shall restrict D on $\mathcal{F}(X)$. Let $A \in \mathcal{F}(X)$ and let $P \in \mathcal{F}(X)$ be a projection with $AP = PA = A$. Replacing A with P in the relation (4) we obtain

$$D(P) = D(P)P + PD(P). \tag{5}$$

A right multiplication of (5) by P gives

$$PD(P)P = 0. \tag{6}$$

Again replacing A with $(A + P)$ in the relation (4), we get

$$\begin{aligned}\sum_{i=0}^{2n+1} \binom{2n+1}{i} D(A^{2n+1-i}P^i) = {}& \left(\sum_{i=0}^{2n} \binom{2n}{i} D(A^{2n-i}P^i)\right)(A+P) \\ &+ \left(\sum_{i=0}^{2n} \binom{2n}{i} A^{2n-i}P^i\right) D(A+P).\end{aligned}$$

Rearranging the above relation and using (4) we get

$$\sum_{i=1}^{2n} f_i(A,P) = 0,$$

where $f_i(A,P)$ stands for the expression of terms involving i factors of P, that is

$$\begin{aligned}f_i(A,P) = {}& \binom{2n+1}{i} D(A^{2n+1-i}P^i) \\ &- \binom{2n}{i}\Big(D(A^{2n-i}P^i)A + (A^{2n-i}P^i)D(A)\Big) \\ &- \binom{2n}{i-1}\Big(D(A^{2n+1-i}P^i)P + (A^{2n+1-i}P^i)D(P)\Big).\end{aligned}$$

Replacing A by $A+2P, A+3P, \ldots, A+2nP$ $2n$ times in the relation (4) and expressing the resulting system of $2n$ homogeneous equations of the variables $f_i(A,P)$, $i = 1, 2, \ldots, 2n$, we see that the coefficient matrix of the system is a Vandermonde matrix

$$\begin{bmatrix} 1 & 1 & \ldots & 1 \\ 2 & 2^2 & \ldots & 2^{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 2n & (2n)^2 & \ldots & (2n)^{2n} \end{bmatrix}.$$

Since the determinant of this matrix is different from zero, it follows immediately that the system has only a trivial solution. In particular

$$\begin{aligned}f_{2n-1}(A,P) = {}& \binom{2n+1}{2n-1} D(A^2) - \binom{2n}{2n-1}\Big(D(A)A + AD(A)\Big) \\ &- \binom{2n}{2n-2}\Big(D(A)P + AD(P)\Big) = 0\end{aligned}$$

and

$$f_{2n}(A,P) = \binom{2n+1}{2n} D(A) - \binom{2n}{2n}\Big(D(P)A + PD(A)\Big) - \binom{2n}{2n-1}\Big(D(A)P + AD(P)\Big) = 0.$$

The above relations reduce to

$$n(2n+1)D(A^2) = 2nD(A)A + 2nAD(A) + n(2n-1)D(A^2)P + n(2n-1)A^2D(P), \tag{7}$$

$$(2n+1)D(A) = D(P)A + PD(A) + 2nD(A)P + 2nAD(P). \tag{8}$$

Thanks to (6), a right multiplication by P in (7) gives

$$D(A^2)P = D(A)A + AD(A)P. \tag{9}$$

Applying (9) in the relation (7), we obtain

$$n(2n+1)D(A^2) = n(2n+1)D(A)A + 2nAD(A) + n(2n-1)\Big(AD(A)P + A^2D(P)\Big). \tag{10}$$

A left multiplication by A in (8) gives

$$AD(A) = AD(A)P + A^2D(P).$$

By applying the above relation in (10), we get

$$D(A^2) = D(A)A + AD(A). \tag{11}$$

From relation (8) one can conclude that D is a linear mapping, which maps $\mathcal{F}(X)$ into itself. By relation (11) D is a Jordan derivation on $\mathcal{F}(X)$. Since $\mathcal{F}(X)$ is prime, it follows that D is a derivation by Herstein's theorem. In view of Theorem D one can conclude that

$$D(A) = [A, B] \tag{12}$$

for all $A \in \mathcal{F}(X)$ and some $B \in \mathcal{L}(X)$. It remains to prove that (12) holds for all $A \in \mathcal{A}(X)$. For this purpose we introduce the mapping $D_1 : \mathcal{A}(X) \to \mathcal{L}(X)$ by $D_1(A) = [A, B]$, where B is from (12) and consider $D_0 = D - D_1$. The mapping D_0 is linear, satisfies the relation (4) and it vanishes on $\mathcal{F}(X)$. Our aim is to prove that D_0 vanishes on $\mathcal{A}(X)$ as well. Let $A \in \mathcal{A}(X)$, let P be a one-dimensional projection and let us introduce $S \in \mathcal{A}(X)$ by $S = A + PAP - (AP + PA)$. We have $SP = PS = 0$. It is easy to see that $D_0(S) = D_0(A)$ and $D_0(S^{2n}) = D_0(A^{2n})$. The relation (4) now leads to

$$\begin{aligned} &D_0(S^{2n})S + S^{2n}D_0(S) \\ &\quad = D_0(S^{2n+1}) = D_0(S^{2n+1} + P) = D_0((S+P)^{2n+1}) \\ &\quad = D_0((S+P)^{2n})(S+P) + (S+P)^{2n}D_0(S+P) \\ &\quad = D_0(S^{2n})S + D_0(S^{2n})P + (S^{2n} + P)D_0(S) \\ &\quad = D_0(S^{2n})S + D_0(S^{2n})P + S^{2n}D_0(S) + PD_0(S). \end{aligned}$$

Therefore,

$$D_0(S^{2n})P + PD_0(S) = 0,$$

which can be written as

$$D_0(A^{2n})P + PD_0(A) = 0.$$

Replacing A with $-A$ in the above relation and comparing the relation so obtained with the above relation, we obtain

$$PD_0(A) = 0$$

for all $A \in \mathcal{A}(X)$. Since P is an arbitrary one-dimensional projection, it now follows that $D_0(A) = 0$ for all $A \in \mathcal{A}(X)$, which completes the proof of the theorem. □

Theorem 4. *Let X be a real or complex Banach space, let $\mathcal{A}(X)$ be a standard operator algebra on X. Suppose that there exists a linear mapping $D : \mathcal{A}(X) \to \mathcal{L}(X)$ satisfying either the relation*

$$D(A^{2n+1}) = D(A^{2n})A - A^{2n}D(A)$$

or the relation

$$D(A^{2n+1}) = D(A)A^{2n} - AD(A^{2n})$$

for all $A \in \mathcal{A}(X)$ and some integer $n \geq 1$. Then $D(A) = 0$ for all $A \in \mathcal{A}(X)$.

Proof. In case when the second relation holds true, then the proof runs similarly, and therefore, it will be omitted. We have

$$D(A^{2n+1}) = D(A^{2n})A - A^{2n}D(A) \tag{13}$$

for all $A \in \mathcal{A}(X)$. First we shall restrict D on $\mathcal{F}(X)$. Let $A \in \mathcal{F}(X)$ and $P \in \mathcal{F}(X)$ be a projection with $AP = PA = A$. From the relation (4) we obtain

$$D(P) = D(P)P - PD(P). \tag{14}$$

A right multiplication by P in the above relation gives

$$PD(P)P = 0$$

and a left multiplication by P in (14) gives, considering the above relation,

$$PD(P) = 0. \tag{15}$$

Putting $A + P$ for A in (13), we get

$$\sum_{i=0}^{2n+1} \binom{2n+1}{i} D(A^{2n+1-i}P^i) = \left(\sum_{i=0}^{2n} \binom{2n}{i} D(A^{2n-i}P^i)\right)(A+P) - \left(\sum_{i=0}^{2n} \binom{2n}{i} A^{2n-i}P^i\right)D(A+P).$$

Rearranging the above relation and considering (13), we obtain

$$\sum_{i=1}^{2n} f_i(A, P) = 0,$$

where $f_i(A, P)$ stands for the expression of terms involving i factors of P, that is

$$\begin{aligned} f_i(A,P) = \tbinom{2n+1}{i} D(A^{2n+1-i}P^i) \\ - \tbinom{2n}{i}\Big(D(A^{2n-i}P^i)A - (A^{2n-i}P^i)D(A)\Big) \\ - \tbinom{2n}{i-1}\Big(D(A^{2n+1-i}P^i)P - (A^{2n+1-i}P^i)D(P)\Big). \end{aligned}$$

Replacing A by $A + 2P$, $A + 3P, \dots, A + 2nP$ $2n$ times in the relation (13) and expressing the resulting system of $2n$ homogeneous equations of variables $f_i(A, P)$, $i = 1, 2, \dots, 2n$, we see that the coefficient matrix of the system is a Vandermonde matrix

$$\begin{bmatrix} 1 & 1 & \dots & 1 \\ 2 & 2^2 & \dots & 2^{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 2n & (2n)^2 & \dots & (2n)^{2n} \end{bmatrix}.$$

Since the determinant of the above matrix is different from zero, it follows immediately that the system has only a trivial solution. In particular

$$\begin{aligned} f_{2n}(A,P) = \tbinom{2n+1}{2n} D(A) - \tbinom{2n}{2n}\Big(D(P)A - PD(A)\Big) \\ - \tbinom{2n}{2n-1}\Big(D(A)P - AD(P)\Big) = 0. \end{aligned}$$

The above relation reduces to

$$(2n+1)D(A) = D(P)A - PD(A) + 2nD(A)P - 2nAD(P)$$

and using (15) in the above relation gives

$$(2n+1)D(A) = D(P)A - PD(A) + 2nD(A)P. \tag{16}$$

Two-sided multiplication by P in the above relation leads to

$$PD(A)P = 0.$$

By left multiplying by P in the relation (16) and considering (15) together with the above relation, we obtain

$$PD(A) = 0. \tag{17}$$

A left multiplication by A in the above relation gives

$$AD(A) = 0. \tag{18}$$

An application (17) in (16) leads to

$$(2n+1)D(A) = D(P)A + 2nD(A)P. \tag{19}$$

A right multiplication by P in the above relation gives

$$D(A)P = D(P)A.$$

Considering the above relation in (19), we obtain

$$D(A) = D(A)P. \tag{20}$$

From the relation (20) one can conclude that D maps $\mathcal{F}(X)$ into itself. We have, therefore, a linear mapping D, which maps $\mathcal{F}(X)$ into itself satisfying the relation (18). Applying Lemma 2, we can conclude that $D(A) = 0$ for all $A \in \mathcal{F}(X)$.

It remains to prove that $D(A) = 0$ holds for all $A \in \mathcal{A}(X)$ as well. The mapping D on $\mathcal{A}(X)$ is linear, satisfies the relation (13) and it vanishes on $\mathcal{F}(X)$. Our aim is to prove that D vanishes on $\mathcal{A}(X)$ as well. Let $A \in \mathcal{A}(X)$, let P be a one-dimensional projection and let us introduce $S \in \mathcal{A}(X)$ by $S = A + PAP - (AP + PA)$. We have $SP = PS = 0$. It is easy to see that $D(S) = D(A)$ and $D(S^{2n}) = D(A^{2n})$. The relation (13) now leads to

$$\begin{aligned} &D(S^{2n})S - S^{2n}D(S) \\ &\quad = D(S^{2n+1}) = D(S^{2n+1} + P) = D((S + P)^{2n+1}) \\ &\quad = D((S + P)^{2n})(S + P) - (S + P)^{2n}D(S + P) \\ &\quad = D(S^{2n})S + D(S^{2n})P - (S^{2n} + P)D(S) \\ &\quad = D(S^{2n})S + D(S^{2n})P - S^{2n}D(S) - PD(S). \end{aligned}$$

Therefore,

$$D(S^{2n})P - PD(S) = 0$$

which can be written as

$$D(A^{2n})P - PD(A) = 0.$$

Replacing A with $-A$ in the above relation and comparing the relation so obtained with the above relation gives

$$PD(A) = 0.$$

Since P is arbitrary one-dimensional projection, it follows from the above relation that $D(A) = 0$ for all $A \in \mathcal{A}(X)$. The proof of the theorem is now complete. □

We are now in the position to prove Theorem 1.

Proof of Theorem 1. In the case when the second system of the relation holds true, then the proof runs similarly, and therefore, it will be omitted. We have

$$\begin{aligned} D(A^{2n+1}) &= D(A^{2n})A + A^{2n}G(A), \\ G(A^{2n+1}) &= G(A^{2n})A + A^{2n}D(A) \end{aligned}$$

for all $A \in \mathcal{A}(X)$. Subtracting the above relations gives

$$T(A^{2n+1}) = T(A^{2n})A - A^{2n}T(A), \tag{21}$$

where $T = D - G$. Using Theorem 4, we can conclude that $T(A) = 0$ for all $A \in \mathcal{A}(X)$, which implies $D = G$. This ascertainment enables us to combine the given two relations into only one relation

$$D(A^{2n+1}) = D(A^{2n})A + A^{2n}D(A)$$

for all $A \in \mathcal{A}(X)$. From Theorem 3 it follows that $D(A) = G(A) = [A, B]$ for all $A \in \mathcal{A}(X)$, and so the proof is complete. □

We conclude the paper with the following purely algebraic conjecture.

Conjecture 5. *Let R be semiprime ring with suitable torsion restrictions and let $D, G : R \to R$ be additive mappings satisfying either the relations*

$$D(x^{2n+1}) = D(x^{2n})x + x^{2n}G(x),$$
$$G(x^{2n+1}) = G(x^{2n})x + x^{2n}D(x)$$

or the relations

$$D(x^{2n+1}) = D(x)x^{2n} + xG(x^{2n}),$$
$$D(x^{2n+1}) = D(x)x^{2n} + xG(x^{2n})$$

for all $x \in R$ and some integer $n \geq 1$. Then D and G are derivations and $D = G$.

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