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Numerical Solution Based on Hat Functions for Solving Nonlinear Stochastic Itô Volterra Integral Equations Driven by Fractional Brownian Motion

B. Hashemi, M. Khodabin and K. Maleknejad

Abstract. This paper presents a numerical method for solving nonlinear stochastic Itô Volterra integral equations driven by fractional Brownian motion with Hurst parameter $H \in (0, 1)$ via of hat functions. Using properties of the generalized hat basis functions and fractional Brownian motion, new stochastic operational matrix of integration is achieved and the nonlinear stochastic equation is transformed into nonlinear system of algebraic equations which by solving it, an approximation solution with high accuracy is obtained. In addition, error analysis of the method is investigated, and by some examples, efficiency and accuracy of the suggested method are shown.

Mathematics Subject Classification. Primary 65C30, 60H35, 65C20; Secondary 60H20, 68U20.

Keywords. Brownian and fractional Brownian motion process, Stochastic integral equation, Hat functions.

1. Introduction

Recently, there is an increasing demand for solving stochastic differential equations and stochastic integral equations numerically. These equations appear in models of various problems in science and engineering events and so on. They are often dependent on a Gaussian white noise which governed by some probability rules and mathematically described as a formal derivative of a Brownian motion process. Such phenomena needs to model using stochastic differential equations or, stochastic Volterra integral equations and stochastic integro-differential equations. Most of them cannot be solved analytically; therefore, numerical computation and analysis will become important [1–8].

Some stochastic differential and integral equations have been caused by fractional Brownian motion and have many applications in models arising in physics, telecommunication networks, and finance [9]. There exist several ways to solve them, pathwise and related techniques, Dirichlet forms, Euler approximations, Malliavin calculus, and Skorohod integral [10–14]; almost all methods have very poor numerical convergence, so it is important to find approximation solutions with reasonable accuracy for them. For example, Ezzati et al. in "Numerical Implementation of Stochastic Operational Matrix Driven by a Fractional Brownian Motion for Solving a Stochastic Differential Equation" have used block pulse functions for solving stochastic differential equations driven by fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$ [15].

In this paper, we consider the following nonlinear stochastic Itô Volterra integral equation which has been caused by a fractional Brownian motion:

$$X(t) = h(t) + \int_0^t f(s)\mu(X(s))ds + \int_0^t g(s)\sigma(X(s))dB^{(H)}(s), \ t \in [0, T], \quad (1)$$

where $B^{(H)}(t)$ is a fractional Brownian motion with Hurst parameter $H \in (0, 1)$, $X(t)$, $h(t)$, $f(t)$, and $g(t)$, for $t \in [0, T]$, are stochastic processes defined on the same probability space (Ω, F, P), $X(t)$ is unknown function, and $\mu(s)$ and $\sigma(s)$ are analytic functions.

We try to solve Eq. (1), using hat functions. Previously in [16], Heydari et al. solved the case which has been caused by simple Brownian motion.

For computing the approximation solution of above equation, we first bring some properties of the generalized hat basis functions, then we get the new operational matrix of stochastic integration driven by fractional Brownian motion and obtain a system of nonlinear algebraic equations. Finally, we look into error analysis of this method and illustrate some examples to show accuracy of the suggested method.

2. Fractional Brownian Motion and its Properties

2.1. Fractional Brownian Motion

A standard fractional Brownian motion $(B^{(H)}(t))_{t\geq 0}$ with Hurst parameter $H \in (0, 1)$ is a continuous Gaussian process with zero mean and a covariance function:

$$Cov(B^{(H)}(s), B^{(H)}(t)) = \frac{1}{2}(s^{2H} + t^{2H} - \mid t - s \mid^{2H}).$$

Fractional Brownian motion has the following properties:

(a) $B^{(H)}(0) = 0$ and $E(B^{(H)}(t)) = 0$ for all $t \geq 0$.
(b) $B^{(H)}$ has homogeneous increments.
(c) $E(B^{(H)}(t)^2) = t^{2H}, t \geq 0$.
(d) $B^{(H)}$ has continuous trajectories.
If $H = 1/2$, we get to a standard Brownian motion [9].

2.2. Fractional Itô Formula

Let $H \in (0,1)$. Assume that $f(s,x) : R \times R \rightarrow R$ belongs to $C^{1,2}(R \times R)$, and assume that the random variables

$$f(t, B^{(H)}(t)), \int_0^t \frac{\partial f}{\partial s}(s, B^{(H)}(s))ds, \int_0^t \frac{\partial^2 f}{\partial x^2}(s, B^{(H)}(s))s^{2H-1}ds,$$

all belong to $L^2(\Omega)$. Then:

$$f(t, B^{(H)}(t)) = f(0,0) + \int_0^t \frac{\partial f}{\partial s}(s, B^{(H)}(s))ds + \int_0^t \frac{\partial f}{\partial x}(s, B^{(H)}(s))dB^H(s)$$
$$+ H \int_0^t \frac{\partial^2 f}{\partial x^2}(s, B^{(H)}(s))s^{2H-1}ds. \tag{2}$$

For more details, see [9].

3. Hat Functions and Their Properties

The family of first $(n+1)$ hat functions on [0, T] are defined as follows [17–20]:

$$\phi_0(t) = \begin{cases} \frac{h-t}{h} & 0 \leq t \leq h, \\ 0 & otherwise, \end{cases}$$

$$\phi_i(t) = \begin{cases} \frac{t-(i-1)h}{h} & (i-1)h \leq t \leq ih, \\ \frac{(i+1)h-t}{h} & ih \leq t \leq (i+1)h, \\ 0 & otherwise, \end{cases}$$

which i=1,2,...,n-1 and $h = \frac{T}{n}$. In addition, we have:

$$\phi_n(t) = \begin{cases} \frac{t-(T-h)}{h} & T-h \leq t \leq T, \\ 0 & otherwise. \end{cases}$$

From the above definitions, we have:

$$\phi_i(kh) = \begin{cases} 1 & i = k, \\ 0 & i \neq k, \end{cases} \tag{3}$$

and

$$\phi_i(t)\phi_k(t) = 0, \mid i-k \mid \geq 2. \tag{4}$$

An arbitrary function $f(t) \in L^2[0,T]$ can be expanded by the generalized hat basis functions as:

$$f(t) \simeq \sum_{i=0}^{n} f_i \phi_i(t) = F^T \Phi(t) = \Phi(t)^T F, \tag{5}$$

where

$$F = [f_0, f_1, ..., f_n]^T, \tag{6}$$

and

$$\Phi(t)=[\phi_0(t),\phi_1(t),...,\phi_n(t)]^T. \tag{7}$$

The coefficients f_i in (5) are given by:

$$f_i=f(ih), i=0,1,...,n. \tag{8}$$

From relation (4), we have:

$$\Phi(t)\Phi(t)^T=\begin{pmatrix} \phi_0^2(t) & \phi_0(t)\phi_1(t) & & & & \\ \phi_0(t)\phi_1(t) & \phi_1^2(t) & \phi_1(t)\phi_2(t) & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & \phi_{n-1}(t)\phi_n(t) \\ & & & & \phi_{n-1}(t)\phi_n(t) & \phi_n^2(t) \end{pmatrix}.$$

According to (3) and expanding elements of $\Phi(t)\Phi(t)^T$ by generalized hat functions, we have:

$$\Phi(t)\Phi(t)^T\simeq\begin{pmatrix} \phi_0(t) & 0 & \cdots & 0 \\ 0 & \phi_1(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \phi_n(t) \end{pmatrix}.$$

Integrating of vector $\Phi(t)$ which is given by (7) yields [21]:

$$\int_0^t \Phi(s)ds\simeq\ P\Phi(t), t\in[0,T], \tag{9}$$

where P is $(n+1)\times(n+1)$, and called operational matrix of integration for the generalized hat basis functions, and is given by the following:

$$P=\frac{h}{2}\begin{pmatrix} 0 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 1 & 2 & \cdots & 2 & 2 \\ 0 & 0 & 1 & \cdots & 2 & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 2 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

4. Stochastic Operational Matrix

Theorem 4.1. *The Itô integral of $\Phi(t)$ which is given by (7) yields:*

$$\int_0^t \Phi(s)dB(s)\simeq\ P_s\Phi(t), \tag{10}$$

where the matrix P_s is $(n+1)\times(n+1)$, and called operational matrix of stochastic integration for the generalized hat functions, and is given by the following:

$$P_s = \begin{pmatrix} 0 & \alpha_0 & \alpha_0 & & \cdots & \alpha_0 & \alpha_0 \\ 0 & B(h)+\beta_1 & \beta_1+\alpha_1 & & \cdots & \beta_1+\alpha_1 & \beta_1+\alpha_1 \\ 0 & 0 & B(2h)+\beta_2 & \beta_2+\alpha_2 & \cdots & \beta_2+\alpha_2 & \beta_2+\alpha_2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & B((n-1)h)+\beta_{n-1} & \beta_{n-1}+\alpha_{n-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & B(T)+\beta_n \end{pmatrix},$$

and

$$\begin{cases} \alpha_i = \frac{1}{h}\int_{ih}^{(i+1)h} B(s)ds, & \mathrm{i} = 0,1,2,...,\mathrm{n}-1, \\ \beta_i = -\frac{1}{h}\int_{(i-1)h}^{ih} B(s)ds, & \mathrm{i} = 1,2,...,\mathrm{n}. \end{cases}$$

Proof. See [16]. □

Theorem 4.2. *Integrating of $\Phi(t)$ which is given by (7), according to fractional Brownian motion, yields:*

$$\int_0^t \Phi(s)dB^{(H)}(s) \simeq P_{sH}\Phi(t), \tag{11}$$

where the matrix P_{sH} is $(n+1)\times(n+1)$, and called operational matrix of stochastic integration driven by fractional Brownian motion for the generalized hat functions, and is given by the following:

$$P_{sH} = \begin{pmatrix} 0 & \alpha_0 & \alpha_0 & \alpha_0 & \cdots & \alpha_0 & \alpha_0 \\ 0 & B^{(H)}(h)+\beta_1 & \beta_1+\alpha_1 & \beta_1+\alpha_1 & \cdots & \beta_1+\alpha_1 & \beta_1+\alpha_1 \\ 0 & 0 & B^{(H)}(2h)+\beta_2 & \beta_2+\alpha_2 & \cdots & \beta_2+\alpha_2 & \beta_2+\alpha_2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & B^{(H)}((n-1)h)+\beta_{n-1} & \beta_{n-1}+\alpha_{n-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & B^{(H)}(T)+\beta_n \end{pmatrix},$$

and

$$\begin{cases} \alpha_i = \frac{1}{h}\int_{ih}^{(i+1)h} B^{(H)}(s)ds, & \mathrm{i} = 0,1,2,...,\mathrm{n}-1, \\ \beta_i = -\frac{1}{h}\int_{(i-1)h}^{ih} B^{(H)}(s)ds, & \mathrm{i} = 1,2,...,\mathrm{n}. \end{cases}$$

Proof. To compute $\int_0^t \phi(s)dB^{(H)}(s)$, choose $X_t = B^{(H)}(t)$ and $f(t,x) = \phi_i(t)\times x$. Then, according to relation (2), we have:

$$Y_t = f(t, B^{(H)}(t)) = \phi_i(t)\times B^{(H)}(t).$$

Therefore

$$d(\phi_i(t)\times B^{(H)}(t)) = B^{(H)}(t)\times \phi_i'(t)dt + \phi_i(t)dB^{(H)}(t).$$

By integrating from 0 to t, we have:

$$\phi_i(t)B^{(H)}(t) - \phi_i(0)B^{(H)}(0) = \int_0^t B^{(H)}(y)\phi_i'(y)dy + \int_0^t \phi_i(y)dB^{(H)}(y).$$

Therefore:

$$\int_0^t \phi_i(y)dB^{(H)}(y) = \phi_i(t)B^{(H)}(t) - \int_0^t B^{(H)}(y)\phi_i'(y)dy. \tag{12}$$

By expanding $\int_0^t \phi_i(y)dB^{(H)}(y)$ in terms of hat functions, we will have:

$$\int_0^t \phi_i(y)dB^{(H)}(y) \simeq \sum_{j=0}^{n} a_{ij}\phi_j(t) = \sum_{j=0}^{n} \left(\int_0^{jh} \phi_i(y)dB^{(H)}(y) \right) \phi_j(t).$$

Using (12), we have:

$$a_{ij} = \int_0^{jh} \phi_i(y)dB^{(H)}(y) = \phi_i(jh)B^{(H)}(jh) - \int_0^{jh} B^{(H)}(y)\phi_i'(y)dy.$$

Using definition and properties of hat functions which has been mentioned in Sect. 3, a_{ij} have the following form:

$$a_{0j} = \begin{cases} 0 & j = 0, \\ \frac{1}{h}\int_0^h B^{(H)}(y)dy & j \geq 1, \end{cases}$$

$$a_{ij} = \begin{cases} 0 & j \leq i-1, \\ B^{(H)}(ih) - \frac{1}{h}\int_{(i-1)h}^{ih} B^{(H)}(y)dy & j = i, \\ -\frac{1}{h}\left(\int_{(i-1)h}^{ih} B^{(H)}(y)dy - \int_{ih}^{(i+1)h} B^{(H)}(y)dy\right) & j \geq i+1 \text{ and } i \neq n, \end{cases}$$

where $i = 1, ..., n$ and $j = 0, 1, ..., n$.

Therefore, by substituting, $\alpha_i = \frac{1}{h}\int_{ih}^{(i+1)h} B^{(H)}(s)ds$ and $\beta_i = -\frac{1}{h}\int_{(i-1)h}^{ih} B^{(H)}(s)ds$, the matrix P_{sH}, and so the Itô integral driven by fractional Brownian motion of $\Phi(x)$ will be obtained. □

In this paper, we will work with matrix P_{sH} and its entries.

5. Some Effective Properties of the Generalized Hat Basis Functions

In this part, we will introduce some effective properties of the generalized hat basis functions, which Heydari et al. have used them to solve nonlinear Itô Volterra integral equations for simple Brownian motion [16].

For any two constant vectors $X^T = [x_0, x_1, ..., x_n]$ and $Y^T = [y_0, y_1, ..., y_n]$, we define $X^T \odot Y^T = [x_0y_0, x_1y_1, ..., x_ny_n]$ and $G(X^T) = [G(x_0), G(x_1),$
$..., G(x_n)]$, for any analytic function G.

Lemma 5.1. *Suppose $X^T\Phi(t)$ and $Y^T\Phi(t)$ be expansions of $X(t)$ and $Y(t)$ by the generalized hat basis functions. Then, we have:*

$$X(t)Y(t) \simeq (X^T \odot Y^T)\Phi(t). \tag{13}$$

Proof. See [16]. □

Corollary 5.2. *Suppose $X^T\Phi(t)$ be the expansion of $X(t)$ by the generalized hat basis functions. Then, for any integer $m \geq 2$, we have:*

$$[X(t)]^m \simeq [x_0^m, x_1^m, ..., x_n^m]\Phi(t). \tag{14}$$

Proof. See [16]. □

Theorem 5.3. *If $X^T\Phi(t)$ be the expansion of $X(t)$ by the generalized hat basis functions and G be an analytic function, and then, we have:*

$$G[X(t)] \simeq G(X^T)\Phi(t). \tag{15}$$

Proof. See [16]. □

Corollary 5.4. *Suppose F and G be two analytic functions, and also $X^T\Phi(t)$ and $Y^T\Phi(t)$ be the expansions of $X(t)$ and $Y(t)$ by the generalized hat basis functions, and then, we have:*

$$F(X(t))G(Y(t)) \simeq (F(X^T) \odot G(Y^T))\Phi(t). \tag{16}$$

Proof. See [16]. □

6. Numerical Method

In this section, we employ the operational matrices of integration and stochastic integration which has been caused by fractional Brownian motion with Hurst parameter $H \in (0, 1)$. Therefore, using hat basis functions and their effective properties, we try to solve the following equation:

$$X(t) = h(t) + \int_0^t f(s)\mu(X(s))ds + \int_0^t g(s)\sigma(X(s))dB^{(H)}(s), \ t \in [0, T]. \tag{17}$$

We approximate $X(t)$, $h(t)$, $f(t)$, and $g(t)$ as follows:

$$X(t) \simeq X^T\Phi(t) = \Phi(t)^T X, \tag{18}$$

$$h(t) \simeq H^T\Phi(t) = \Phi(t)^T H, \tag{19}$$

and

$$f(t) \simeq F^T\Phi(t) = \Phi(t)^T F, \tag{20}$$

$$g(t) \simeq G^T\Phi(t) = \Phi(t)^T G, \tag{21}$$

where X, H, F, and G are the generalized hat functions coefficients vectors.

By results from the previous section, we have:

$$\mu(X(t)) \simeq \mu(X^T)\Phi(t), \tag{22}$$

$$\sigma(X(t)) \simeq \sigma(X^T)\Phi(t), \tag{23}$$

and

$$f(t)\mu(X(t)) \simeq (F^T \odot \mu(X^T))\Phi(t), \tag{24}$$

$$g(t)\sigma(X(t)) \simeq (G^T \odot \sigma(X^T))\Phi(t). \tag{25}$$

By substituting above relations and operational matrices of integration in Eq. (17), we have:

$$X^T\Phi(t) \simeq H^T\Phi(t) + (F^T \odot \mu(X^T))P\Phi(t) + (G^T \odot \sigma(X^T))(P_{sH})\Phi(t). \tag{26}$$

If we replace $\simeq$ by $=$, we get the following system of nonlinear algebraic equations in which we try to solve it:

$$X^T - (F^T \odot \mu(X^T))P - (G^T \odot \sigma(X^T))(P_{sH}) = H^T. \tag{27}$$

Therefore, the approximation solution of Eq. (17) is $X \simeq X^T\Phi(t)$.

7. Error Analysis

If we approximate the unknown error function $e_n(t) = X(t) - X_n(t)$, where $X_n(t)$ is an approximation solution of Eq. (17), using the proposed method for the following nonlinear stochastic integral equation:

$$\begin{aligned} e_n(t) = -\epsilon_n(t) &+ \int_0^t f(s)(\mu(X(s)) - \mu(X_n(s)))ds \\ &+ \int_0^t g(s)(\sigma(X(s)) - \sigma(X_n(s)))dB^{(H)}(s), \end{aligned} \tag{28}$$

we can estimate $e_n(t)$, where $\epsilon_n(t)$ is the residual function, and can be obtained from the following equation:

$$\epsilon_n(t) = X_n(t) - h(t) - \int_0^t f(s)\mu(X_n(s))ds - \int_0^t g(s)\sigma(X_n(s))dB^{(H)}(s),\ t \in [0,T].$$

To estimate $e_n(t)$ from Eq. (28), we should approximate $\mu(X(t)) - \mu(X_n(t))$ and $\sigma(X(t)) - \sigma(X_n(t))$, because $X(t)$ is unknown.

Since μ and σ are analytic functions, using Taylor's theorem, we have

$$\begin{aligned}\mu(X(t))-\mu(X_n(t)) &= \mu(X_n(t)+e_n(t))-\mu(X_n(t))\\ &= \mu'(X_n(t))\times e_n(t)+\frac{1}{2!}\mu''(X_n(t))e_n(t)^2\\ &\quad+\frac{1}{3!}\mu'''(\theta_n(t))e_n(t)^3,\end{aligned}$$

and

$$\sigma(X(t))-\sigma(X_n(t)) = \sigma'(X_n(t))e_n(t)+\frac{1}{2!}\sigma''(X_n(t))e_n(t)^2+\frac{1}{3!}\sigma'''(\delta_n(t))e_n(t)^3,$$

where $\theta_n(t)=X_n(t)+\theta e_n(t)$, $\delta_n(t)=X_n(t)+\delta e_n(t)$, and $\theta,\delta\in(0,T)$.

By substituting above relations in Eq. (28), we have a nonlinear stochastic integral equation with unknown function $e_n(t)$, which by applying the proposed method, we can approximate it.

8. Numerical Examples

To clarify the method, we illustrate the following examples which their exact solutions are exist. Note that n is the number of basis functions and m is the number of iterations.

Example 8.1. Consider the following nonlinear stochastic Itô Volterra integral equation which has been caused by fractional Brownian motion and has exact solution:

$$\begin{aligned}X(t) &= X_0-2\times H\times a^2\int_0^t s^{2H-1}X(s)(1-X(s)^2)ds\\ &\quad+a\times\int_0^t (1-X^2(s))dB^{(H)}(s),\ t\in(0,1).\end{aligned}$$

The exact solution of above equation is:

$$X(t)=tanh(aB^{(H)}(t)+arctanh(X_0)).$$

For $H=0.5$, this equation has been given in [5]. For different values of H and with 200 iterations, the absolute error of approximation solutions for $n=32$ is given in Tables 1, 2, and 3. This example is solved for $a=\frac{1}{30}$ and $X_0=\frac{1}{10}$.

The exact and approximation solutions of the Example 8.1 for $n=32$, $t=0.05$, and $H=0.5$ with 200 iterations are given in Fig. 1, and for $H=0.8$, $n=64$ with 500 iterations are shown in Fig. 2.

Example 8.2. Consider the following nonlinear stochastic Itô Volterra integral equation which has been caused by fractional Brownian motion. When $H=0.5$, this equation is illustrated in [5]:

Table 1. Error mean, $\bar{X}_E$, error standard deviation, S_E, and confidence interval for error mean of Example 8.1 with Hurst parameter $H = 0.2$

t	$\bar{X}_E$	S_E	%95 confidence interval for error mean	
			Lower	Upper
0.05	2.040843×10^{-3}	6.240495×10^{-4}	4.607550×10^{-4}	9.033127×10^{-4}
0.1	2.623988×10^{-3}	8.251740×10^{-4}	6.370481×10^{-4}	1.222236×10^{-3}
0.15	3.195517×10^{-3}	9.966029×10^{-4}	7.282005×10^{-4}	1.434961×10^{-3}
0.2	3.812141×10^{-3}	1.170226×10^{-3}	8.300452×10^{-4}	1.659933×10^{-3}

Table 2. Error mean, $\bar{X}_E$, error standard deviation, S_E, and confidence interval for error mean of Example 8.1 with Hurst parameter $H = 0.5$

t	$\bar{X}_E$	S_E	%95 confidence interval for error mean	
			Lower	Upper
0.05	3.210683×10^{-5}	1.014270×10^{-5}	6.743902×10^{-6}	1.393679×10^{-5}
0.1	6.638530×10^{-5}	2.054621×10^{-5}	1.386075×10^{-5}	2.843149×10^{-5}
0.15	8.861528×10^{-5}	2.773538×10^{-5}	1.904823×10^{-5}	3.871731×10^{-5}
0.2	1.209729×10^{-4}	3.842619×10^{-5}	2.675356×10^{-5}	5.400423×10^{-5}

Table 3. Error mean, $\bar{X}_E$, error standard deviation, S_E, and confidence interval for error mean of Example 8.1 with Hurst parameter $H = 0.8$

t	$\bar{X}_E$	S_E	%95 confidence interval for error mean	
			Lower	Upper
0.05	1.277309×10^{-7}	3.960305×10^{-8}	5.699309×10^{-8}	8.507835×10^{-8}
0.1	2.842930×10^{-7}	8.753508×10^{-8}	1.291663×10^{-7}	1.912435×10^{-7}
0.15	5.783858×10^{-7}	1.891093×10^{-7}	3.106045×10^{-7}	4.447140×10^{-7}
0.2	8.224517×10^{-7}	2.593554×10^{-7}	4.144255×10^{-7}	5.983523×10^{-7}

$$X(t) = X_0 - H \times a^2 \int_0^t tanh(X(s)) s^{2H-1} sech^2(X(s)) ds$$
$$+ a \int_0^t sech(X(s)) dB^{(H)}(s),\ t \in (0,1).$$

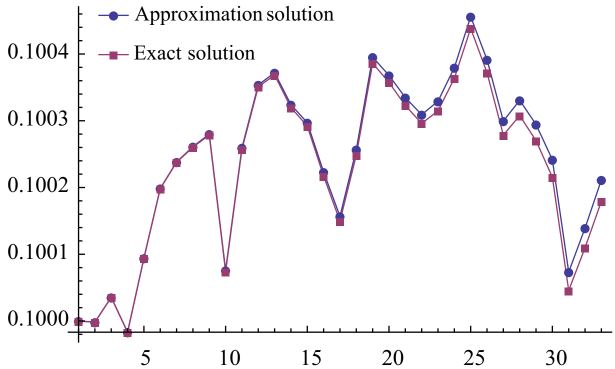

Figure 1. Exact and approximation solutions of Example 8.1 for $n = 32$, $m = 200$, $H = 0.5$, and $t = 0.05$

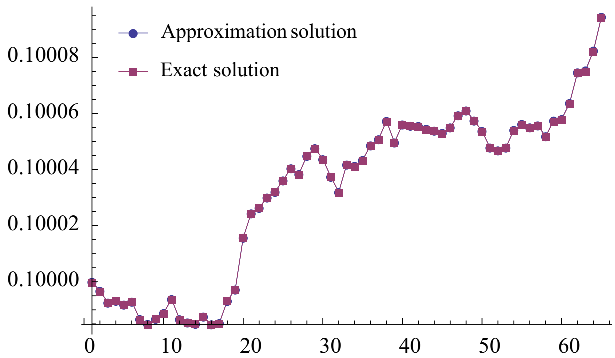

Figure 2. Exact and approximation solutions of Example 8.1 for $n = 64$, $m = 500$, $H = 0.8$, and $t = 0.05$

Table 4. Error mean, $\bar{X}_E$, error standard deviation, S_E, and confidence interval for error mean of Example 8.2 with Hurst parameter $H = 0.2$

t	$\bar{X}_E$	S_E	%95 confidence interval for error mean	
			Lower	Upper
0.05	1.085271×10^{-3}	3.367596×10^{-4}	2.486759×10^{-4}	4.874955×10^{-4}
0.1	1.316461×10^{-3}	4.079752×10^{-4}	3.098707×10^{-4}	5.991941×10^{-4}
0.15	1.603519×10^{-3}	5.030140×10^{-4}	3.653977×10^{-4}	7.221197×10^{-4}
0.2	1.736776×10^{-3}	5.482654×10^{-4}	3.936551×10^{-4}	7.824681×10^{-4}

The exact solution of above equation is:

$$X(t) = arcsinh(aB^H(t) + sinh(X_0)).$$

Table 5. Error mean, $\bar{X}_E$, error standard deviation, S_E, and confidence interval for error mean of Example 8.2 with Hurst parameter $H = 0.5$

t	$\bar{X}_E$	S_E	%95 confidence interval for error mean	
			Lower	Upper
0.05	1.499915×10^{-5}	4.643697×10^{-6}	3.196107×10^{-6}	6.489274×10^{-6}
0.1	3.020395×10^{-5}	9.367891×10^{-6}	6.135273×10^{-6}	1.277869×10^{-5}
0.15	4.306922×10^{-5}	1.386794×10^{-5}	9.253781×10^{-6}	1.908850×10^{-5}
0.2	6.165456×10^{-5}	1.908316×10^{-5}	1.324612×10^{-5}	2.677931×10^{-5}

Table 6. Error mean, $\bar{X}_E$, error standard deviation, S_E, and confidence interval for error mean of Example 8.2 with Hurst parameter $H = 0.8$

t	$\bar{X}_E$	S_E	%95 confidence interval for error mean	
			Lower	Upper
0.05	4.067660×10^{-8}	1.311433×10^{-8}	2.024607×10^{-8}	2.954635×10^{-8}
0.1	1.726966×10^{-7}	5.879228×10^{-8}	9.222035×10^{-8}	1.339140×10^{-7}
0.15	2.418395×10^{-7}	7.841807×10^{-8}	1.108158×10^{-7}	1.664275×10^{-7}
0.2	4.282123×10^{-7}	1.375553×10^{-7}	2.301637×10^{-7}	3.277137×10^{-7}

For different values of H and 200 iterations, the absolute error of approximation solutions for $n = 32$ is given in Tables 4, 5, and 6. In addition, in this equation, $a = \frac{1}{30}$ and $X_0 = \frac{1}{10}$.

The exact and approximation solutions of the Example 8.2 for $n = 32$, $t = 0.05$, and $H = 0.5$ with 200 iterations are given in Fig. 3, and for $H = 0.8$, $n = 64$ with 500 iterations are shown in Fig. 4.

9. Conclusion

Since it may be hard or impossible to find exact solution of nonlinear stochastic Itô Volterra integral equations, specially some type of them which has been caused by fractional Brownian motion, we tried to solve them numerically. Previously, these type of equations for simple Brownian motion has been solved numerically in [16]. Using stochastic operational matrix of integration for the generalized hat basis functions, we transformed our nonlinear stochastic Itô Volterra integral equation into a nonlinear system of algebraic equations. By solving this system, we obtained an approximation solution. Finally, we looked into error analysis of the method and with some examples, and showed accuracy and efficiency of the suggested method.

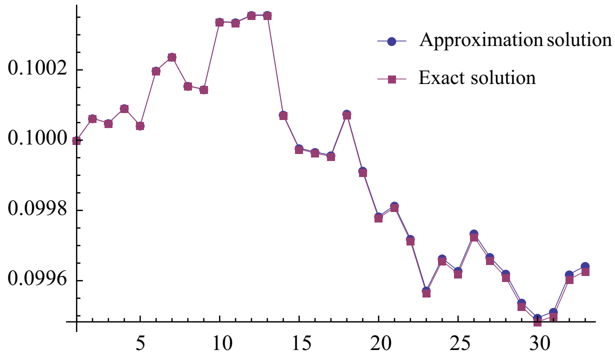

Figure 3. Exact and approximation solutions of Example 8.2 for $n = 32$, $m = 200$, $H = 0.5$, and $t = 0.05$

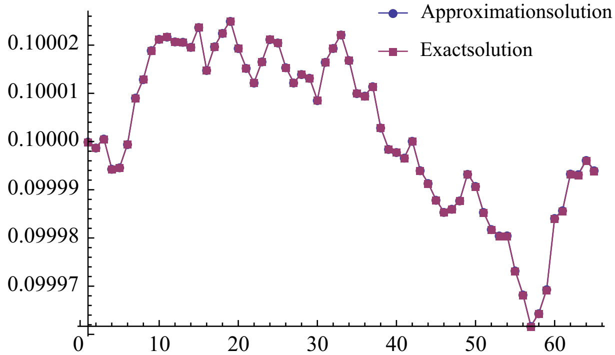

Figure 4. Exact and approximation solutions of Example 8.2 for $n = 64$, $m = 500$, $H = 0.8$, and $t = 0.05$

References

[1] Maleknejad, K., Khodabin, M., Rostami, M.: Numerical solutions of stochastic Volterra integral equations by a stochastic operational matrix based on block puls functions. Math. Comput. Model. **55**, 791–800 (2012)

[2] Khodabin, M., Maleknejad, K., Rostami, M., Nouri, M.: Numerical solution of stochastic differential equations by second order Runge-Kutta methods. Math. Comput. Model. **53**, 1910–1920 (2011)

[3] Jankovic, S., Ilic, D.: One linear analytic approximation for stochastic integro-differential equations. Acta Math. Sci. **30**, 1073–1085 (2010)

[4] Cortes, J., Jodar, L., Villafuerte, L.: Mean square numerical solution of random differential equations: facts and possibilities. Comput. Math. Appl. **53**, 1098–1106 (2007)

[5] Kloeden, P.E., Platen, E.: Numerical solution of stochastic differential equations. In: Applications of Mathematics. Springer, Berlin (1999)

[6] Murge, M., Pachpatte, B.: Successive approximations for solutions of second order stochastic integro differential equations of Ito type. Indian J. Pure. Appl. Math. **21**(3), 260–274 (1990)

[7] Cortes, J.C., Jodar, L., Villafuerte, L.: Numerical solution of random differential equations: a mean square approach. Math. Comput. Model. **45**, 757–765 (2007)

[8] Khodabin, M., Maleknejad, K., Rostami, M., Nouri, M.: Numerical approach for solving stochastic Volterra-Fredholm integral equations by stochastic operational matrix. Comput. Math. Appl. **64**, 1903–1913 (2012)

[9] Biagini, F., Hu, Y., Øksendal, B., Zhang, T.: Stochastic Calculus for Fractional Brownian Motion and Applications. Springer, London (2008)

[10] Bertoin, J.: Sur une int*é*grale pour les processus *à* α-variation born*é*e. Ann. Prob. **17**(4), 1277–1699 (1989)

[11] Guerra, J., Nualart, D.: Stochastic differential equations driven by fractional Brownian motion and standard Brownian motion. Stoch. Anal. Appl. **26**(5), 1053–1075 (2008)

[12] Lisei, H., So*ó*s, A.: Approximation of stochastic diferential equations driven by fractional Brownian motion. In: Seminar on Stochastic Analysis, Random Fields and Applications Progress in Probability, vol. 59, pp. 227–241 (2008)

[13] Mishura, Y., Shevchenko, G.: The rate of convergence for Euler approximations of solutions of stochastic differential equations driven by fractional Brownian motion. Stochastic **80**(5), 489–511 (2008)

[14] Russo, F., Vallois, P.: Forward, backward and symmetric stochastic integration. Prob. Theory Relat. Fields **97**(3), 403–421 (1993)

[15] Ezzati, R., Khodabin, M., Sadati, Z.: Numerical Implementation of Stochastic Operational Matrix Driven by a Fractional Brownian Motion for Solving a Stochastic Differential Equation, Hindawi Publishing Corporation Abstract and Applied Analysis Volume (2014), Article ID 523163, p. 11

[16] Heydari, M.H., Hooshmandasl, M.R., Cattani, C., MaalekGhaini, F.M.: An efficient computational method for solving nonlinear stochastic Itô-Volterra integral equations: Application for stochastic problems in physics. J. Comput. Phys. **283**, 148–168 (2015)

[17] Deb, A., Dasgupta, A., Sarkar, G.: A new set of orthogonal functions and its applications to the analysis of dynamic systems. J. Franklin Inst. **343**(1), 1–26 (2006)

[18] Maleknejad, K., Almasieh, H., Roodaki, M.: Triangular functions (TF) method for the solution of Volterra-Fredholm integral equations. Commun. Nonlinear Sci. Numer. Simul. **15**(11), 3293–3298 (2009)

[19] Babolian, E., Masouri, Z., Hatamzadeh-Varmazyar, S.: Numerical solution of nonlinear Volterra-Fredholm integro-differential equations via direct method using triangular functions. Comput. Math. Appl. **58**, 239–247 (2009)

[20] Babolian, E., Mokhtari, R., Salmani, M.: Using direct method for solving variational problems via triangular orthogonal functions. Appl. Math. Comput. **202**, 452–464 (2008)

[21] Tripathi, M.P., Baranwal, V.K., Pandey, R.K., Singh, O.P.: A new numerical algorithm to solve fractional differential equations based on operational matrix of generalized hat functions. Commun. Nonlinear Sci. Numer. Simul. **18**, 1327–1340 (2013)

B. Hashemi, M. Khodabin and K. Maleknejad
Department of Mathematics
Karaj Branch
Islamic Azad University
Karaj
Iran
e-mail: m-khodabin@kiau.ac.ir

B. Hashemi
e-mail: hashemi_bentolhoda@payammail.ir

K. Maleknejad
e-mail: maleknejad@iust.ac.ir

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