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Higher Order Tangent Bundles

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Abstract. The tangent bundle T^kM of order k, of a smooth Banach manifold M consists of all equivalent classes of curves that agree up to their accelerations of order k. For a Banach manifold M and a natural number k, first we determine a smooth manifold structure on T^kM which also offers a fiber bundle structure for (π_k, T^kM, M). Then we introduce a particular lift of linear connections on M to geometrize T^kM as a vector bundle over M. More precisely based on this lifted nonlinear connection we prove that T^kM admits a vector bundle structure over M if and only if M is endowed with a linear connection. As a consequence, applying this vector bundle structure we lift Riemannian metrics and Lagrangians from M to T^kM. In addition, using the projective limit techniques, we declare a generalized Fréchet vector bundle structure for $T^\infty M$ over M.

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1. Introduction

Higher order tangent bundles T^kM of a smooth manifold M as the space of all equivalent classes of curves that agree up to their accelerations of order k, is a natural generalization of the notion of tangent bundle TM. Higher order geometry had witnessed a wide interest due to the works of Bucataru, Crampin etc., Dodson and Galanis, de León and Rodrigues, Miron, Morimoto and others [4,7,8,13,15,16]. The geometry of T^kM in the finite dimensional case is developed by Miron and his school [15]. They studied higher order Lagrangians and also prolongation of Riemannian metrics, Finsler structures and Lagrangians to T^kM.

However, even for the case of $n = 2$, constructing a vector bundle (for abbreviation v.b.) structure on T^2M over M is not as evident as in the case of TM. More precisely sometimes it is impossible to define a v.b. structure on T^2M. Dodson and Galanis [8] proved that for a Banach manifold M, T^2M can be thought of as a Banach vector bundle over M if and only if M is endowed with a linear connection. The author proved the same result in a different way to geometrize the bundle of accelerations with more tools like second order covariant derivative, exponential mapping and an appropriate second order Lie bracket [18]. In this paper, to geometrize higher order tangent bundles, first we introduce an special lifted connection which will plays a pivotal role in our main theorem. Then we prove that for any $k \in \mathbb{N}, T^kM$ can be thought of as a v.b. over M with the structure group $GL(\mathbb{E}^k)$ if and only if M admits a linear connection. Furthermore, this result considerably eases constructing a v.b. structure on $\pi_{ji} : T^jM \longrightarrow T^iM$ for $j > i$. We shall also show that if for some $k \geq 2, (\pi_k, T^kM, M)$ becomes a v.b. isomorphic to $\oplus_{i=1}^k TM$, then for any $n \in \mathbb{N} \cup \{\infty\}, T^nM$ admits a v.b. structure over M. More precisely in the case of infinite order, $T^\infty M$ becomes a Fréchet manifold which may be thought of as a generalized v.b. over M. Moreover, the structure group becomes a generalized Fréchet lie group which represents the advantage of using projective limit techniques.

Another old problem in geometry is that of prolongation of Riemannian and Lagrangian structures to the tangent bundles T^kM. These problems can also be solved as a consequence of our main theorem. Finally, using the restricted symplectic group and the classical Lagrangian of electrodynamics we propose two examples to support our theory. However, for more examples we refer to [19,20].

Through this paper all the maps and manifolds are assumed to be smooth, but except in Sect. 4, a lesser degree of differentiability can be assumed. Whenever partition of unity is necessary, we assume that our manifolds are partitionable [12,18].

Most of the results of this paper are novel even for the case that M is a finite dimensional manifold.

2. Preliminaries

Let M be a manifold, possibly infinite dimensional, modeled on the Banach space $\mathbb{E}$. For any $x_0 \in M$ define

$$C_{x_0} := \{\gamma : (-\epsilon, \epsilon) \longrightarrow M \ ; \ \gamma(0) = x_0 \ \text{ and } \ \gamma \ \text{ is smooth}\}.$$

As a natural extension of the tangent bundle TM define the following equivalence relation. The curves $\gamma_1, \gamma_2 \in C_{x_0}$ are said to be k-equivalent, denoted by $\gamma_1 \approx_{x_0}^k \gamma_2$, if and only if $\gamma_1^{(j)}(0) = \gamma_2^{(j)}(0)$ for $1 \leq j \leq k$. Define $T_{x_0}^kM := C_{x_0}/\approx_{x_0}^k$ and the **tangent bundle of order** k **or** k**-osculating bundle** of M to be $T^kM := \bigcup_{x\in M} T_x^kM$. Denote by $[\gamma, x_0]_k$ the representative of the equivalence class containing γ and define the canonical projection $\pi_k : T^kM \longrightarrow M$ which projects $[\gamma, x_0]_k$ onto x_0.

Let $\mathcal{A} = \{(U_\alpha, \psi_\alpha)\}_{\alpha\in I}$ be a C^∞ atlas for M. For any $\alpha \in I$ define

$$\begin{aligned} \Psi_\alpha^k : {\pi_k}^{-1}(U_\alpha) &\longrightarrow \psi_\alpha(U_\alpha) \times \mathbb{E}^k \\ [\gamma, x_0]_k &\longmapsto \left((\psi_\alpha \circ \gamma)(0), (\psi_\alpha \circ \gamma)'(0), \ldots, \frac{1}{k!}(\psi_\alpha \circ \gamma)^{(k)}(0)\right) \end{aligned}$$

Theorem 2.1. *The family* $\mathcal{B} = \{(\pi_k{}^{-1}(U_\alpha), \Psi_\alpha^k)\}_{\alpha \in I}$ *declares a smooth manifold structure on* T^kM *which models it on* $\mathbb{E}^{k+1}$.

Proof. Clearly Ψ_α^k is well defined and $\bigcup_{\alpha\in I} \pi_k{}^{-1}(U_\alpha) = T^kM$. Ψ_α^k is surjective. In fact, for any $(x, \xi_1, \dots, \xi_k) \in \psi_a(U_\alpha) \times \mathbb{E}^k$ the class $[\gamma, \psi_\alpha^{-1}(x)]_k$, with $\gamma := \psi_\alpha^{-1} \circ \bar{\gamma}$ and $\bar{\gamma}(t) = x + t\xi_1 + \cdots + t^k\xi_k$, is mapped to $(x, \xi_1, \dots, \xi_k)$ via Ψ_α^k. It is easy to show that Ψ_α^k is also injective.

For any $\alpha, \beta \in I$ with $U_{\beta\alpha} := U_\beta \cap U_\alpha \neq \emptyset$, the overlap map

$$\Psi_{\beta\alpha}^k := \Psi_\beta^k \circ {\Psi_\alpha^k}^{-1} : \psi_\alpha(U_{\beta\alpha}) \times \mathbb{E}^k \longrightarrow \psi_\beta(U_{\beta\alpha}) \times \mathbb{E}^k$$

is given by

$$\begin{aligned}
\Psi_{\beta\alpha}^k(x, \xi_1, \dots, \xi_k) &= \Psi_\beta^k([\gamma, x_0]_k) \\
&= \left((\psi_\beta \circ \gamma)(0), (\psi_\beta \circ \gamma)'(0), \dots, \frac{1}{k!}(\psi_\beta \circ \gamma)^{(k)}(0) \right) \\
&= \left((\psi_\beta \circ \psi_\alpha^{-1} \circ \bar{\gamma})(0), (\psi_\beta \circ \psi_\alpha^{-1} \circ \bar{\gamma})'(0), \dots, \frac{1}{k!}(\psi_\beta \circ \psi_\alpha^{-1} \circ \bar{\gamma})^{(k)}(0) \right) \\
&= \Bigg(\psi_{\beta\alpha}(x), \mathrm{d}\psi_{\beta\alpha}(x)\xi_1, \dots, \frac{1}{k!} \Big\{ \mathrm{d}\psi_{\beta\alpha}(x)[\bar{\gamma}^{(k)}(0)] \\
&\quad + \sum_{j_1+j_2=k} a_{(j_1,j_2)}^k \mathrm{d}^2\psi_{\beta\alpha}(x)[\bar{\gamma}^{(j_1)}(0), \bar{\gamma}^{(j_2)}(0)] \\
&\quad + \cdots + \mathrm{d}^k\psi_{\beta\alpha}(x)(\bar{\gamma}'(0), \dots, \bar{\gamma}'(0)] \Big\} \Bigg) \\
&= \Bigg(\psi_{\beta\alpha}(x), \mathrm{d}\psi_{\beta\alpha}(x)\xi_1, \mathrm{d}\psi_{\beta\alpha}(x)(\xi_2) + \frac{1}{2}\mathrm{d}^2\psi_{\beta\alpha}(x)(\xi_1, \xi_1), \dots, \\
&\quad \times \frac{1}{k!} \Big\{ \mathrm{d}\psi_{\beta\alpha}(x)[k!\xi_k] + \sum_{j_1+j_2=k} a_{(j_1,j_2)}^k \mathrm{d}^2\psi_{\beta\alpha}(x)[j_1!\xi_{j_1}, j_2!\xi_{j_2}] \\
&\quad + \cdots + \mathrm{d}^k\psi_{\beta\alpha}(x)(\xi_1, \dots, \xi_1) \Big\} \Bigg)
\end{aligned}$$

where $\psi_{\beta\alpha} = \psi_\beta \circ \psi_\alpha^{-1}$ and $\bar{\gamma}(t) = x + t\xi_1 + \cdots + t^k\xi_k$ as before. Moreover, we used the following explicit formula for the chain rule of order k

$$\begin{aligned}
(\psi_{\beta\alpha} \circ \bar{\gamma})^{(k)}(0) &= \mathrm{d}^k(\psi_{\beta\alpha} \circ \bar{\gamma})(0)(1, \dots, 1) \\
&= \sum_{i=1}^k \sum \frac{k!}{j_1! \cdots j_i! m_1! \cdots m_k!} \\
&\quad \mathrm{d}^i\psi_{\beta\alpha}(\bar{\gamma}(0))[\bar{\gamma}^{(j_1)}(0), \dots, \bar{\gamma}^{(j_i)}(0)]
\end{aligned} \tag{1}$$

where the second sum is over all (ordered) i-tuples $(j_1, \dots, j_i)$ of positive integers, such that $j_1 + \cdots + j_i = k$ and m_1 of the numbers $l_1, \dots, l_i$ are equal to $1, m_2$ are equal to 2 and so on ([3, p. 234], [14, p. 359]). The coefficient $\frac{k!}{j_1! \cdots j_i! m_1! \cdots m_k!}$ will henceforth be denoted by $a_{(j_1,\dots,j_i)}^k$. $\square$

Due to the transition functions of the bundle (π_k, T^kM, M), we can see that generally it is a smooth fiber bundle.

To compute the local forms for the change of charts of TT^kM on overlaps, we remind some facts about fiber bundles. Let $p : E \longrightarrow M$ be a smooth Banach fiber bundle with fibers diffeomorphic to the Banach manifold F and the Banach spaces $\mathbb{E}$ and $\mathbb{B}$ are the model spaces for F and M respectively. Suppose that $\Phi = (\phi, \bar{\phi}) : E|_U \longrightarrow \phi(U) \times \bar{\phi}(E|_U) \subseteq \phi(U) \times \mathbb{E}$ be a local trivialization where (U, ϕ) is a chart of M and let $(\Psi = (\psi, \bar{\psi}), V)$ be another local trivialization with $U \cap V \neq \emptyset$. Then $\Psi \circ \Phi^{-1}(x, \xi) = ((\psi \circ \phi^{-1})(x), G_{\Psi\Phi}(x, \xi))$ where $G_{\Psi\Phi} : U \cap V \times \bar{\phi}(E|_{U\cap V}) \subseteq U \cap V \times \mathbb{E} \longrightarrow \mathbb{E}$ is smooth. The canonical induced trivialization for TE is

$$T(\Psi \circ \Phi^{-1})(x, \xi; y, \eta) = \big((\psi \circ \phi^{-1})(x), G_{\Psi\Phi}(x, \xi), d(\psi \circ \phi^{-1})(x)y, \partial_1 G_{\Psi\Phi}(x, \xi)y + \partial_2 G_{\Psi\Phi}(x, \xi)\eta\big). \tag{2}$$

for any $(x, \xi, y, \eta) \in \phi(U \cap V) \times \bar{\phi}(E|_{U\cap V}) \times \mathbb{B} \times \mathbb{E}$. (Throughout this paper the symbol ∂_i denotes the partial derivative with respect to the i-th variable.) Now using the transition functions for the bundle $\pi_k : T^kM \longrightarrow M$ we can compute the transformation rule of natural charts of TT^kM. For any $u = (x, \xi_1, \ldots, \xi_k) \in U_\alpha \times \mathbb{E}^k$ and $(y, \eta_1, \ldots, \eta_k) \in \mathbb{E}^{k+1}$, we have

$$T\Psi^k_{\beta\alpha}(u; y, \eta_1, \ldots, \eta_k) = \Big(\Psi^k_{\beta\alpha}(u); \mathrm{d}\psi_{\beta\alpha}(x)y\ , \bar{\eta}_1, \ldots, \bar{\eta}_k)\Big) \tag{3}$$

where

$$\begin{aligned}
\bar{\eta}_i &= \frac{1}{i!}\{\mathrm{d}\psi_{\beta\alpha}(x)(i!\eta_i) + \sum_{j_1+j_2=i} a^i_{(j_1,j_2)}[\mathrm{d}^2\psi_{\beta\alpha}(x)(j_1!\eta_{j_1}, j_2!\xi_{j_2}) \\
&\quad + \mathrm{d}^2\psi_{\beta\alpha}(x)(j_1!\xi_{j_1}, j_2!\eta_{j_2})] + \cdots + i\mathrm{d}^i\psi_{\beta\alpha}(x)(\xi_1, \xi_1, \ldots, \xi_1, \eta_1) \\
&\quad + \mathrm{d}^2\psi_{\beta\alpha}(x)(i!\xi_i, y) + \sum_{j_1+j_2=i} a^i_{(j_1,j_2)}\mathrm{d}^3\psi_{\beta\alpha}(x)(j_1!\xi_{j_1}, j_2!\xi_{j_2}, y) \\
&\quad + \cdots + \mathrm{d}^{i+1}\psi_{\beta\alpha}(x)(\xi_1, \xi_1, \ldots, \xi_1, y)\} \\
&= \frac{1}{i!}\frac{\partial^{i+1}}{\partial s \partial t^i}(\psi_{\beta\alpha} \circ \bar{c})(t, s)|_{t=s=0}
\end{aligned}$$

and

$$\begin{aligned}
\bar{c} : (-\epsilon, \epsilon)^2 &\longrightarrow \psi_\alpha(U_\alpha) \subseteq \mathbb{E} \\
(t, s) &\longmapsto x + sy + \sum_{j=1}^{i} t^j(\xi_j + \eta_j).
\end{aligned}$$

3. Tangent Bundle of Order *k* for Banach Manifolds

This section includes two parts. In the first part for any linear connection ∇ on M, in the sense of Vilms [21], we determine a v.b. morphism $K : TT^kM \longrightarrow \oplus_{i=1}^k TM$ which may be thought as an special lift of connections. This kind of lift, named connection maps by Bucataru [4], induces nonlinear connections on T^kM. Then using K as a key, we determine a v.b. structure on $\pi_k : T^kM \longrightarrow M$ which is followed with a suitable converse.

3.1. Connection Maps in Higher Order Geometry

Consider the $C^\infty(T^kM)$-linear map $J : \mathfrak{X}(T^kM) \longrightarrow \mathfrak{X}(T^kM)$ such that locally on a chart $(\pi_k^{-1}(U_\alpha), \Psi_\alpha^k)$ is given by

$$J_\alpha(u; y, \eta_1, \ldots, \eta_k) = (u; 0, y, \eta_1, \ldots, \eta_{k-1}).$$

for any $u = (x, \xi_1, \ldots, \xi_k) \in T^kM$ and every $(u; y, \eta_1, \ldots, \eta_k) \in T_uT^kM$.

Definition 3.1. A connection map on T^kM is a vector bundle morphism

$$K = (\overset{1}{K}, \overset{2}{K}, \ldots, \overset{k}{K}) : TT^kM \longrightarrow \Big(\oplus_{i=1}^k TM, \oplus_{i=1}^k \tau_M, \oplus_{i=1}^k M\Big)$$

such that for any $1 \leq a \leq k-1$, $\overset{k}{K} \circ J^a = \overset{k-a}{K}$ and $\overset{k}{K} \circ J^k = \pi_{k*}$

Bucataru defined this connection map in the finite dimensional context [4]. Note that

$$\overset{a}{K} = \overset{k}{K} \circ J^{k-a} = \overset{k}{K} \circ J^{k-a-1} \circ J = \overset{a+1}{K} \circ J$$

and $\overset{a}{K} \circ J^a = (\overset{k}{K} \circ J^{k-a}) \circ J^a = \pi_{k*}$.

Lemma 3.2. *Locally on a chart* $(\pi_k^{-1}(U_\alpha), \Psi_\alpha^k)$ *the connection map*

$$\oplus_{i=1}^k \Psi_\alpha^1 \circ K \circ T\Psi_\alpha^{k\,-1} := K_\alpha = (\overset{1}{K}_\alpha, \ldots, \overset{k}{K}_\alpha)$$

at $(u; y, \eta_1, \ldots, \eta_k) \in T_uT^kM$ *is given by*

$$\begin{aligned} &K|_\alpha(u; y, \eta_1, \ldots, \eta_k) \\ &= \bigoplus_{i=1}^k \Big(x, \eta_i + \overset{1}{M}_\alpha (u)\eta_{i-1} + \overset{2}{M}_\alpha (u)\eta_{i-2} + \cdots + \overset{i}{M}_\alpha (u)y\Big) \end{aligned} \tag{4}$$

Proof. Since K is bundle morphism there are local maps $\overset{i}{M}_\alpha : U_\alpha \times \mathbb{E}^k \longrightarrow L(\mathbb{E}, \mathbb{E})$, $1 \leq i \leq k$, such that

$$K_\alpha(u; y, 0, \ldots, 0) = (x, \overset{1}{M}_\alpha (u)y) \oplus (x, \overset{2}{M}_\alpha (u)y) \oplus \cdots \oplus (x, \overset{k}{M}_\alpha (u)y).$$

Moreover, due to the facts $\overset{a}{K} \circ J^a = \pi_{k*}$ and $\overset{a}{K} = \overset{a+1}{K} \circ J$, we get

$$\begin{aligned} K_\alpha(u; 0, \eta_1, 0, \ldots, 0) &= K_\alpha \circ J(u; \eta_1, 0, \ldots, 0) \\ &= (x, \eta_1) \oplus (x, \overset{1}{M} (u)\eta_1) \oplus \cdots \oplus (x, \overset{k-1}{M} (u)\eta_1) \end{aligned}$$

and likewise

$$K_\alpha(u; 0, 0, \eta_2, 0, \ldots, 0) = (x, 0) \oplus (x, \eta_2) \oplus (x, \overset{1}{M} (u)\eta_2) \cdots \oplus (x, \overset{k-2}{M} (u)\eta_2).$$

which completes the proof. □

For any $\alpha, \beta \in I$ with $U_{\beta\alpha} \neq \emptyset$, the compatibility condition for M_α^i and M_β^i, $1 \leq i \leq k$, on the overlaps comes from the fact $\oplus_{i=1}^k T\psi_{\beta\alpha} \circ K_\alpha = K_\beta \circ T\Psi_{\beta\alpha}^k$. We apply equality (3) and the local form of K to obtain

$$\begin{aligned} &\mathrm{d}\psi_{\beta\alpha}(x)[\eta_i + \overset{1}{M}_\alpha (u)\eta_{i-1} + \cdots + \overset{i}{M}_\alpha (u)y] \\ &\quad = \bar{\eta}_i + \overset{1}{M}_\beta (\bar{u})\bar{\eta}_{i-1} + \overset{2}{M}_\beta (\bar{u})\bar{\eta}_{i-2} + \cdots + \overset{i}{M}_\beta (\bar{u})\bar{y} \end{aligned} \tag{5}$$

for any $u \in T^kM$ and any $(u, y, \eta_1, \ldots, \eta_k) \in T_uT^kM$. Moreover, $\bar{y} = \mathrm{d}\psi_{\beta\alpha}(x)y$ and $\bar{\eta}_1, \ldots, \bar{\eta}_k$ are as in the Eq. (3).

Theorem 3.3. *Let ∇ be a (linear) connection on M with the local components $\{\Gamma_\alpha\}_{\alpha\in I}$. There exists an induced connection map on T^kM with the following local components.*

$$\begin{aligned}
&\overset{1}{M}_\alpha (x,\xi_1)y = \Gamma_\alpha(x,\xi_1)y\\
&\overset{2}{M}_\alpha (x,\xi_1,\xi_2)y = \frac{1}{2}\Big(\sum_{i=1}^{2} \partial_i \overset{1}{M}_\alpha (x,\xi_1)(y,i\xi_i) + \overset{1}{M}_\alpha (x,\xi_1)[\overset{1}{M}_\alpha (x,\xi_1)y]\Big),\\
&\vdots\\
&\overset{k}{M}_\alpha (x,\xi_1,\ldots,\xi_k)y = \frac{1}{k}\Big(\sum_{i=1}^{k} \partial_i \overset{k-1}{M}_\alpha (x,\xi_1,\ldots,\xi_{k-1})(y,i\xi_i)\\
&\qquad\qquad + \overset{1}{M}_\alpha (x,\xi_1)[\overset{k-1}{M}_\alpha (x,\xi_1,\ldots,\xi_{k-1})y]\Big)
\end{aligned}$$

The proof of the compatibility condition for $\overset{i}{M}_\alpha$ and $\overset{i}{M}_\beta$ on overlaps can be found in [19].

Note that kernel of K is a distribution, say $H\pi_k$, on T^kM complementary to the canonical vertical distribution $V\pi_k$. The horizontal distribution $H\pi_k$ is called the nonlinear connection associated to K [15].

3.2. T^kM as a Vector Bundle

For $k \geq 2$, the bundle structure defined in Theorem 2.1 is quite far from being a v.b. due to the complicated nonlinear transition functions. Here we propose a v.b. structure on $\pi_k : T^kM \longrightarrow M$ which makes it a smooth v.b. isomorphic to k copies of TM. The converse of the problem is also true, i.e., a v.b. structure on T^kM isomorphic to $\oplus_{i=1}^k TM$, for $k \geq 2$, yields a linear connection on M. Moreover, it will be shown that if for some integer $k \geq 2, T^kM$ becomes a v.b over M with the before-mentioned property then T^iM also admits a v.b. structure over M for any $i \in \mathbb{N}$. These v.b. structures simplifies the study of higher tangent bundles. For example as a consequence, we propose a lifted Riemannian metric to T^kM which only depends to the original given metric on the base manifold M.

Theorem 3.4. *Let ∇ be a (linear) connection on M and K be the induced connection map introduced in Theorem 3.3. The following trivializations define a vector bundle structure on $\pi_k : T^kM \longrightarrow M$ with the structure group $GL(\mathbb{E}^k)$.*

$$\begin{aligned}
\Phi_\alpha^k : \pi_k^{-1}(U_\alpha) &\longrightarrow \psi_\alpha(U_\alpha) \times \mathbb{E}^k\\
[\gamma, x]_k &\longmapsto (\gamma_\alpha(0), \gamma'_\alpha(0), z_\alpha^2([\gamma,x]_k), \ldots, z_\alpha^k([\gamma,x]_k))
\end{aligned}$$

where $\gamma_\alpha = \psi_\alpha \circ \gamma$ *and*

$$\begin{aligned} z_\alpha^2([\gamma,x]_k) &= \frac{1}{2}\Big\{\frac{1}{1!}\gamma_\alpha''(0) + \overset{1}{M}_\alpha [\gamma_\alpha(0),\gamma_\alpha'(0)]\gamma_\alpha'(0)\Big\},\ldots, \\ z_\alpha^k([\gamma,x]_k) &= \frac{1}{k}\Big\{\frac{1}{(k-1)!}\gamma_\alpha^{(k)}(0) + \frac{1}{(k-2)!}\overset{1}{M}_\alpha [\gamma_\alpha(0),\gamma_\alpha'(0)]\gamma_\alpha^{(k-1)}(0) \\ &\quad + \cdots + \overset{k-1}{M}_\alpha [\gamma_\alpha(0),\gamma_\alpha'(0),\ldots,\frac{1}{(k-1)!}\gamma_\alpha^{(k-1)}(0)]\gamma_\alpha'(0)\Big\}. \end{aligned}$$

Moreover, for any $(x,\xi_1,\ldots\xi_k) \in \psi_\alpha(U_{\alpha\beta}) \times \mathbb{E}^k$ *we have*

$$\Phi_\beta^k \circ {\Phi_\alpha^k}^{-1}(x,\xi_1,\xi_2,\ldots,\xi_k) = \Big(\psi_{\beta\alpha}(x), \mathrm{d}\psi_{\beta\alpha}(x)\xi_1,\ldots,\mathrm{d}\psi_{\beta\alpha}(x)\xi_k\Big).$$

Proof. Clearly for any $\alpha \in I$, Φ_α^k is well defined and injective.

For any $(x,\xi_1,\ldots,\xi_k) \in \psi_\alpha(U_\alpha)\times\mathbb{E}^k$ we show that there exits a curve γ in M such that $\Phi_\alpha^k([\gamma,x]_k) = (x,\xi_1,\ldots,\xi_k)$. If $\bar\gamma_2(t) = x + t\xi_1 + \frac{t^2}{2}\{2\xi_2 - \overset{1}{M}_\alpha (x,\xi_1)\xi_1\}$ then $z_\alpha^2([\gamma_2,x]_k) = \xi_2$ where $\gamma_2(t) = \psi_\alpha^{-1}\circ\bar\gamma_2(t)$. Now by induction we assume that for $i-1<k$ there exists $\bar\gamma_{i-1}$, a polynomial of degree $i-1$, such that $\gamma_{i-1} = \psi_\alpha^{-1}\circ\bar\gamma_{i-1}$ and $z_\alpha^j([\gamma_j,x]_k) = \xi_j$ for $2\le j\le i-1$. Now $\bar\gamma_i$ is defined by setting

$$\begin{aligned} \bar\gamma_i(t) = \bar\gamma_{i-1}(t) + \frac{t^i}{i}\Big\{ & i\xi_i - \frac{1}{(i-2)!}\overset{1}{M}_\alpha (x,\xi_1)\bar\gamma_{i-1}^{(i-1)}(0) \\ & - \cdots \overset{i-1}{M}_\alpha (x,\xi_1,\frac{1}{2!}\bar\gamma_2^{(2)}(0),\ldots,\frac{1}{(i-1)!}\bar\gamma_{i-1}^{(i-1)}(0))\xi_1\Big\} \end{aligned}$$

and $\gamma_i(t) = \psi_\alpha^{-1}\circ\bar\gamma_i(t)$. Set $\gamma = \gamma_k$. As a result, $z^k([\gamma,x]_k) = \xi_k$ which means that $\Phi_\alpha^k([\gamma,x]_k) = (x,\xi_1,\ldots,\xi_k)$. Since $\mathrm{proj}_1\circ\Phi_\alpha^k = \pi_k$ it follows that T^kM is a fiber bundle.

For any $\alpha,\beta\in I$ with $U_{\beta\alpha}\neq\emptyset$, we prove that $\Phi_{\beta\alpha}^k := \Phi_\beta^k\circ{\Phi_\alpha^k}^{-1}$ induces a linear isomorphism between fibers. In fact, we have

$$\begin{aligned} &\Phi_{\beta\alpha}^k(x,\xi_1,\xi_2,\ldots,\xi_k) = \Phi_\beta^k([\gamma,x]_k) \\ &= \Big((\psi_\beta\circ\gamma)(0),(\psi_\beta\circ\gamma)'(0),z_\beta^2([\gamma,x]_k),\ldots,z_\beta^k([\gamma,x]_k)\Big). \end{aligned}$$

Step 1. Since $\psi_\beta\circ\gamma = \psi_\beta\circ\psi_\alpha^{-1}\circ\psi_\alpha\circ\gamma = \psi_{\beta\alpha}\circ\bar\gamma$, for any $2\le i\le k$, we get

$$\begin{aligned} iz_\beta^i([\gamma,x]_k) &= \frac{(\psi_{\beta\alpha}\circ\bar\gamma)^{(i)}(0)}{(i-1)!} + \overset{1}{M}_\beta (\psi_{\beta\alpha}(x),(\psi_{\beta\alpha}\circ\bar\gamma)'(0)) \\ &\quad\times\frac{(\psi_{\beta\alpha}\circ\bar\gamma)^{(i-1)}(0)}{(i-2)!} \\ &\quad + \cdots + \overset{i-1}{M}_\beta\left(\psi_{\beta\alpha}(x),\ldots,\frac{(\psi_{\beta\alpha}\circ\bar\gamma)^{(i-1)}(0)}{(i-1)!}\right)(\psi_{\beta\alpha}\circ\bar\gamma)'(0). \end{aligned}$$

Using the chain rule formula (1), we conclude that

$$
\begin{aligned}
iz_{\beta}^{i}([\gamma,x]_k) &= \frac{(\psi_{\beta\alpha}\circ\bar{\gamma})^{(i)}(0)}{(i-1)!} + \overset{1}{M}_{\beta}\left(\psi_{\beta\alpha}(x),(\psi_{\beta\alpha}\circ\bar{\gamma})'(0)\right)\\
&\quad\times\frac{(\psi_{\beta\alpha}\circ\bar{\gamma})^{(i-1)}(0)}{(i-2)!}\\
&\quad+\cdots+\overset{i-1}{M}_{\beta}\left(\psi_{\beta\alpha}(x),\ldots,\frac{(\psi_{\beta\alpha}\circ\bar{\gamma})^{(i-1)}(0)}{(i-1)!}\right)(\psi_{\beta\alpha}\circ\bar{\gamma})'(0)\\
&= \frac{1}{(i-1)!}\Big\{\mathrm{d}\psi_{\beta\alpha}(x)(\bar{\gamma}^{(i)}(0))\\
&\quad+\sum_{j_1+j_2=i}a^{i}_{(j_1,j_2)}\mathrm{d}^2\psi_{\beta\alpha}(x)[\bar{\gamma}^{(j_1)}(0),\bar{\gamma}^{(j_2)}(0)]\\
&\quad+\cdots+\mathrm{d}^i\psi_{\beta\alpha}(x)[\bar{\gamma}'(0),\ldots,\bar{\gamma}'(0)]\Big\}\\
&\quad+\overset{1}{M}_{\beta}\left(\psi_{\beta\alpha}(x),(\psi_{\beta\alpha}\circ\bar{\gamma})'(0)\right)\frac{(\psi_{\beta\alpha}\circ\bar{\gamma})^{(i-1)}(0)}{(i-2)!}\\
&\quad+\cdots+\overset{i-1}{M}_{\beta}\left(\psi_{\beta\alpha}(x),\ldots,\frac{(\psi_{\beta\alpha}\circ\bar{\gamma})^{(i-1)}(0)}{(i-1)!}\right)(\psi_{\beta\alpha}\circ\bar{\gamma})'(0)\\
&= \mathrm{d}\psi_{\beta\alpha}(x)\Bigg[i\xi_i-\overset{1}{M}_{\alpha}(x,\xi_1)\frac{\bar{\gamma}^{(i-1)}}{(i-2)!}-\cdots\\
&\quad-\overset{i-1}{M}_{\alpha}\left(x,\xi_1,\ldots,\frac{\bar{\gamma}^{(i-1)}}{(i-1)!}\right)\bar{\gamma}'(0)\Bigg]\\
&\quad\times\frac{1}{(i-1)!}\left\{\sum_{j_1+j_2=i}a^{i}_{(j_1,j_2)}\mathrm{d}^2\psi_{\beta\alpha}(x)[\bar{\gamma}^{(j_1)}(0),\bar{\gamma}^{(j_2)}(0)]\right.\\
&\quad\left.+\cdots+\mathrm{d}^i\psi_{\beta\alpha}(x)[\bar{\gamma}'(0),\ldots,\bar{\gamma}'(0)]\right\}\\
&\quad+\overset{1}{M}_{\beta}\left(\psi_{\beta\alpha}(x),(\psi_{\beta\alpha}\circ\bar{\gamma})'(0)\right)\frac{(\psi_{\beta\alpha}\circ\bar{\gamma})^{(i-1)}(0)}{(i-2)!}\\
&\quad+\cdots+\overset{i-1}{M}_{\beta}\left(\psi_{\beta\alpha}(x),\ldots,\frac{(\psi_{\beta\alpha}\circ\bar{\gamma})^{(i-1)}(0)}{(i-1)!}\right)(\psi_{\beta\alpha}\circ\bar{\gamma})'(0)
\end{aligned}
$$

Step 2. Setting $x=\bar{\gamma}(0),\xi_1=\bar{\gamma}'(0),\ldots,\xi_{i-1}=\frac{\bar{\gamma}^{(i-1)}}{(i-1)!},y=\bar{\gamma}'(0),\eta_1=\frac{\bar{\gamma}^{(2)}(0)}{1!},\ldots,\eta_{i-1}=\frac{\bar{\gamma}^{(i)}}{(i-1)!}$ and

$$\bar{c}_i(t,s)=\bar{\gamma}(0)+s\bar{\gamma}'(0)+\sum_{l=1}^{i-1}\frac{t^l}{l!}(\bar{\gamma}^{(l)}(0)+s\bar{\gamma}^{(l+1)}(0)),$$

then, Eq. (5) implies that

$$\mathrm{d}\psi_{\beta\alpha}(x)\Big[\overset{1}{M}_\alpha\ (x,\xi_1)\frac{\bar{\gamma}^{(i-1)}}{(i-2)!}+\cdots+\overset{i-1}{M}_\alpha\ (x,\xi_1,\ldots,\frac{\bar{\gamma}^{(i-1)}}{(i-1)!})\bar{\gamma}'(0)\Big]$$
$$=-\mathrm{d}\psi_{\beta\alpha}(x)\frac{\bar{\gamma}^{(i)}(0)}{(i-1)!}+\frac{\partial^i}{\partial s\partial t^{i-1}}\frac{(\psi_{\beta\alpha}\circ\bar{c}_i)}{(i-1)!}(0,0)$$
$$+\ \overset{1}{M}_\beta\Big((\psi_{\beta\alpha}\circ\bar{c}_i)(0,0),\frac{\partial}{\partial t}(\psi_{\beta\alpha}\circ\bar{c}_i)(0,0)\Big)\frac{\partial^{i-1}}{\partial s\partial t^{i-2}}\frac{(\psi_{\beta\alpha}\circ\bar{c}_i)}{(i-2)!}(0,0)$$
$$+\cdots+\ \overset{i-1}{M}_\beta\Big((\psi_{\beta\alpha}\circ\bar{c}_i)(0,0),\ldots,\frac{\partial^{i-1}}{\partial t^{i-1}}\frac{(\psi_{\beta\alpha}\circ\bar{c}_i)}{(i-1)!}(0,0)\Big)\frac{\partial}{\partial s}(\psi_{\beta\alpha}\circ\bar{c}_i)(0,0).$$

It is not hard to check that for any $1\le l\le i-1$, $\frac{\partial^l}{\partial s\partial t^{l-1}}(\psi_{\beta\alpha}\circ\bar{c}_i)(0,0)=(\psi_{\beta\alpha}\circ\bar{\gamma})^{(l)}(0)$ and $\frac{\partial^l}{\partial t^l}(\psi_{\beta\alpha}\circ\bar{c}_i)(0,0)=(\psi_{\beta\alpha}\circ\bar{\gamma})^{(l)}(0)$.

These last two equations yield

$$\mathrm{d}\psi_{\beta\alpha}(x)\Big[-\overset{1}{M}_\alpha\ (x,\xi_1)\frac{\bar{\gamma}^{(i-1)}}{(i-2)!}-\cdots-\overset{i-1}{M}_\alpha\ \Big(x,\xi_1,\ldots,\frac{\bar{\gamma}^{(i-1)}}{(i-1)!}\Big)\bar{\gamma}'(0)\Big]$$
$$=+\mathrm{d}\psi_{\beta\alpha}(x)\frac{\bar{\gamma}^{(i)}(0)}{(i-1)!}-\frac{(\psi_{\beta\alpha}\circ\bar{\gamma})^{(i)}(0)}{(i-1)!}$$
$$-\ \overset{1}{M}_\beta\Big(\psi_{\beta\alpha}(x),(\psi_{\beta\alpha}\circ\bar{\gamma})'(0)\Big)\frac{(\psi_{\beta\alpha}\circ\bar{\gamma})^{(i-1)}(0)}{(i-2)!}$$
$$-\cdots-\ \overset{i-1}{M}_\beta\Big(\psi_{\beta\alpha}(x),\ldots,\frac{(\psi_{\beta\alpha}\circ\bar{\gamma})^{(i-1)}(0)}{(i-1)!}\Big)(\psi_{\beta\alpha}\circ\bar{\gamma})'(0)$$
$$=-\frac{1}{(i-1)!}\Bigg\{\sum_{j_1+j_2=i}a^i_{(j_1,j_2)}\mathrm{d}^2\psi_{\beta\alpha}(x)[\bar{\gamma}^{(j_1)}(0),\bar{\gamma}^{(j_2)}(0)]$$
$$+\cdots+\mathrm{d}^i\psi_{\beta\alpha}(x)[\bar{\gamma}'(0),\ldots,\bar{\gamma}'(0)]\Bigg\}$$
$$-\ \overset{1}{M}_\beta\Big(\psi_{\beta\alpha}(x),(\psi_{\beta\alpha}\circ\bar{\gamma})'(0)\Big)\frac{(\psi_{\beta\alpha}\circ\bar{\gamma})^{(i-1)}(0)}{(i-2)!}$$
$$-\cdots-\ \overset{i-1}{M}_\beta\Big(\psi_{\beta\alpha}(x),\ldots,\frac{(\psi_{\beta\alpha}\circ\bar{\gamma})^{(i-1)}(0)}{(i-1)!}\Big)(\psi_{\beta\alpha}\circ\bar{\gamma})'(0)$$

Step 3. As a consequence of steps 1 and 2, we get

$$iz^i_\beta([\gamma,x]_k)=i\mathrm{d}\psi_{\beta\alpha}(x)\xi_i;\quad 2\le i\le k$$

that is

$$\Phi^k_{\beta\alpha}(x,\xi_1,\xi_2,\ldots,\xi_k)=\Big(\psi_{\beta\alpha}(x),\mathrm{d}\psi_{\beta\alpha}(x)\xi_1,\ldots,\mathrm{d}\psi_{\beta\alpha}(x)\xi_k\Big).$$

This last means that for any $\alpha,\beta\in I$ with $U_{\beta\alpha}\neq\emptyset$

$$\Phi^k_{\beta\alpha}:U_{\beta\alpha}\longrightarrow GL(\mathbb{E}^k)$$
$$x\longmapsto\Big(\mathrm{d}\psi_{\beta\alpha}(x)(.),\ldots,\mathrm{d}\psi_{\beta\alpha}(x)(.)\Big).$$

is smooth. As a result, the family of trivializations $\{(\pi_k^{-1}(U_\alpha),\Phi^k_\alpha)\}_{\alpha\in I}$ provides a vector bundle structure for $\pi_k:T^kM\longrightarrow M$ with the fibers isomorphic to $\mathbb{E}^k$. Moreover, since $\pi_k:T^kM\longrightarrow M$ and $\oplus^k_{i=1}\pi_1:\oplus^k_{i=1}TM\longrightarrow M$

have the same transition functions and fibers, then they are isomorphic vector bundles over M. □

The v.b. structure which is proposed in Theorem 3.4 is affective in the following sense:

Proposition 3.5. *Suppose that for some $k \geq 2, \pi_k : T^k M \longrightarrow M$ admits a v.b. structure isomorphic to $\oplus_{i=1}^{k} TM$. Then $\pi_{k-1} : T^{k-1}M \longrightarrow M$ also possesses a v.b. structure isomorphic to $\oplus_{i=1}^{k-1} TM$.*

Proof. Let $\{(\pi_k{}^{-1}(U_\alpha), \Phi_\alpha^k)\}_{\alpha\in I}$ be a family of trivializations for $\pi_k : T^k M \longrightarrow M$ induced by the atlas $\mathcal{A} = \{(U_\alpha, \psi_\alpha)\}_{\alpha\in I}$ of M as in Theorem 3.4. Then for any $[\gamma, x]_k \in T_x^k M$, we have

$$\Phi_\alpha^k([\gamma, x]_k) = \big(\psi_\alpha(x), (\psi_\alpha \circ \gamma)'(0), z_\alpha^2([\gamma, x]_k), \ldots, z_\alpha^k([\gamma, x]_k)\big).$$

We claim that $\{(\pi_{k-1}{}^{-1}(U_\alpha), \Phi_\alpha^{k-1})\}_{\alpha\in I}$ defines a v.b. structure on $\pi_{k-1} : T^{k-1}M \longrightarrow M$ where

$$\Phi_\alpha^{k-1}([\gamma, x]_k) = \big(\psi_\alpha(x), (\psi_\alpha \circ \gamma)'(0), z_\alpha^2([\gamma, x]_k), \ldots, z_\alpha^{k-1}([\gamma, x]_{k-1})\big).$$

We show that Φ_α^{k-1} is bijective. In fact, suppose that $\Phi_\alpha^{k-1}([\gamma_1, x]_{k-1}) = \Phi_\alpha^{k-1}([\gamma_2, x]_{k-1})$ then consider representatives of the classes $[\gamma_1, x]_{k-1}$ and $[\gamma_2, x]_{k-1}$ such that $\gamma_1^{(k)}(0) = \gamma_2^{(k)}(0)$. Since $\Phi_\alpha^k([\gamma_1, x]_k) = \Phi_\alpha^k([\gamma_2, x]_k)$ then injectivity of Ψ_α^k yields $[\gamma_1, x]_k = [\gamma_2, x]_k$ which means that $[\gamma_1, x]_{k-1} = [\gamma_2, x]_{k-1}$.

Suppose that $(x, \xi_1, \ldots, \xi_{k-1}) \in \psi_\alpha(U_\alpha) \times \mathbb{E}^{k-1}$ be an arbitrary element. Since Φ_α^k is bijective, there exists $[\gamma, x]_k \in T_x^k M$ such that

$$\Phi_\alpha^k([\gamma, x]_k) = (x, \xi_1, \ldots, \xi_{k-1}, 0).$$

Clearly $\Phi_\alpha^{k-1}([\gamma, x]_{k-1}) = (x, \xi_1, \ldots, \xi_{k-1})$, that is, Φ_α^{k-1} is surjective. Finally,

$$\begin{aligned} \Phi_{\beta\alpha}^{k-1} : \psi_\alpha(U_{\beta\alpha}) &\longrightarrow GL(\mathbb{E}^{k-1}) \\ x &\longmapsto \Big(\underbrace{\mathrm{d}\psi_{\beta\alpha}(x)(.), \ldots, \mathrm{d}\psi_{\beta\alpha}(x)(.)}_{(k-1)-times}\Big) \end{aligned}$$

is smooth. According to proposition 1.2 page 45 of [12], we deduce that $\pi_{k-1} : T^{k-1}M \longrightarrow M$ admits a v.b. structure isomorphic to $\oplus_{i=1}^{k-1} TM$. □

If we restrict our attention to C^k-partitionable manifolds (see e.g. [18]) then, we have the following inverse for Theorem 3.4.

Theorem 3.6. *Suppose that $k \geq 2$. If $\pi_k : T^k M \longrightarrow M$ admits a v.b. structure isomorphic to $\oplus_{i=1}^{k} TM$, then a linear connection on M can be defined.*

Proof. For $k > 2$ one can iterate Lemma 3.5 and conclude that $\pi_2 : T^2 M \longrightarrow M$ admits a v.b. structure isomorphic to $TM \oplus TM$. Then according to [8] Theorem 3.4 or [18] Theorem 2.3 there exists a linear connection on M. □

Corollary 3.7. *i. For $k \geq 2, \pi_k : T^k M \longrightarrow M$ admits a v.b. structure isomorphic to $\oplus_{i=1}^{k} TM$ if and only if M is endowed with a linear connection.*

ii. *If for some $k \geq 2, \pi_k$ becomes a v.b. isomorphic to $\oplus_{i=1}^{k} TM$ then for every $i \in \mathbb{N}$ the tangent bundle T^iM also admits a v.b. structure isomorphic to $\oplus_{j=1}^{i} TM$.*

3.3. Lifting of a Riemannian Metric

Invoking Theorem 3.4, we introduce an special lift of a given Riemannian metric g from the base manifold M to its higher order tangent bundle T^kM. Let $\{(U_\alpha, \psi_\alpha)\}_{\alpha \in I}$ be an atlas for M. Denote by $g_\alpha : TU_\alpha \times TU_\alpha \longrightarrow \mathbb{R}$ the local representative of the metric g restricted to the chart (U_α, ψ_α). Fix $k \in \mathbb{N}$ and consider the v.b. trivializations introduced in Theorem 3.4. For every $\alpha \in I$ define the bilinear symmetric form

$$G_\alpha^k : \pi_k^{-1}(U_\alpha) \times \pi_k^{-1}(U_\alpha) \longrightarrow \mathbb{R}$$

mapping $\Big([\gamma_1, x]_k, [\gamma_2, x]_k\Big)$ to

$$\sum_{i=1}^{k} g_\alpha(x)\Big(\mathrm{proj}_i \circ \Phi_\alpha^k([\gamma_1, x]_k), \mathrm{proj}_i \circ \Phi_\alpha^k([\gamma_2, x]_k)\Big)$$

where proj_i stands for the projection to the $(i+1)$'th factor.

Due to the transition functions $\Phi_{\beta\alpha}^k$ the family $\{G_\alpha\}_{\alpha \in I}$ defines a Riemannian metric on T^kM.

Remark 3.8. In the case that M is modeled on a Hilbert manifold (or a self dual Banach space [12]), we deal with Riemannian metrics. But if we go one step further then we will loose the definiteness condition of our metrics.

Remark 3.9. Let M be a Hilbert manifold. As a result of Theorem 3.4 and theorem 3.1, chapter VII [12] we can assume that a system of local trivializations $\{(\Phi_\alpha^k, \pi_k^{-1}(U_\alpha))\}_{\alpha \in I}$ consists only orthogonal trivializations that is the transition maps take values in the orthogonal (or Hilbert) group

$$\mathbb{O}(\mathbb{E}^k) = \{h \in GL(\mathbb{E}^k); \langle hv, hw \rangle = \langle v, w \rangle; v, w \in \mathbb{E}^k\}$$

(see also [20]).

3.4. Lifting of Lagrangians to Higher Order Tangent Bundles

In this section, using Theorem 3.4, we introduce a lift for Lagrangian form the base manifold to its higher order tangent bundles. To this end, we first review the concepts of Lagrangian and Lagrangian vector field from [6,15].

Let M be smooth manifold modeled on the Banach space $\mathbb{E}$. A Lagrangian on M is a smooth map $L : TM \longrightarrow \mathbb{R}$ and the associated fiber derivative is the map

$$FL : TM \longrightarrow T^*M$$

where $FL(v)w = \frac{\mathrm{d}}{\mathrm{d}t} L(v + tw)|_{t=0}$ for any $v, w \in T_xM$.

Definition 3.10. A bilinear continuous map $B : \mathbb{E} \times \mathbb{E} \longrightarrow \mathbb{R}$ is called weakly nondegenerate if for any $y \in \mathbb{E}$ the map $B^b : \mathbb{E} \longrightarrow \mathbb{E}^*$; $B^b(y)z = B(y, z)$ is injective. We call B nondegenerate (or strongly nondegenerate) if B^b is an isomorphism [6].

Note that if $\mathbb{E}$ is a finite dimensional Banach space, then there is no difference between strong and weak nondegeneracy.

In a chart $(U_\alpha, \psi^1_\alpha)$ of TM, let L_α represent L, that is, $L_\alpha = L \circ {\psi^1_\alpha}^{-1}$. The Lagrangian L is called (weakly) nondegenerate if for any chart $(U_\alpha, \psi^1_\alpha), \partial^2_2 L_\alpha(x,y) : L^2_{sym}(\mathbb{E},\mathbb{E}) \longrightarrow \mathbb{R}$ is (weakly) nondegenerate where ∂_2 denotes the partial derivative with respect to the second variable.

In finite dimensions this reads

$$rank(g_{ij}(x,y)) = rank\big(\frac{1}{2}\frac{\partial^2 L_\alpha(x,y)}{\partial y^i \partial y^j}\big)_{1\leq i,j\leq dim(M)} = dim(M)$$

where $\psi^1_\alpha = (x^i, y^i)_{1\leq i\leq n}$ is a local chart of TM.

We define the action of L by $A : TM \longrightarrow \mathbb{R}, A(v) = FL(v)v$ and the energy of L is $E = A - L$. Locally, we have

$$E_\alpha(x,y) = \partial_2 L_\alpha(x,y)y - L_\alpha(x,y).$$

Definition 3.11. The vector field $Z_E \in \mathfrak{X}(TM)$ locally defined by

$$\begin{aligned} Z_E : TM|_{U_\alpha} &\longrightarrow TTM|_{U_\alpha} \\ (x,y) &\longmapsto (x,y,y,2Z_\alpha(x,y)) \end{aligned}$$

is called a Lagrangian vector filed for L ([6]) where

$$Z_\alpha(x,y) = \frac{1}{2}\big[\partial^2_2 L_\alpha(x,y)\big]^{-1}\big(\partial_1 L_\alpha(x,y) - \partial_1(\partial_2 L_\phi(x,y)y)\big)$$

It is easily seen that Z_E is a second order vector field and the family $\{\overset{1}{M}_\alpha = \partial_2 Z_a\}_{\alpha\in I}$ defines a connection on M. Then theorem 3.4 guarantees that (π_k, T^kM, M) admits a vector bundle structure. A (weakly) nondegenerate lagrangian of order k on M is a differentiable map $L^k : T^kM \longrightarrow M$ for which $\partial^2_{k+1}L_\alpha : L^2_{sym}(\mathbb{E},\mathbb{E}) \longrightarrow \mathbb{R}, \alpha \in I$, is (weakly) nondegenerate.

Let L be a nondegenerate Lagrangian on M. Then L^k is a nondegenerate Lagrangian of order where

$$\begin{aligned} L^k_\alpha : \pi_k^{-1}(U_\alpha) &\longrightarrow \mathbb{R} \\ [\gamma, x]_k &\longmapsto \sum_{i=1}^{k} L_\alpha\big(\gamma_\alpha(0), \mathrm{proj}_i \circ \Phi^k_\alpha([\gamma,x]_k)\big); \quad \alpha \in I \end{aligned}$$

(see also [17] and [15] for different lifted Lagrangians).

4. Infinite Order Tangent Bundle

For any $x \in M$ and $\gamma_1, \gamma_2 \in C_x$ define the **infinite equivalence relation** denoted by $\approx^\infty$ as follows

$$\gamma_1 \approx^\infty_x \gamma_2 \quad \text{if and only if for any } \ k \in \mathbb{N}, \quad \gamma_1 \approx^k_x \gamma_2.$$

The equivalence class containing γ is called an infinite tangent vector at x and is denoted by $[\gamma, x]_\infty$. Alternatively, we may set $[\gamma,x]_\infty = \cap_{k=1}^\infty [\gamma,x]_k$ where the intersection in non-empty since γ belongs to $[\gamma,x]_k$ for any $k \in \mathbb{N}$. The **infinite tangent space** at $x, T^\infty_x M$ is defined to be $T^\infty_x M := C_x/\approx^\infty_x$. The infinite tangent bundle to M is denoted by $T^\infty M$ where $T^\infty M := \cup_{x\in M} T^\infty_x M$.

The canonical projection $\pi_\infty : T^\infty M \longrightarrow M$ projects the equivalence class $[\gamma, x]_\infty$ onto x. If no confusion can rise, we write $T^\infty M$ for both $T^\infty M$ as a manifold and $T^\infty M$ as a bundle over M.

We now propose a generalized Fréchet manifold (v.b.) structure for $T^\infty M$ that will be of aid in considering $T^\infty M$ as the projective limit of Banach manifolds (v.b.'s) $T^k M$.

There are natural difficulties with Fréchet manifolds, bundles and even spaces. For example, the pathological structure of the general linear group on Fréchet spaces puts in the question defining a v.b. structure for $T^\infty M$ [2,9]. Moreover, there are serious drawbacks in the study of differential equations on Fréchet manifolds [1,11]. To overcome these difficulties, we will use the projective limit tools to endow $T^\infty M$ with a reasonable manifold and v.b. structure.

First, we give some hints about a wide class of Fréchet manifolds, i.e., those which may be considered as projective limits of Banach manifolds (For more details see [8] and the references therein). Let $\{M^i, \phi^{ji}\}_{i,j\in\mathbb{N}}$ be a projective family of Banach manifolds where the model spaces $\{\mathbb{E}^i\}_{i,\in\mathbb{N}}$, respectively, also form a projective system of Banach spaces with the given connecting morphisms $\{\rho^{ji} : \mathbb{E}^j \longrightarrow \mathbb{E}^i;\ j \geq i\}_{i,j\in\mathbb{N}}$. Elements of $M := \varprojlim M^i$ consist of all threads $(x_i)_{i\in\mathbb{N}} \in \prod_{i=1}^\infty M^i$ where $\phi^{ji}(x_j) = x_i$ for all $j \geq i$. Suppose that for every thread $(x_i)_{i\in\mathbb{N}} \in \varprojlim M^i$ there exists a projective system of charts $\{U^i, \phi^i\}_{i\in\mathbb{N}}$, such that $x_i \in U^i$ and $\varprojlim U^i$ is open in $M := \varprojlim M^i$. Then M admits a Fréchet manifold structure on $\mathbb{F}$ with the corresponding charts $\{\varprojlim U^i, \varprojlim \phi^i)\}$. Furthermore, for any $i \in \mathbb{N}$ we have the natural projections $\phi_i : M \longrightarrow M^i;\ (x_k)_{k\in\mathbb{N}} \longmapsto x_i$ and $\rho_i : (e_k)_{k\in\mathbb{N}} \longmapsto e_i$.

In our case for any natural number i, $M^i := T^i M$ and $\mathbb{E}^i := \overbrace{\mathbb{E} \times \mathbb{E} \cdots \times \mathbb{E}}^{\text{i+1 times}}$ with the usual product norm $\| \ . \ \|_i$. For $j \geq i$, the connecting morphism $\phi^{ji} : T^j M \longrightarrow T^i M$ maps the class $[\gamma, x]_j$ onto $[\gamma, x]_i$ and $\rho^{ji} : \mathbb{E}^j \longrightarrow \mathbb{E}^i$ is just projection to the first $i+1$ factors. The canonical projective systems of charts are $\{\big({\pi_i}^{-1}(U_\alpha), \Phi^i_\alpha\big)\}_{i\in\mathbb{N}}$ rising from Theorem 3.4. Consequently, $T^\infty M$ admits a **smooth Fréchet manifold structure** modeled on the Fréchet space $\mathbb{F} = \varprojlim \mathbb{E}^i \subseteq \prod_{k=1}^\infty$. Note that $\mathbb{F}$ is a Fréchet space with the associated metric

$$d(x, y) = \sum_{i=1}^\infty \frac{\| x_i - y_i \|_i}{2^i(1 + \| x_i - y_i \|_i)}$$

where $x, y \in \mathbb{F} := \varprojlim \mathbb{E}^i, x_i = \rho_i(x), y_i = \rho_i(y)$ and $\rho_i : \mathbb{F} \longrightarrow \mathbb{E}^i;\ (x_k)_{k\in\mathbb{N}} \longmapsto x_i$ is the canonical projection.

In a further step, we will try to supply $\pi_\infty : T^\infty M \longrightarrow M$ with a generalized v.b. structure. Suppose that for any $i \in \mathbb{N}, (\pi_i, E^i, M)$ be a Banach v.b. on M with the fibers of type $\mathbb{E}^i$ where $\{\mathbb{E}^i, \rho^{ji}\}_{i,j\in\mathbb{N}}$ also forms a projective system of Banach spaces. With these notations we state the following definition from [9].

Definition 4.1. The system $\{(\pi_i, E^i, M), f^{ji}\}_{i,j\in\mathbb{N}}$ is called a **strong projective system of Banach v.b.'s** over the same basis M if;

(i) $\{E^i, f^{ji}\}_{i,j\in\mathbb{N}}$ is a projective system of Banach manifolds.

(ii) For any $(x^i)_{i\in\mathbb{N}} \in F := \varprojlim E^i$, there exists a projective system of trivializations $\tau^i : {\pi_i}^{-1}(U) \longrightarrow U \times \mathbb{E}^i$ of (E^i, π_i, M), such that $x^i \in U \subseteq M$ and $(id_U \times \rho^{ji}) \circ \tau^j = \tau^i \circ f^{ji}$ for all $j \geq i$.

Now the projective systems of v.b's is defined by setting $E^i := T^iM$, $\tau^i_\alpha := (\psi^{-1}_\alpha \times id_{\mathbb{E}^i}) \circ \Phi^i_\alpha$ and ϕ^{ji}, ρ^{ji} as in the previous part.

For any $i \in \mathbb{N}$ define the Banach Lie group

$$\mathcal{H}^0_i(\mathbb{E}^i) = \{(l_1, \ldots, l_i) \in \prod_{j=1}^{i} \mathcal{L}(\mathbb{E}^j, \mathbb{E}^j);\ \rho^{jk} \circ l_j = l_k \circ \rho^{jk} \quad \text{for all} \ \ k \leq j \leq i\}$$

Using proposition 1.2 of [9], we conclude that $\pi_\infty : T^\infty M \longrightarrow M$ admits a generalized v.b. structure over M with fibers isomorphic to the Fréchet space $\mathbb{F} = \varprojlim \mathbb{E}^i$ and the structure group $\mathcal{H}^0(\mathbb{F}) := \varprojlim \mathcal{H}^0_i(\mathbb{E}^i)$.

Remark 4.2. In the case where M is a finite dimensional manifold, $T^\infty M$ becomes a Fréchet v.b. over M with fibers isomorphic to the known Fréchet space $\mathbb{R}^\infty$.

Example 4.3. In this example, we introduce the restricted symplectic group $Sp_2(\mathcal{H})$ ([10]) and we propose a vector bundle structure for $\big(\pi_k, T^kSp_2(\mathcal{H}), Sp_2(\mathcal{H}\big)$ for $k \in \mathbb{N} \cup \{\infty\}$. Let $(\mathcal{H}, \langle,\rangle)$ be an infinite dimensional real Hilbert space and J be a complex structure on $\mathcal{H}$. The symplectic group $Sp(\mathcal{H})$ is defined by

$$Sp(\mathcal{H}) = \{g \in GL(\mathcal{H}) : g^* Jg = J\}.$$

The Lie algebra of $Sp(\mathcal{H})$ is

$$\mathfrak{sp}(\mathcal{H}) = \{x \in \mathcal{B}(\mathcal{H});\ xJ = -Jx^*\}$$

Denote by $\mathcal{B}_2(\mathcal{H})$ the Hilbert?-Schmidt class $B_2(\mathcal{H}) = \{g \in \mathcal{B}(\mathcal{H}) : Tr(g^*g) < \infty\}$ where Tr is the usual trace and $\mathcal{B}(\mathcal{H})$ is the set of all bounded linear operators on $\mathcal{H}$. Define the restricted symplectic group to be

$$Sp_2(\mathcal{H}) = \{g \in Sp(\mathcal{H}) : g - 1 \in B_2(\mathcal{H})\}$$

Then the Lie algebra of $Sp_2(\mathcal{H})$ is $\mathfrak{sp}_2(\mathcal{H}) = \{x \in B_2(\mathcal{H}) : xJ = -Jx^*\}$ which is a closed subspaces of $B_2(\mathcal{H})$, and hence a Hilbert space [10]. Moreover, for any $g \in Sp_2(\mathcal{H})$,

$$(TSp_2(\mathcal{H}))_g = g\mathfrak{sp}_2(\mathcal{H}) \subset B_2(\mathcal{H})$$

is an inner product space endowed with the left invariant Riemannian metric

$$\langle v, w\rangle_g = \langle g^{-1}v, g^{-1}w\rangle = Tr((gg^*)^{-1}vw^*); \quad v, w \in T_gSp_2(\mathcal{H}) \tag{6}$$

However, the Riemannian connection on $Sp_2(\mathcal{H})$ is given by the local form (Christoffel symbol)

$$2g^{-1}\Gamma_g(gx, gy) = xy + yx + x^*y + y^*x - xy^* - yx^*$$

for any $g \in Sp_2(\mathcal{H})$ and $x, y \in \mathfrak{sp}_2(\mathcal{H})$.

As a consequence, for any $k \in \mathbb{N}, \pi_k : T^kSp_2(\mathcal{H}) \longrightarrow Sp_2(\mathcal{H})$ admits a vector bundle structure with fibers isomorphic to $\mathfrak{sp}_2(\mathcal{H})^k$ and the structure group $GL(\mathfrak{sp}_2(\mathcal{H}))$. Since the base manifold is a Riemannian manifolds, for $k \in \mathbb{N}$, the above vector bundle can be considered as a vector bundle with $\mathbb{O}(\mathfrak{sp}_2(\mathcal{H})^k)$ as its structure group (Remark 3.9).

Moreover, $\pi_\infty : T^\infty Sp_2(\mathcal{H}) \longrightarrow Sp_2(\mathcal{H})$ becomes a generalized vector bundle with fibers isomorphic to $\mathfrak{sp}_2(\mathcal{H})^\infty = \varprojlim \mathfrak{sp}_2(\mathcal{H})^i$ and the structure group $\mathcal{H}^0(\mathfrak{sp}_2(\mathcal{H})^\infty)$.

Example 4.4. Let (M, g) be a pseudo Riemannian manifold. Consider the Lagrangian

$$L(x, y) = mcg(x)(y, y) + \frac{2e}{m}A(x)y; \quad (x, y) \in T_xM$$

where $0 \neq m, c, e$ are known physical constants and A is a 1-form on M. L is known as the classical Lagrangian of electrodynamics and g is called the gravitational potential and A is the electromagnetic potential [15].

Following the formalism of Sect. 3.4 and also [15] we consider the Lagrangian

$$\begin{aligned} L^k : T^kM &\longrightarrow \mathbb{R} \\ [\gamma, x]_k &\longmapsto mcg(x)\big(\mathrm{proj}_k \circ \Phi^k_\alpha([\gamma, x]_k), \mathrm{proj}_k \circ \Phi^k_\alpha([\gamma, x]_k)\big) \\ &\qquad + \frac{2e}{m}A(x)[\mathrm{proj}_k \circ \Phi^k_\alpha([\gamma, x]_k)] \end{aligned}$$

Clearly, L^k is a nondegenerate Lagrangian of order k on M which is known as the Lagrangian of electrodynamics of order k [15].

We remind that a k-semispray on M is a map $S : T^kM \longrightarrow TT^kM$, such that for any $u = (x, \xi_1, \ldots, \xi_k) \in T^kM$ the map $S_\alpha := T\Psi^k_\alpha \circ S \circ {\Psi^k_\alpha}^{-1}$ is given by

$$S_\alpha(u) = \Big(u; \xi_1, 2\xi_2, \ldots, k\xi_k, (k+1)G_\alpha(u)\Big)$$

where $G_\alpha : \pi_k^{-1}(U_\alpha) \longrightarrow \mathbb{E}$ is a differentiable map. The family $\{G_\alpha\}_{\alpha \in I}$ are local components of the semispray S.

For $U \subseteq M$ identify $\pi_k^{-1}(U)$ with its image $\Psi^k_\alpha(\pi_k^{-1}(U))$. For the Lagrangian L^k consider the k-semispray S with the local components

$$\begin{aligned} G(u) = \frac{1}{2(k+1)mc} g(x)^{-1}\{&\partial_k L^k(u) - \partial_1\partial_{k+1}L^k(u)\xi_1 \\ &- \partial_2\partial_{k+1}L^k(u)2\xi_2 - \cdots - \partial_k\partial_{k+1}\partial^2 L^k(u)k\xi_k\} \end{aligned}$$

where $g(x)$ is considered as a map from T_xM to T_x^*M (see e.g. [5] for the finite dimensional case).

Now the curve $c : (-\epsilon, \epsilon) \longrightarrow M$ is a motion of the Lagrangian system (T^kM, L^k) if c satisfies the system of $k+1$ order differential equation

$$\frac{\mathrm{d}^{k+1}}{\mathrm{d}t^{k+1}}c(t) = (k+1)G\Big(c(t), c'(t), \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}c(t)2, \ldots, \frac{1}{k!}\frac{\mathrm{d}^k}{\mathrm{d}t^k}c(t)\Big).$$

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