Mediterr. J. Math. (2017) 14:6
DOI 10.1007/s00009-016-0804-7
1660-5446/17/010001-14
published online December 27, 2016

Mediterranean Journal of Mathematics

On Szász–Mirakyan Operators Preserving e^{2ax}, $a > 0$

Tuncer Acar, Ali Aral and Heiner Gonska

Abstract. A modification of Szász–Mirakyan operators is presented that reproduces the functions 1 and e^{2ax}, $a > 0$ fixed. We prove uniform convergence, order of approximation via a certain weighted modulus of continuity, and a quantitative Voronovskaya-type theorem. A comparison with the classical Szász–Mirakyan operators is given. Some shape preservation properties of the new operators are discussed as well. Using a natural transformation, we also present a uniform error estimate for the operators in terms of the first- and second-order moduli of smoothness.

Mathematics Subject Classification. 41A25, 41A36.

Keywords. Szász–Mirakyan operators, King operators, Weighted modulus of continuity, Uniform convergence.

1. Introduction

The mappings which are nowadays called Szász–Mirakyan operators were introduced independently by the two authors mentioned and Favard between 1941 and 1950 (see [10,14,18]). For $x \in [0, \infty)$ and $f : [0, \infty) \to \mathbb{R}$ for which the right-hand side is absolutely convergent, they are defined by

$$S_n f(x) = S_n(f; x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) e^{-nx} \frac{(nx)^k}{k!}, \quad n \in \mathbb{N}. \tag{1.1}$$

The operators S_n have many properties similar to those of the classical Bernstein operators given for $f \in C[0, 1]$, say. In particular, both are positive, linear and, for $i = 0, 1$, reproduce the functions $e_i(x) = x^i$. In 2003, King[13] introduced a sequence of positive linear operators which modify the Bernstein operators and preserve the test functions e_0 and e_2 on $[0, 1]$. King's approach was further investigated by several authors which we do not cite in this text.

King-type modifications of Szász–Mirakyan operators were also considered. For example, Duman and Özarslan [9] constructed Szász–Mirakyan-type operators which reproduce the test functions e_0 and e_2 on $[0,\infty)$, and quite a general approach is described by Aral et al. in [4].

Motivated by the above-mentioned papers, we propose to construct Szász–Mirakyan-type operators which reproduce the functions e_0 and e^{2ax}, $a>0$ fixed and we formulate a sufficient condition under which the new operators perform better than S_n.

For functions $f\in C\left[0,\infty\right)$, such that the right-hand side below is absolutely convergent, we introduce operators as

$$R_{a,n}^{*}\left(f;x\right):=R_{n}^{*}\left(f;x\right):=e^{-n\alpha_n\left(x\right)}\sum_{k=0}^{\infty}\frac{\left(n\alpha_n\left(x\right)\right)^k}{k!}f\left(\frac{k}{n}\right), \tag{1.2}$$

$x\geq 0, n\in\mathbb{N}$, such that the conditions

$$R_{n}^{*}\left(e^{2at};x\right)=e^{2ax} \tag{1.3}$$

are satisfied for all x and all n. The operators R_n^* are linear, positive and preserve the constant functions. Note that for $\alpha_n\left(x\right)=x$, the operators (1.2) reduce to the classical Szász–Mirakyan operators (1.1). However, this case will not be included in our considerations.

Using (1.2) and (1.3), we explicitly require

$$e^{2ax}=e^{n\alpha_n\left(x\right)\left(e^{2a/n}-1\right)}$$

which is the case for

$$\alpha_n\left(x\right)=\frac{2ax}{n\left(e^{2a/n}-1\right)}. \tag{1.4}$$

Thus, the operator (1.2) can be rewritten in the form:

$$\begin{aligned}R_{n}^{*}\left(f;x\right)&=e^{-n\alpha_n\left(x\right)}\sum_{k=0}^{\infty}\frac{\left(n\alpha_n\left(x\right)\right)^k}{k!}f\left(\frac{k}{n}\right)\\&=e^{-\frac{2ax}{\left(e^{2a/n}-1\right)}}\sum_{k=0}^{\infty}\frac{\left(2ax\right)^k}{k!\left(e^{2a/n}-1\right)^k}f\left(\frac{k}{n}\right)\\&=S_n\left(f,\varphi_n\left(x\right)\right),\end{aligned} \tag{1.5}$$

where

$$\varphi_n\left(x\right):=\left(S_n\left(e^{2at}\right)\right)^{-1}\circ e^{2ax}.$$

Note that in the excellent paper [3], Aldaz and Render introduced linear positive operators which preserve the same exponential functions. In addition, general King-type operators which preserve some exponential functions were studied by Birou [6].

Quantitative Voronovskaya theorems for various types of operators have been studied intensively in the last decade. This kind of results is useful to describe the rate of pointwise convergence and the error of approximation simultaneously. In the recent paper [2], Acar et al. proved quantitative forms of Voronvskaya's theorem on unbounded intervals for general linear positive operators by means of a weighted modulus of smoothness. Moreover, a similar theorem for generalized Szász–Mirakyan operators was proved via different weighted moduli of smoothness by Acar et al. [1].

The paper is organized as follows. In Sect. 2, we give some lemmas which will be necessary to prove our main results. Section 3 contains the proof of uniform convergence of the operators and also a statement concerning the degree of this uniform convergence. A quantitative Voronovskaya-type theorem for R_n^* is given in Sect. 4. In Sect. 5, we present some shape preserving properties of the operators (1.2) and we compare the operators R_n^* with the classical ones. In the last Sect. 6, considering an isomorphism between $(C^*[0,\infty), \|\cdot\|_{[0,\infty)})$ and $(C[0,1], \|\cdot\|_{[0,1]})$, we present a uniform estimate for R_n^* in terms of the first- and second-order moduli of smoothness.

2. Preliminary Results

We give the following lemmas without proofs, since they are similar to the corresponding results for Szász–Mirakyan operators and require some elementary calculations.

Lemma 1. *Let $a \geq 0$. Then, we have*

$$\begin{aligned} R_n^*\left(e^{at};x\right) &= e^{n\alpha_n(x)\left(e^{a/n}-1\right)} \\ &= e^{\frac{2ax}{\left(e^{2a/n}-1\right)}\left(e^{a/n}-1\right)} \\ &= e^{\frac{2ax}{\left(e^{a/n}+1\right)}}. \end{aligned} \tag{2.1}$$

Lemma 2. *We have*

$$\begin{aligned} &R_n^*\left(e_0;x\right)=1,\ \ R_n^*\left(e_1;x\right)=\alpha_n\left(x\right), \\ &R_n^*\left(e_2;x\right)=\alpha_n^2\left(x\right)+\frac{\alpha_n\left(x\right)}{n}. \end{aligned}$$

Lemma 3. *Let $\varphi_x^k(t) := (t-x)^k$, $k = 0,1,2,\ldots$ Then*

$$R_n^*\left(\varphi_x^0\left(t\right);x\right)=1, \tag{2.2}$$

$$R_n^*\left(\varphi_x^1\left(t\right);x\right)=\alpha_n\left(x\right)-x \tag{2.3}$$

$$R_n^*\left(\varphi_x^2\left(t\right);x\right)=\left(\alpha_n\left(x\right)-x\right)^2+\frac{\alpha_n\left(x\right)}{n}. \tag{2.4}$$

Moreover, considering equality (1.4), we find

$$\lim_{n\to\infty} n\left(\frac{2ax}{n\left(e^{2a/n}-1\right)}-x\right)=-ax, \tag{2.5}$$

$$\lim_{n\to\infty} n\left(\left(\frac{2ax}{n\left(e^{2a/n}-1\right)}-x\right)^2+\frac{2ax}{n^2\left(e^{2a/n}-1\right)}\right)=x. \tag{2.6}$$

3. A Quantitative Result

Here, we explore the rate of uniform convergence of the operators R_n^* on $[0,\infty)$. The use of the unweighted Chebyshev norm makes sense in $C^*[0,\infty)$. This is the (small) subspace of $C[0,\infty)$ of all real-valued continuous functions on $[0,\infty)$ with the property that $lim_{x\to\infty} f(x)$ exists and is finite, endowed with the uniform norm.

In 1970, Boyanov and Veselinov [7] showed that uniform convergence of any sequence of positive linear operators in the above setting can be checked as follows.

Theorem 1. *The sequence $A_n : C^*[0,\infty) \to C^*[0,\infty)$ of positive linear operators satisfies the conditions*

$$\lim_{n\to\infty} A_n\left(e^{-kt};x\right) = e^{-kt}, \qquad k=0,1,2,$$

uniformly in $[0,\infty)$, if and only if

$$\lim_{n\to\infty} A_n\left(f;x\right) = f\left(x\right)$$

uniformly in $[0,\infty)$, for all $f\in C^*[0,\infty)$.

The two authors mentioned applied their theorem to Szász–Mirakyan and Baskakov operators.

To obtain an estimate for the rate of convergence in the above theorem, we will use the following modulus of continuity:

$$\omega^*\left(f;\delta\right) := \sup_{\substack{x,t>0 \\ |e^{-x}-e^{-t}|\le\delta}} \left|f\left(t\right)-f\left(x\right)\right|,$$

which is well defined for every $\delta \ge 0$ and every function $f \in C^*[0,\infty)$ (see Holhoş [12]). The modulus $\omega^*\left(\cdot;\delta\right)$ has the property:

$$\left|f\left(t\right)-f\left(x\right)\right| \le \left(1+\frac{\left(e^{-x}-e^{-t}\right)^2}{\delta^2}\right)\omega^*\left(f;\delta\right), \delta>0. \tag{3.1}$$

For more details on $\omega^*\left(\cdot;\delta\right)$, we refer the reader to [12]. There also the following statement can be found.

Theorem 2. *If a sequence of positive linear operators $A_n : C^*[0,\infty) \to C^*$ $[0,\infty)$ satisfy the equalities:*

$$\left\|A_n e_0 - 1\right\|_{[0,\infty)} = \alpha_n,$$
$$\left\|A_n\left(e^{-t}\right) - e^{-t}\right\|_{[0,\infty)} = \beta_n,$$
$$\left\|A_n\left(e^{-2t}\right) - e^{-2t}\right\|_{[0,\infty)} = \gamma_n,$$

then

$$\left\|A_n f - f\right\|_{[0,\infty)} \le 2\omega^*\left(f;\sqrt{\alpha_n+2\beta_n+\gamma_n}\right),\ f\in C^*[0,\infty).$$

Remark 1. The assumption that $\alpha_n,\beta_n,\gamma_n$ tend to zero as n goes to infinity (made in [12]) is not needed.

The following theorem deals with uniform convergence of R_n^*.

Theorem 3. *For $f \in C^*[0,\infty)$, we have*

$$\|R_n^* f - f\|_{[0,\infty)} \le 2\omega^*\left(f; \sqrt{2\beta_n + \gamma_n}\right),$$

where

$$\beta_n = \left\|R_n^*\left(e^{-t}\right) - e^{-t}\right\|_{[0,\infty)},$$
$$\gamma_n = \left\|R_n^*\left(e^{-2t}\right) - e^{-2t}\right\|_{[0,\infty)}.$$

Moreover, β_n and γ_n tend to zero as n goes to infinity, so that $R_n^* f$ converges uniformly to f.

Proof. The inequality immediately follows from Theorem 2. Taking definition (1.5) and equality (1.4) into account, for $\lambda \ge 0$, one can write as

$$\begin{aligned} R_n^*\left(e^{-\lambda t}; x\right) &= e^{-n\alpha_n(x)} \sum_{k=0}^{\infty} \frac{(n\alpha_n(x))^k}{k!} e^{-\frac{\lambda k}{n}} \\ &= e^{-n\alpha_n(x)} e^{n\alpha_n(x) e^{-\frac{\lambda}{n}}} \\ &= e^{n\alpha_n(x)\left(e^{-\lambda/n} - 1\right)} \\ &= e^{\frac{2ax}{\left(e^{2a/n}-1\right)}\left(e^{-\lambda/n}-1\right)} \\ &= e^{-\frac{2ax}{e^{\lambda/n}}\left(\frac{e^{\lambda/n}-1}{e^{2a/n}-1}\right)}. \end{aligned}$$

Take $\lambda = 1$ first. Using the inequality

$$\frac{u-v}{\ln u - \ln v} < \frac{u+v}{2} \qquad \text{for } \ 0 < v < u,$$

we have

$$e^{-xu_n} - e^{-x} < \frac{1-u_n}{2}\left(xe^{-xu_n} + xe^{-x}\right),$$

where $u_n = \frac{2a}{e^{1/n}}\left(\frac{e^{1/n}-1}{e^{2a/n}-1}\right)$. On the other hand, since

$$\max_{x>0} xe^{-bx} = \frac{1}{eb}$$

for every $b > 0$, we can write as

$$\begin{aligned} e^{-xu_n} - e^{-x} &< \frac{(1-u_n)}{2}\left(\frac{1}{eu_n} + \frac{1}{e}\right) \\ &= \frac{\left(1-u_n^2\right)}{2eu_n}. \end{aligned}$$

Thus

$$\left\|R_n^*\left(e^{-t}; x\right) - e^{-x}\right\|_{[0,\infty)} = \beta_n < \frac{\left(1-u_n^2\right)}{2eu_n} \to 0 \tag{3.2}$$

as $n \to \infty$.

For $\lambda = 2$, we have

$$\begin{aligned} e^{-xv_n} - e^{-2x} &< \frac{2-v_n}{2}\left(xe^{-xu_n} + xe^{-2x}\right) \\ &< \frac{(2-v_n)}{2}\left(\frac{1}{eu_n} + \frac{1}{2e}\right) \\ &= \frac{\left(4-v_n^2\right)}{4ev_n}, \end{aligned}$$

where $v_n = \frac{2a}{e^{2/n}}\left(\frac{e^{2/n}-1}{e^{2a/n}-1}\right)$. Therefore

$$\left\|R_n^*\left(e^{-2t};x\right) - e^{-2x}\right\|_{[0,\infty)} = \gamma_n < \frac{\left(4-v_n^2\right)}{4ev_n} \to 0 \tag{3.3}$$

as $n \to \infty$. Hence, by Theorem 2 (or Theorem 1), the proof is complete. □

4. A Quantitative Voronovskaya-Type Theorem

We will now examine the asymptotic behavior of the operators R_n^* by proving a quantitative Voronovskaya theorem.

Theorem 4. *Let $f, f'' \in C^*[0,\infty)$. Then, the inequality*

$$\begin{aligned} &\left|n\left[R_n^*(f;x) - f(x)\right] + axf'(x) - \frac{x}{2}f''(x)\right| \\ &\quad \leq |p_n(x)|\,|f'(x)| + |q_n(x)|\,|f''(x)| + 2\left(2q_n(x) + x + r_n(x)\right)\omega^*\left(f'';1/\sqrt{n}\right) \end{aligned}$$

holds for any $x \in [0,\infty)$, where

$$\begin{aligned} p_n(x) &:= nR_n^*\left(\varphi_x^1(t);x\right) + ax, \\ q_n(x) &:= \frac{1}{2}\left(nR_n^*\left(\varphi_x^2(t);x\right) - x\right), \\ r_n(x) &= n^2\sqrt{R_n^*\left(\left(e^{-x}-e^{-t}\right)^4;x\right)}\sqrt{R_n^*\left((t-x)^4;x\right)}. \end{aligned}$$

Proof. By the Taylor expansion of f at the point $x \in \mathbb{R}^+$, we can write as

$$f(t) = f(x) + f'(x)(t-x) + \frac{f''(x)}{2}(t-x)^2 + h(t,x)(t-x)^2, \tag{4.1}$$

where

$$h(t,x) := \frac{f''(\eta) - f''(x)}{2}$$

and η is a number between x and t. If we apply the operator R_n^* to both sides of equality (4.1), we immediately have

$$\begin{aligned} &\left|R_n^*(f;x) - f(x) - f'(x)R_n^*\left(\varphi_x^1(t);x\right) - \frac{f''(x)}{2}R_n^*\left(\varphi_x^2(t);x\right)\right| \\ &\quad \leq \left|R_n^*\left(h(t,x)\varphi_x^2(t),x\right)\right|. \end{aligned}$$

Considering the equalities (2.2)–(2.4), we can write as

$$\begin{aligned}&\left|n\left[R_n^*\left(f;x\right)-f\left(x\right)\right]+axf'\left(x\right)-\frac{x}{2}f''\left(x\right)\right|\\&\quad\leq\left|nR_n^*\left(\varphi_x^1\left(t\right);x\right)+ax\right|\left|f'\left(x\right)\right|+\frac{1}{2}\left|nR_n^*\left(\varphi_x^2\left(t\right);x\right)-x\right|\left|f''\left(x\right)\right|\\&\qquad+\left|nR_n^*\left(h\left(t,x\right)\varphi_x^2\left(t\right);x\right)\right|.\end{aligned}$$

Put $p_n\left(x\right):=nR_n^*\left(\varphi_x^1\left(t\right);x\right)+ax$ and $q_n\left(x\right):=\frac{1}{2}\left(nR_n^*\left(\varphi_x^2\left(t\right);x\right)-x\right).$ Hence

$$\begin{aligned}&\left|n\left[R_n^*\left(f;x\right)-f\left(x\right)\right]+axf'\left(x\right)-\frac{x}{2}f''\left(x\right)\right|\\&\quad\leq\left|p_n\left(x\right)\right|\left|f'\left(x\right)\right|+\left|q_n\left(x\right)\right|\left|f''\left(x\right)\right|+\left|nR_n^*\left(h\left(t,x\right)\varphi_x^2\left(t\right);x\right)\right|.\end{aligned}$$

Note that, by the equalities (2.5) and (2.6), $p_n\left(x\right)\to0$, $q_n\left(x\right)\to0$ as $n\to\infty$ at any point $x\in\mathbb{R}^+$. To complete the proof, we must estimate the last term $\left|nR_n^*\left(h\left(t,x\right)\varphi_x^2\left(t\right),x\right)\right|$. Using the inequality (3.1), we get

$$\left|h\left(t,x\right)\right|\leq\left(1+\frac{\left(e^{-x}-e^{-t}\right)^2}{\delta^2}\right)\omega^*\left(f'';\delta\right).$$

If $\left|e^{-x}-e^{-t}\right|\leq\delta$, then $\left|h\left(t,x\right)\right|\leq2\omega^*\left(f'';\delta\right)$. If $\left|e^{-x}-e^{-t}\right|>\delta$, then $\left|h\left(t,x\right)\right|\leq2\frac{\left(e^{-x}-e^{-t}\right)^2}{\delta^2}\omega^*\left(f'';\delta\right)$. Therefore, we have $\left|h\left(t,x\right)\right|\leq2\left(1+\frac{\left(e^{-x}-e^{-t}\right)^2}{\delta^2}\right)\omega^*\left(f'';\delta\right)$. Using this, we obtain

$$\begin{aligned}nR_n^*\left(\left|h\left(t,x\right)\right|\varphi_x^2\left(t\right),x\right)&\leq2n\omega^*\left(f'';\delta\right)R_n^*\left(\left(t-x\right)^2;x\right)\\&\quad+\frac{2n}{\delta^2}\omega^*\left(f'';\delta\right)R_n^*\left(\left(e^{-x}-e^{-t}\right)^2\left(t-x\right)^2;x\right).\end{aligned}$$

Applying the Cauchy–Schwarz inequality, we infer

$$\begin{aligned}nR_n^*\left(\left|h\left(t,x\right)\right|\varphi_x^2\left(t\right),x\right)&\leq2n\omega^*\left(f'';\delta\right)R_n^*\left(\left(t-x\right)^2;x\right)\\&\quad+\frac{2n}{\delta^2}\omega^*\left(f'';\delta\right)\sqrt{R_n^*\left(\left(e^{-x}-e^{-t}\right)^4;x\right)}\sqrt{R_n^*\left(\left(t-x\right)^4;x\right)}.\end{aligned}$$

Choosing $\delta=1/\sqrt{n}$ and using the notation

$$r_n\left(x\right):=\sqrt{n^2R_n^*\left(\left(e^{-x}-e^{-t}\right)^4;x\right)}\sqrt{n^2R_n^*\left(\left(t-x\right)^4;x\right)}$$

we arrive at

$$\begin{aligned}&\left|n\left[R_n^*\left(f,x\right)-f\left(x\right)\right]+axf'\left(x\right)-\frac{x}{2}f''\left(x\right)\right|\\&\quad\leq\left|p_n\left(x\right)\right|\left|f'\left(x\right)\right|+\left|q_n\left(x\right)\right|\left|f''\left(x\right)\right|+2\left(2q_n\left(x\right)+x+r_n\left(x\right)\right)\omega^*\left(f'';1/\sqrt{n}\right)\end{aligned}$$

which was our claim. □

Remark 2. Direct calculations give

$$\lim_{n\to\infty} n^2 R_n^*\left((t-x)^4;x\right) = \frac{22}{3}a^2x^4 + 4a^2x^2\left(\frac{6}{a}x + \frac{7}{4a^2} + \frac{5}{2}x^2\right) -2ax\left(\frac{2}{a}x + \frac{2}{3}ax^3 + 3x^2\right) - 8a^3x^3\left(\frac{2}{a}x + \frac{9}{4a^2}\right).$$

Furthermore, using Mathematica, the following was obtained:

$$\begin{aligned}
&\lim_{n\to\infty} n^2 R_n^*\left(\left(e^{-t}-e^{-x}\right)^4;x\right)\\
&= e^{-4x}\left(2a^2x^2\left(\frac{4}{a}+2\right)^2 - 2ax\left(\frac{2}{3}a + \frac{16}{3a} + 4\right)\right)\\
&\quad - 4e^{-4x}\left(2a^2x^2\left(\frac{1}{4a}+\frac{1}{2}\right)^2 - 2ax\left(\frac{1}{6}a + \frac{1}{12a} + \frac{1}{4}\right)\right)\\
&\quad - 4e^{-4x}\left(2a^2x^2\left(\frac{9}{4a}+\frac{3}{2}\right)^2 - 2ax\left(\frac{1}{2}a + \frac{9}{4a} + \frac{9}{4}\right)\right)\\
&\quad + 6e^{-4x}\left(2a^2x^2\left(\frac{1}{a}+1\right)^2 - 2ax\left(\frac{1}{3}a + \frac{2}{3a} + 1\right)\right).
\end{aligned}$$

An immediate consequence of the last remark is

Corollary 1. *Let $f, f'' \in C^*[0,\infty)$. Then, the inequality*

$$\lim_{n\to\infty} n\left[R_n^*(f,x) - f(x)\right] = -axf'(x) + \frac{x}{2}f''(x)$$

holds for any $x \in [0,\infty)$.

5. Comparison with Classical Szász–Mirakyan Operators

In this section, we compare the operators R_n^* with classical Szász–Mirakyan operators. The results obtained in this section show that the new operators present a better approximation under certain conditions, such as generalized convexity. A function $f \in C[0,\infty)$ is said to be strictly $(1,\varphi)$ convex if

$$\begin{vmatrix} 1 & 1 & 1 \\ \varphi(x_0) & \varphi(x_1) & \varphi(x_2) \\ f(x_0) & f(x_1) & f(x_2) \end{vmatrix} > 0, \qquad 0 \le x_0 < x_1 < x_2 < \infty.$$

This is equivalent to $f \circ \varphi^{-1}$ being strictly convex in the classical sense. For this concept cf. Ziegler [20] (his remark on p. 426 is important!), his earlier paper [19], and the very instructive thesis of Bessenyei [5], see Th. 2.7, p. 34, in particular.

A function $f \in C^2[0,\infty)$ (the space of twice continuously differentiable functions) is strictly $(1,\varphi)$ convex with respect to $\varphi(x) = e^{2ax}$, $a > 0$, if and only if

$$f''(x) > 2af'(x), \quad x > 0. \tag{5.1}$$

This follows immediately from the definition in the limiting case $x_0 = x_1 = x_2 = x$.

By Corollary 1 and (5.1), we have the following

Corollary 2. *If the function $f \in C^2[0,\infty)$ is strictly $(1,\varphi)$ convex with respect to $\varphi(x) = e^{2ax}$, $a > 0$, then for all $x \geq 0$, there exists $n_0 = n_0(x) \in \mathbb{N}$, such that for $n \geq n_0$, there holds:*

$$f(x) \leq R_n^*(f,x).$$

On the other hand, we recall the following theorem of Cheney and Sharma [8] (see also Stancu [17]).

Theorem 5. 1. *If $f \in C[0,\infty)$ is convex, then $f(x) \leq S_n(f,x) \quad x \geq 0$.*
2. *If $f \in C[0,\infty)$ is convex, then $S_{n+1}(f,x) \leq S_n(f,x) \quad x \geq 0,\ n \geq 1$.*
3. *If f is decreasing (increasing), then $S_n(f)$ is decreasing (increasing).*

Theorem 6. *Let $f \in C[0,\infty)$ be decreasing and convex. Then, for each $x \geq 0$, there exists $n_1 = n_1(x) \in \mathbb{N}$, such that for $n \geq n_1$, the inequalities*

$$R_n^*(f,x) \geq R_{n+1}^*(f,x) \geq f(x)$$

hold.

Proof. Using 2. and 3. of Theorem 5 and recalling that

$$\varphi_n(x) := \left(S_n\left(e^{2at}\right)\right)^{-1} \circ e^{2ax}$$

(see 1.5), one has $\varphi_n(x) \leq \varphi_{n+1}(x)$ and hence

$$\begin{aligned} R_n^*(f,x) - R_{n+1}^*(f,x) &= S_n(f,\varphi_n(x)) - S_{n+1}(f,\varphi_{n+1}(x)) \\ &= [S_n(f,\varphi_n(x)) - S_{n+1}(f,\varphi_n(x))] \\ &\quad + [S_{n+1}(f,\varphi_n(x)) - S_{n+1}(f,\varphi_{n+1}(x))] \\ &\geq 0. \end{aligned}$$

This means

$$R_n^*(f,x) \geq R_{n+1}^*(f,x), \quad x \geq 0.$$

Since f is decreasing and convex, by Corollary 1, there exists $n_1 = n_1(x) \in \mathbb{N}$, such that for $n \geq n_1$, we get

$$R_{n+1}^*(f;x) \geq f(x).$$

This completes the proof. □

Theorem 7. *Let $f \in C[0,\infty)$ be increasing and strictly $(1,\varphi)$ convex with respect to $\varphi(x) = e^{2ax}$, $a > 0$. Then*

$$f(x) \leq R_n^*(f,x) \leq S_n(f,x), \qquad x \geq 0.$$

Proof. From the remark of Ziegler in [20, p. 426], we know that

$$f(x) \leq R_n^*(f,x), \qquad x \geq 0, n \geq 1,$$

because the function f is $(1,\varphi)$ convex with respect to $\varphi(x) = e^{2ax}$, $a > 0$. Since $\varphi(x) = e^{2ax}$ is convex and by Theorem 5, 1, we have

$$S_n(\varphi) \geq \varphi. \tag{5.2}$$

Since $(S_n(\varphi))^{-1}$ is increasing, using (5.2), we get

$$(S_n(\varphi))^{-1} \circ S_n(\varphi) \geq (S_n(\varphi))^{-1} \circ \varphi.$$

Thus

$$x \geq \left((S_n(\varphi))^{-1} \circ \varphi\right)(x),$$

and hence

$$R_n^*(f,x) \leq S_n(f,x).$$

□

6. One Further Uniform Estimate

In this section, employing a technique developed by Gonska [11] and Păltănea [15], we present another quantitative result for the uniform convergence of the operators R_n^* in terms of the first- and second-order moduli of smoothness. Very recently, Păltănea at al. [16] have obtained quantitative results on the degree of approximation using a suitable transformation which reduces the approximation problem on $[0,\infty)$ to that one on $[0,1]$. We will use a similar approach adopted to our situation.

The spaces $(C^*[0,\infty), \|\cdot\|_{[0,\infty)})$ and $(C[0,1], \|\cdot\|_{[0,1]})$ are isometrically isomorphic. Define

$$\psi(y) := e^{-y},\ y \in [0,\infty),$$

and let

$$T : C[0,1] \to C^*[0,\infty)$$

be given by

$$T(f)(y) = f^*(y) = f(\psi(y)),\ f \in C[0,1], y \in [0,\infty).$$

with the observation

$$\lim_{t\to\infty} f^*(t) = \lim_{t\to\infty} f(\psi(t)) = f(0).$$

Clearly, T is linear and bijective. Moreover, for all $f \in C[0,1]$, one has

$$\|Tf\|_{[0,\infty)} = \sup_{t\in[0,\infty)} |f(\psi(t))| = \|f\|_{[0,1]}.$$

Hence, T is isometric with

$$T^{-1}(f^*) = f^* \circ \psi^{-1},\ \text{for } f^* \in C^*[0,\infty).$$

We recall here a general quantitative result involving the first- and second-order moduli of smoothness. Such estimates were first established by Gonska (see [11]) and later refined by Păltănea as far as the constants are concerned. Păltănea's result (see [15, Corollary 2.2.1]) reads as follows.

Theorem 8. *Let $K = [a,b]$, $K' \subset K$ and for $i \in \mathbb{N} \cup \{0\}$, $x \in \mathbb{R}$, we consider the ith monomial $e_i(x) := x^i$. If $L : C(K) \to C(K')$ is a positive linear* *operator, then for* $f \in C(K)$, $x \in K'$, *and each* $0 < h \le \frac{1}{2}length(K)$, *the following holds:*

$$|L(f;x) - f(x)| \le |L(e_0;x) - 1|\,|f(x)| + \frac{1}{h}\,|L(e_1 - x;x)|\,\omega_1(f;h) \\ + \left[L(e_0;x) + \frac{1}{2h^2}L\left((e_1 - x)^2;x\right)\right]\omega_2(f;h),$$

where $\omega_1(f;h)$ *and* $\omega_2(f;h)$ *are the first-order modulus of continuity and the second-order modulus of continuity is given by*

$$\omega_1(f;h) = \sup\{|f(x) - f(y)| : x, y \in [0,1], |x - y| \le h\},$$

$$\omega_2(f;h) = \sup\left\{\left|f(x) - 2f\left(\frac{x+y}{2}\right) + f(y)\right| : x, y \in [0,1], |x - y| \le 2h\right\},$$

respectively.

Remark 3. The condition $h \le \frac{1}{2}\text{length}(K)$ in the above can be eliminated for operators which preserve linear functions. For $K = K' = [0,1]$ and $Le_0 = e_0$, this implies

$$\|Lf - f\|_{[0,1]} \le \frac{1}{h}\,\|Le_1 - e_1\|_{[0,1]}\,\omega_1(f;h) \\ + \left[1 + \frac{1}{2h^2}\left(2\,\|Le_1 - e_1\|_{[0,\infty)} + \|Le_2 - e_2\|_{[0,1]}\right)\right]\omega_2(f;h)$$

for $0 < h \le \frac{1}{2}$.

The above uniform estimate follows from

$$|L(e_1 - x;x)| = |L(e_1;x) - xL(1;x)| \\ \le \|Le_1 - e_1\|_{[0,\infty)}$$

and

$$L\left((e_1 - x)^2;x\right) = L(e_2;x) - 2xL(e_1;x) + x^2L(e_0;x) \\ \le \|Le_2 - e_2\|_{[0,\infty)} + 2\,\|Le_1 - e_1\|_{[0,\infty)}.$$

Let $S^* : C^*[0,\infty) \to C^*[0,\infty)$ be a positive linear operator reproducing the constant functions. Then, $L : T^{-1} \circ S^* \circ T : C[0,1] \to C[0,1]$ is a positive linear operator to which the uniform version of P ältănea's theorem is applicable. This leads to the following.

Theorem 9. *If* S^* *and L are as given above, then for all* $f^* \in C^*[0,\infty)$ *and all* $0 < h \le \frac{1}{2}$, *the following inequality holds:*

$$\|S^*f^* - f^*\|_{[0,\infty)} \\ \le \frac{1}{h}\,\|S^*(\psi) - \psi\|_{[0,\infty)}\,\omega_1(f;h)_{[0,1]} \\ + \left[1 + \frac{1}{2h^2}\left(\left\|S^*(\psi^2) - \psi^2\right\|_{[0,\infty)} + 2\,\|S^*(\psi) - \psi\|_{[0,\infty)}\right)\right] \\ \times\omega_2(f;h)_{[0,1]}. \tag{6.1}$$

Here, $f = f^* \circ \psi^{-1}$.

Proof. For $L : T^{-1} \circ S^{*} \circ T$, the quantities from Remark 3 can be rewritten as follows:

(i).

$$\begin{aligned}\|Lf - f\|_{[0,1]} &= \left\|\left(T^{-1} \circ S^{*}\right)(f \circ \psi) - T^{-1}(f \circ \psi)\right\|_{[0,1]} \\ &= \left\|T^{-1}(S^{*}f^{*}) - T^{-1}(f^{*})\right\|_{[0,1]} \\ &= \left\|(S^{*}f^{*}) \circ \psi^{-1} - f^{*} \circ \psi^{-1}\right\|_{[0,1]} \\ &= \|S^{*}f^{*} - f^{*}\|_{[0,\infty)}.\end{aligned}$$

(ii). Using the fact $T^{-1}(f^{*}) = f^{*} \circ \psi^{-1} = f$, $T(f) = f^{*} = f \circ \psi$, we get

$$\begin{aligned}\|Le_1 - e_1\|_{[0,1]} &= \left\|\left(T^{-1} \circ S^{*} \circ T\right)(e_1) - e_1\right\|_{[0,1]} \\ &= \left\|\left(T^{-1} \circ S^{*}\right)(\psi) - e_1\right\|_{[0,1]} \\ &= \left\|S^{*}(\psi) \circ \psi^{-1} - \left(T^{-1} \circ T\right)(e_1)\right\|_{[0,1]} \\ &= \left\|S^{*}(\psi) \circ \psi^{-1} - T^{-1}(\psi)\right\|_{[0,1]} \\ &= \left\|S^{*}(\psi) \circ \psi^{-1} - \psi \circ \psi^{-1}\right\|_{[0,1]} \\ &= \|S^{*}(\psi) - \psi\|_{[0,\infty)}.\end{aligned}$$

(iii).

$$\begin{aligned}\|Le_2 - e_2\|_{[0,1]} &= \left\|\left(T^{-1} \circ S^{*}\right)(T(e_2)) - T^{-1}(T(e_2))\right\|_{[0,1]} \\ &= \left\|\left(T^{-1} \circ S^{*}\right)\left(\psi^2\right) - T^{-1}\left(\psi^2\right)\right\|_{[0,1]} \\ &= \left\|S^{*}\left(\psi^2\right) \circ \psi^{-1} - \psi^2 \circ \psi^{-1}\right\|_{[0,1]} \\ &= \left\|S^{*}\left(\psi^2\right) - \psi^2\right\|_{[0,\infty)}.\end{aligned}$$

Hence, we have

$$\begin{aligned}\|S^{*}f^{*} - f^{*}\|_{[0,\infty)} \leq{}& \frac{1}{h}\|S^{*}(\psi) - \psi\|_{[0,\infty)}\,\omega_1(f;h)_{[0,1]} \\ &+ \left[1 + \frac{1}{2h^2}\left(\left\|S^{*}\left(\psi^2\right) - \psi^2\right\|_{[0,\infty)} + 2\|S^{*}(\psi) - \psi\|_{[0,\infty)}\right)\right] \\ &\times \omega_2(f;h)_{[0,1]}.\end{aligned}$$

□

Remark 4. In the statement of the theorem, the quantities $\omega_1(f;h)_{[0,1]}$ and $\omega_2(f;h)_{[0,1]}$ may be rewritten as follows:

$$\omega_1(f;h)_{[0,1]} = \omega^{*}(f^{*};h)$$

from above. Moreover

$$\begin{aligned}\omega_2(f;h)_{[0,1]} &= \sup\left\{|f(x-s) - 2f(x) + f(x+s)| : x \pm s \in [0,1], |s| \leq h\right\} \\ &= \sup\left\{\left|\left(f^{*} \circ \psi^{-1}\right)(x) - 2\left(f^{*} \circ \psi^{-1}\right)\left(\frac{x+y}{2}\right)\right.\right. \\ &\quad \left.\left. + \left(f^{*} \circ \psi^{-1}\right)(y)\right| : |x-y| \leq 2h, h \leq 1/2\right\}\end{aligned}$$

$$= \sup \left\{ \left| f^*(t) - 2f^*\left(\ln 2 - \ln\left(e^{-t} + e^{-s}\right) \right) + f^*(s) \right| \right.$$
$$\left. : |\psi(t) - \psi(s)| \le 2h, h \le 1/2 \right\}.$$

Note that in the above, since $|x - y| \le 1$ and $|e^{-t} - e^{-s}| \le e^{-t} + e^{-s} \le 2$, then $\ln 2 - \ln(e^{-t} + e^{-s}) \ge 0$, so f^* is defined there.

From Theorem 9, choosing $h = \sqrt{\frac{1}{2}(\gamma_n + 2\beta_n)}$, there for n large enough, and recalling (3.2) and (3.3), we arrive at the following.

Corollary 3. *For all $f^* \in C^*[0, \infty)$ $(f = f^* \circ \psi^{-1})$ and n large enough, we have*

$$\|R_n^* f^* - f^*\|_{[0,\infty)}$$
$$\le \omega_1\left(f; \sqrt{\frac{1}{2}(\gamma_n + 2\beta_n)}\right)_{[0,1]} + 2\omega_2\left(f; \sqrt{\frac{1}{2}(\gamma_n + 2\beta_n)}\right)_{[0,1]}.$$

References

[1] Acar, T.: Asymptotic formulas for generalized Szász-Mirakyan operators. Appl. Math. Comput. **263**, 223–239 (2015)

[2] Acar, T., Aral, A., Raşa, I.: The new forms of Voronovskaya's theorem in weighted spaces. Positivity (2015). doi:10.1007/s11117-015-0338-4

[3] Aldaz, J.M., Render, H.: Optimality of generalized Bernstein operators. J. Approx. Theory **162**, 1407–1416 (2010)

[4] Aral, A., Inoan, D., Raşa, I.: On the generalized Szász–Mirakyan operators. Results Math. **65**, 441–452 (2014)

[5] Bessenyei, M.: Hermite-Hadamard-type inequalities for generalized convex functions, Dissertation, University of Debrecen (2004)

[6] Birou, M.: A note about some general King-type operators. Ann. Tiberiu Popoviciu Semin. Funct. Equ. Approx. Convexity **12**, 3–16 (2014)

[7] Boyanov, B.D., Veselinov, V.M.: A note on the approximation of functions in an infinite interval by linear positive operators. Bull. Math. Soc. Sci. Math. Roum **14**(62), 9–13 (1970). (no.1)

[8] Cheney, E.W., Sharma, A.: Bernstein power series. Can. J. Math. **16**, 241–252 (1964)

[9] Duman, O., Özarslan, M.A.: Szász-Mirakyan type operators providing a better error estimation. Appl. Math. Lett. **20**, 1184–1188 (2007)

[10] Favard, J.: Sur les multiplicateurs d'interpolation. J. Math. Pures Appl. **23**(9), 219–247 (1944)

[11] Gonska, H.: Quantitative Korovkin-type theorems on simultaneous approximation. Math. Z. **186**, 419–433 (1984)

[12] Holhoş, A.: The rate of approximation of functions in an infinite interval by positive linear operators. Studia Univ. "Babeş-Bolyai", Mathematica **55**(2), 133–142 (2010)

[13] King, J.P.: Positive linear operators which preserve x^2, Acta. Math. Hung. **99**, 203–208 (2003)

[14] Mirakyan, G.M.: Approximation of continuous functions with the aid of polynomials (Russian). Dokl. Akad. Nauk SSSR **31**, 201–205 (1941)

[15] Păltănea, R.: Optimal estimates with moduli of continuity. Result Math. **32**, 318–331 (1997)

[16] Păltănea, R., Smuc, M.: General estimates of the weighted approximation on interval $[0, \infty)$ using moduli of continuity. Bull. Transilv. Univ. Braşov Ser. III **8**(57), 93–108 (2015). (No.2)

[17] Stancu, D.D.: Approximation of functions by a new class of linear polynomial operators. Rev. Roum. Math. Pures Appl. **13**, 1173–1194 (1968)

[18] Szász, O.: Generalization of S. Bernsteins polynomials to the infinite interval. J. Res. Nat. Bur. Stand. **45**, 239–245 (1950)

[19] Ziegler, Z.: Generalized convexity cones. Pac. J. Math. **17**, 561–580 (1966)

[20] Ziegler, Z.: Linear approximation and generalized convexity. J. Approx. Theory **1**, 420–443 (1968)

Tuncer Acar and Ali Aral
Faculty of Science and Arts, Department of Mathematics
Kirikkale University
Yahsihan, 71450 Kirikkale
Turkey
e-mail: tunceracar@ymail.com

Ali Aral
e-mail: aliaral73@yahoo.com

Heiner Gonska
Faculty of Mathematics
University of Duisburg-Essen
Forsthausweg 2
47057 Duisburg
Germany
e-mail: heiner.gonska@uni-due.de

Received: July 10, 2016.
Revised: November 5, 2016.
Accepted: November 24, 2016.