Mediterr. J. Math. 13 (2016), 4763–4782
DOI 10.1007/s00009-016-0774-9
1660-5446/16/064763-20
published online July 20, 2016

Mediterranean Journal of Mathematics

Spectral Analysis for a Singular Differential System with Integral Boundary Conditions

Fenglong Sun, Lishan Liu, Xinguang Zhang and Yonghong Wu

Abstract. In this paper, by constructing a cone $K_1 \times K_2$ in the Cartesian product space $C[0,1] \times C[0,1]$, and using spectral analysis of the relevant linear operator for the corresponding differential system, some properties of the first eigenvalue corresponding to the relevant linear operator are obtained, and the fixed-point index of nonlinear operator in the $K_1 \times K_2$ is calculated explicitly and the existence of at least one positive solution or two positive solutions of the singular differential system with integral boundary conditions is established.

Mathematics Subject Classification. 34B16, 34B18, 34B40.

Keywords. Multiple positive solutions, singular differential system, spectral analysis, fixed-point index, cone.

1. Introduction

Integral boundary value problems arise in different areas of applied mathematics and physics such as heat conduction, chemical engineering, underground water flow, thermo-elasticity, and plasma physics. In mathematics context, these phenomena can be reduced to some model for the nonlocal problems with integral boundary conditions. On the other hand, integral boundary conditions can cover other kinds of nonlocal boundary conditions, such as three-point boundary conditions and multi-point boundary conditions (see [1–7]), as special cases. Hence, boundary value problems with integral boundary conditions constitute a very interesting and important class of problems and have received a great deal of attention, see [8–20] and the references therein.

In [2], using the fixed-point theorem in cones, Ma and Wang studied the existence of at least one positive solution for the following three-point non-singular BVP

$$\begin{cases} u''(t) + a(t)u'(t) + b(t)u(t) + h(t)f(u) = 0, & t \in (0,1), \\ u(0) = 0, & \alpha u(\eta) = u(1), \end{cases}$$

where $0 < \eta < 1$, $0 < \alpha\varphi_1(\eta) < 1$ (φ_1 will be given in Sect. 2), $h \in C([0,1],[0,+\infty))$ and $h(t) \not\equiv 0$, $f \in C([0,+\infty),[0,+\infty))$ is either superlinear or sublinear. Recently, by applying the fixed-point index theorems, Liu et al. [17] studied the existence of positive solutions for the following singular second-order integral boundary value problem under some weaker conditions concerning the first eigenvalue corresponding to the relevant linear operator

$$\begin{cases} u''(t) + a(t)u'(t) + b(t)u(t) + h(t)f(u) = 0, & t \in (0,1), \\ u(0) = \int_0^1 g(s)u(s)\mathrm{d}s, & u(1) = \int_0^1 h(s)u(s)\mathrm{d}s, \end{cases}$$

where $g, h \in L^1(0,1)$ are nonnegative, $h(t) \not\equiv 0$ is allowed to be singular at $t = 0, 1$ and $f \in C((0,+\infty),[0,+\infty))$ may be singular at $u = 0$.

To study the case of systems of equation using the fixed-point theorem in cones, Cheng [21,22] established a product formula for computing the fixed-point index of the system of equations, especially inhomogeneous system of equations with different nonlinear features can be solved by this method. Following this strategy, Liu et al. [23] established the conditions for the existence of at least one or at least two positive solutions for the following singular impulsive BVP

$$\begin{cases} -u''(t) = h_1(t)f_1(t, u(t), v(t)), & t \in J' \\ -v''(t) = h_2(t)f_2(t, v(t), u(t)), & t \in J' \\ -\Delta u'|_{t=t_k} = I_{1,k}(u(t_k)), & k = 1, 2, \ldots, m, \\ -\Delta v'|_{t=t_k} = I_{2,k}(v(t_k)), & k = 1, 2, \ldots, m, \\ \alpha u(0) - \beta u'(0) = 0,\ \alpha v(0) - \beta v'(0) = 0, & \\ \gamma u(1) + \delta u'(1) = 0,\ \gamma v(1) + \delta v'(1) = 0, & \end{cases}$$

with various kinds of nonlinear feature of f_1, f_2.

Motivated by the work mentioned above, in this paper, we are concerned with the multiplicity of positive solutions for the following system of singular differential equations

$$\begin{cases} u''(t) + a_1(t)u'(t) + b_1(t)u(t) + c_1(t)f_1(t, u(t), v(t)) = 0, & t \in (0,1), \\ v''(t) + a_2(t)v'(t) + b_2(t)v(t) + c_2(t)f_2(t, v(t), u(t)) = 0, & t \in (0,1), \\ u(0) = \int_0^1 g_1(s)u(s)\mathrm{d}s, & u(1) = \int_0^1 h_1(s)u(s)\mathrm{d}s, \\ v(0) = \int_0^1 g_2(s)v(s)\mathrm{d}s, & v(1) = \int_0^1 h_2(s)v(s)\mathrm{d}s, \end{cases} \tag{1.1}$$

where $a_i \in C([0,1],\mathbb{R})$, $b_i \in C([0,1],(-\infty,0))$, $f_i \in C([0,1] \times (0,+\infty) \times (0,+\infty),\ [0,+\infty))$, $c_i \in C((0,1),[0,+\infty))$ and $g_i, h_i \in L^1[0,1]$ are nonnegative for $i = 1, 2$. In this paper, $c_i(t) \not\equiv 0$ is allowed to be singular at $t = 0, 1$ and $f_i(t,x,y)$ may be singular at $x = 0$ or $y = 0$. We are mainly interested in handling the singularity of f_1, f_2 on second and third variables, to overcome this difficulty, we shall do spectral analysis for the relevant linear operator of the corresponding differential system, and then construct a Cartesian product cone $K \times K$, and compute the fixed-point index in $K \times K$ under some conditions on f_i concerning the first eigenvalue corresponding to the relevant linear operator. Based on the properties of the fixed-point index, the existence of at least one or at least two positive solutions for the singular differential system (1.1) is established.

The paper is organized as follows. In Sect. 2, we give some preliminaries and establish several lemmas. In Sect. 3, the main results are formulated and proved. In Sect. 4, we give two examples.

2. Preliminaries and Lemmas

In this section, we present some preliminaries and lemmas that are useful to the proof of our main results.

Let $E = C([0,1]) = \{u \mid u : [0,1] \to \mathbb{R}$ is continuous$\}$ be a Banach space with the norm $\|u\| = \max_{t\in[0,1]} |u(t)|$ and $P = \{u \in C([0,1],[0,+\infty)) \mid u(t) \geq 0,\ t \in [0,1]\}$ be a cone in E. Clearly, $E \times E$ is also a Banach space with norm $\|(u,v)\| = \max\{\|u\|,\|v\|\}$ for any $(u,v) \in E \times E$. A function $u \in C([0,1]) \cap C^2((0,1))$ is said to be a positive solution of BVP (1.1) if it satisfies the BVP (1.1) and $u(t) > 0, v(t) > 0$ for $t \in (0,1)$.

Lemma 2.1 [2]. *Assume that $a_i \in C([0,1]), b_i \in C([0,1],(-\infty,0))$. Let $\varphi_{1,i}$ and $\varphi_{2,i}$ be the unique solution of BVP*

$$\begin{cases} \varphi''_{1,i}(t) + a_i(t)\varphi'_{1,i}(t) + b_i(t)\varphi_{1,i}(t) = 0, \\ \varphi_{1,i}(0) = 0, \quad \varphi_{1,i}(1) = 1, \end{cases}$$

and

$$\begin{cases} \varphi''_{2,i}(t) + a_i(t)\varphi'_{2,i}(t) + b_i(t)\varphi_{2,i}(t) = 0, \\ \varphi_{2,i}(0) = 1, \quad \varphi_{2,i}(1) = 0, \end{cases}$$

respectively. Then $\varphi_{1,i}$ is strictly increasing on $[0,1]$, while $\varphi_{2,i}$ is strictly decreasing on $[0,1]$ $(i = 1,2)$.

For convenience in presentation, we now list some assumptions and lemmas which are used throughout the paper.

(H_1) $a_i \in C([0,1]),\ b_i \in C([0,1],(-\infty,0)),\ i = 1,2;$
(H_2) $g_i, h_i \in L^1([0,1])$ are nonnegative, and $k_{1,i} > 0, k_{4,i} > 0, k_i > 0$, where

$$\begin{array}{ll} k_{1,i} = 1 - \int_0^1 \varphi_{2,i}(s)g_i(s)\mathrm{d}s, & k_{2,i} = \int_0^1 \varphi_{1,i}(s)g_i(s)\mathrm{d}s, \\ k_{3,i} = \int_0^1 \varphi_{2,i}(s)h_i(s)\mathrm{d}s, & k_{4,i} = 1 - \int_0^1 \varphi_{1,i}(s)h_i(s)\mathrm{d}s, \\ k_i = k_{1,i}k_{4,i} - k_{2,i}k_{3,i}, & i = 1,2. \end{array}$$

Lemma 2.2 [17]. *Assume that (H_1) and (H_2) hold. Then for any $y \in C((0,1)) \cap L^1((0,1))$, $i = 1,2$, the BVP*

$$\begin{cases} u''_i(t) + a_i(t)u'(t) + b_i(t)u(t) + y(t) = 0, \quad t \in (0,1), \\ u_i(0) = \int_0^1 g_i(s)u_i(s)\mathrm{d}s, \quad u_i(1) = \int_0^1 h_i(s)u_i(s)\mathrm{d}s, \end{cases} \tag{2.1}$$

has a unique solution u_i that can be expressed in the form

$$u_i(t) = \int_0^1 H_i(t,s)y(s)\mathrm{d}s, \quad t \in [0,1], \tag{2.2}$$

where

$$H_i(t,s) = G_i(t,s)p_i(s) + \frac{\varphi_{1,i}(t)k_{3,i} + \varphi_{2,i}(t)k_{4,i}}{k_i}\int_0^1 G_i(\tau,s)p_i(s)g_i(\tau)\mathrm{d}\tau$$
$$+ \frac{\varphi_{1,i}(t)k_{1,i} + \varphi_{2,i}(t)k_{2,i}}{k_i}\int_0^1 G_i(\tau,s)p_i(s)h_i(\tau)\mathrm{d}\tau,$$
$$p_i(t) = \exp\left(\int_0^t a_i(s)\mathrm{d}s\right),$$
$$G_i(t,s) = \frac{1}{\rho_i}\begin{cases}\varphi_{1,i}(t)\varphi_{2,i}(s), & 0 \le t \le s \le 1,\\ \varphi_{1,i}(s)\varphi_{2,i}(t), & 0 \le s \le t \le 1,\end{cases} \quad \rho_i = \varphi'_{1,i}(0). \tag{2.3}$$

Furthermore, $u_i(t) \ge 0$ *on* $[0,1]$ *provided that* $y_i(t) \ge 0$ *on* $(0,1)$.

Lemma 2.3 [17]. *Suppose that* (H_1) *and* (H_2) *hold, then for any* $t, s \in [0,1]$, $i = 1,2$, *we have*

$$0 \le G_i(t,s) \le G_i(s,s),\ 0 \le H_i(t,s) \le \mathscr{H}_i(s), \tag{2.4}$$
$$H_i(t,s) \ge \gamma_i(t)\mathscr{H}_i(s), \tag{2.5}$$

where $\gamma_i(t) = \min\{\phi_{1,i}(t), \phi_{2,i}(t)\}$, $t \in [0,1]$, *and*

$$\mathscr{H}_i(s) = G_i(s,s)p_i(s) + \frac{k_{3,i}+k_{4,i}}{k_i}\int_0^1 G_i(\tau,s)p_i(s)g_i(\tau)\mathrm{d}\tau$$
$$+ \frac{k_{1,i}+k_{2,i}}{k_i}\int_0^1 G_i(\tau,s)p_i(s)h_i(\tau)\mathrm{d}\tau.$$

Since $c_i \in C((0,1),[0,+\infty))$ and $c_i(t) \not\equiv 0$, there exists $t_{0,i} \in (0,1)$ such that $c_i(t_{0,i}) > 0$, $i = 1,2$. Choose $\delta \in (0, \frac{1}{2})$ such that $t_{0,i} \in (\delta, 1-\delta)$, then we have

$$H_i(t,s) \ge \gamma_\delta \mathscr{H}_i(s), \quad t \in [\delta, 1-\delta], s \in [0,1],$$

where

$$0 < \gamma_\delta = \min_{i\in\{1,2\}} \min_{t\in[\delta,1-\delta]} \{\phi_{1,i}(t), \phi_{2,i}(t)\} = \min_{i\in\{1,2\}} \min\{\phi_{1,i}(\delta), \phi_{2,i}(1-\delta)\} < 1.$$

Let

$$K = \{u \in P \mid u(t) \ge \gamma(t)\|u\|, t \in [0,1]\},$$

where $\gamma(t) = \min\{\gamma_1(t), \gamma_2(t)\}$. Then K is a subcone of P. It is easy to verify that for any $u \in K$, we have $\min_{t\in[\delta,1-\delta]} u(t) \ge \gamma_\delta\|u\|$. For any $r > 0$, let $K_r = \{u \in K \mid \|u\| < r\}, \partial K_r = \{u \in K \mid \|u\| = r\}$ and $\bar{K}_r = \{u \in K \mid \|u\| \le r\}$.

To deal with the singularity of c_i and f_i, we list here two more assumptions:

(H_3) $c_i \in C((0,1),[0,+\infty))$, $c_i(t) \not\equiv 0$ and $\int_0^1 \mathscr{H}_i(s)c_i(s)\mathrm{d}s < +\infty$, $i = 1,2$.

(H_4) $c_i \in C((0,1),[0,+\infty))$, $c_i(t) \not\equiv 0$, $f_i \in C([0,1]\times(0,\infty)\times(0,\infty),[0,+\infty))$ and for any $0 < r_i < R_i < +\infty$, $i = 1,2$,

$$\lim_{n\to+\infty} \sup_{(u,v)\in \bar{K}_{R_1}\setminus K_{r_1}\times \bar{K}_{R_2}\setminus K_{r_2}} \int_{e(n)} \mathscr{H}_i(s)c_i(s)f_i(s,u(s),v(s))\mathrm{d}s = 0,$$

where $e(n) = [0, \frac{1}{n}] \cup [\frac{n-1}{n}, 1]$.

It is easy to see that (H_4) is strong enough to imply (H_3). In fact, from (H_4) it follows that for any $0 < r < R < +\infty$ by taking $u(t) \equiv v(t) \equiv R \in \bar{K}_R \backslash K_r$, we have

$$\lim_{n\to+\infty} \int_{e(n)} \mathscr{H}_i(s)c_i(s)f_i(s,R,R)\mathrm{d}s = 0,$$

then $\int_{e(n)} \mathscr{H}_i(s)c_i(s)f_i(s,R,R)\mathrm{d}s < +\infty$. Since $f_i \in C([0,1],(0,+\infty),(0,+\infty),\mathbb{R}^+)$ and $c_i \in C[\frac{1}{n},\frac{n-1}{n}]$, we have

$$\int_0^1 \mathscr{H}_i(s)c_i(s)\mathrm{d}s < +\infty,$$

i.e., that (H_4) implies (H_3).

For any $(u,v) \in (K\backslash\{0\}) \times (K\backslash\{0\})$, we can define mappings $A_v : K\backslash\{0\} \to P$, $B_u : K\backslash\{0\} \to P$ and $T : K\backslash\{0\} \times K\backslash\{0\} \to P \times P$ as follows

$$A_v(u)(t) = \int_0^1 H_1(t,s)c_1(s)f_1(s,u(s),v(s))\mathrm{d}s, \tag{2.6}$$

$$B_u(v)(t) = \int_0^1 H_2(t,s)c_2(s)f_2(s,v(s),u(s))\mathrm{d}s, \tag{2.7}$$

$$T(u,v)(t) = (A_v(u)(t), B_u(v)(t)), \quad t \in [0,1]. \tag{2.8}$$

Also we can define mapping $T_i : E \to E$ as

$$(T_i u)(t) = \int_0^1 H_i(t,s)c_i(s)u(s)\mathrm{d}s, \quad t \in [0,1],\ i = 1,2. \tag{2.9}$$

It is well known that if (u,v) solves the operator equation $(u,v) = T(u,v)$, then (u,v) is a positive solution of system (1.1).

For any $\tau : 0 < \tau < \delta$, we define $T_{\tau,i} : E \to E$:

$$(T_{\tau,i}u)(t) = \int_\tau^{1-\tau} H_i(t,s)c_i(s)u(s)\mathrm{d}s, \quad \text{for all } t \in [0,1],\ u \in E,\ i = 1,2\ldots \tag{2.10}$$

Lemma 2.4 [17]. *Suppose that (H_1)–(H_3) are satisfied, then for the operators T_i defined by (2.9) and $T_{\tau,i}$ defined by (2.10),*

(i) *$T_i : K \to K$ is completely continuous linear operators;*
(ii) *the spectral radius $r(T_i) \neq 0$, T_i has positive eigenfunction corresponding to its first eigenvalue $\lambda_{1,i} = (r(T_i))^{-1}$; and $T_{\tau,i}$ has positive eigenfunction corresponding to its first eigenvalue $\lambda_{\tau,i} = (r(T_{\tau,i}))^{-1}$;*
(iii) *there exists an eigenvalue $\widetilde{\lambda}_{1,i}$ of T_i such that $\lambda_{\tau,i} \to \widetilde{\lambda}_{1,i}$, as $\tau \to 0^+$.*

Lemma 2.5. *If $(H_1),(H_2)$ and (H_4) hold, then*

(i) *for any $R > r > 0$ and $v \in K\backslash\{0\}$, $A_v : \bar{K}_R\backslash K_r \to P$ is completely continuous;*
(ii) *for any $R > r > 0$ and $u \in K\backslash\{0\}$, $B_u : \bar{K}_R\backslash K_r \to P$ is completely continuous;*
(iii) *$T : (K\backslash\{0\}) \times (K\backslash\{0\}) \to P \times P$ is completely continuous.*

Proof. (i) First, for any $r>0$, and $v\in K\backslash\{0\}$, we will show

$$\sup_{u\in\partial K_r}\int_0^1 \mathscr{H}_1(s)c_1(s)f_1(s,u(s),v(s))\mathrm{d}s<+\infty. \tag{2.11}$$

At the same time, this implies $A_v: K\backslash\{0\}\to P$ is well defined.

In fact, by (H_4), for any fixed $v\in K\backslash\{0\}$, there exists a natural number l such that

$$\sup_{u\in\partial K_r}\int_{e(l)} \mathscr{H}_1(s)c_1(s)f_1(s,u(s),v(s))\mathrm{d}s<1.$$

If $u\in\partial K_r$, then $\gamma_l r\le u(t)\le r$ for $t\in[\frac{1}{l},\frac{l-1}{l}]$, where $\gamma_l=\min_{t\in[\frac{1}{l},\frac{l-1}{l}]}\{\phi_{1,1}(t),\phi_{2,1}(t)\}>0$. Let $M_l=\max\{f_1(t,u(t),v(t))\,|\,t\in[\frac{1}{l},\frac{l-1}{l}],\ u\in\partial K_r\}$, then we have

$$\begin{aligned}
&\sup_{u\in\partial K_r}\int_0^1 \mathscr{H}_1(s)c_1(s)f_1(s,u(s),v(s))\mathrm{d}s\\
&\quad\le \sup_{u\in\partial K_r}\int_{e(l)} \mathscr{H}_1(s)c_1(s)f_1(s,u(s),v(s))\mathrm{d}s\\
&\qquad+\sup_{u\in\partial K_r}\int_{\frac{1}{l}}^{\frac{l-1}{l}} \mathscr{H}_1(s)c_1(s)f_1(s,u(s),v(s))\mathrm{d}s\\
&\quad\le 1+M_l\int_0^1 \mathscr{H}_1 c_1(s)\mathrm{d}s<+\infty,
\end{aligned} \tag{2.12}$$

i.e., (2.11) holds.

Next, for any $v\in K\backslash\{0\}$, $A_v:\bar{K}_R\backslash K_r\to P$ is continuous. Let $u_n,u_0\in\bar{K}_R\backslash K_r$ and $\|u_n-u_0\|\to 0(n\to\infty)$. For any $\epsilon>0$, by (H_4) there exists a natural number $m>0$ such that

$$\sup_{u\in\bar{K}_R\backslash K_r}\int_{e(m)} \mathscr{H}_1(s)c_1(s)f_1(s,u(s),v(s))\mathrm{d}s<\frac{\epsilon}{4}. \tag{2.13}$$

Set $\gamma_m=\min_{t\in[\frac{1}{m},\frac{m-1}{m}]}\{\phi_{1,1}(t),\phi_{2,1}(t)\}$, then $\gamma_m r\le u_0(t)\le R, \gamma_m r\le u_n(t)\le R, t\in[\frac{1}{m},\frac{m-1}{m}]$. Let $a_1=\min\{v(t)\,|\,t\in[\frac{1}{m},\frac{m-1}{m}]\}$, $a_2=\max\{v(t)\,|\,t\in[\frac{1}{m},\frac{m-1}{m}]\}$, since $f_1(t,x,y)$ is uniformly continuous w.r.t. x on $[\frac{1}{m},\frac{m-1}{m}]\times[\gamma_m r,R]\times[a_1,a_2]$, we have

$$\lim_{n\to+\infty}|f_1(t,u_n(t),v(t))-f_1(t,u_0(t),v(t))|=0$$

holds uniformly on $t\in[\frac{1}{m},\frac{m-1}{m}]$. Then the Lebesgue-dominated convergence theorem yields that

$$\lim_{n\to+\infty}\int_{\frac{1}{m}}^{\frac{m-1}{m}} \mathscr{H}_1(s)c_1(s)|f_1(s,u_n(s),v(s)-f_1(s,u_0(s),v(s))|\mathrm{d}s\to 0.$$

Thus, for the above $\epsilon>0$, there exists a natural number N such that for $n>N$, we have

$$\int_{\frac{1}{m}}^{\frac{m-1}{m}} \mathscr{H}_1(s)c_1(s)|f_1(s,u_n(s),v(s)-f_1(s,u_0(s),v(s))|\mathrm{d}s<\frac{\epsilon}{2}. \tag{2.14}$$

It follows from (2.4), (2.13) and (2.14) that when $n > N$,

$$\|A_v u_n - A_v u_0\| \le 2 \sup_{u\in \bar{K}_R\setminus K_r} \int_{e(m)} \mathscr{H}_1(s)c_1(s)f_1(s,u(s),v(s))\mathrm{d}s$$
$$+ \int_{\frac{1}{m}}^{\frac{m-1}{m}} \mathscr{H}_1(s)c_1(s)|f_1(s,u_n(s),v(s)-f_1(s,u_0(s),v(s))|\mathrm{d}s$$
$$< 2\times \frac{\epsilon}{4} + \frac{\epsilon}{2} = \epsilon.$$

Therefore, $A_v : \bar{K}_R\setminus K_r \to P$ is continuous.

Assume that $B \subset \bar{K}_R\setminus K_r$ is a bounded set. Then for any $u \in B$, $r \le \|u\| \le R$, from (H_4) and from (2.12), $A_v(B)$ is uniformly bounded. By the Arzela–Ascoli Theorem, we only need to show $A_v(B)$ is equicontinuous.

For any $\epsilon > 0$, from (H_4), there exists a natural number k such that

$$\sup_{u\in \bar{K}_R\setminus K_r} \int_{e(k)} \mathscr{H}_1(s)c_1(s)f_1(s,u(s),v(s))\mathrm{d}s < \frac{\epsilon}{4}.$$

Let $\gamma_k = \min_{t\in[\frac{1}{k},\frac{k-1}{k}]}\{\phi_{1,1}(t),\phi_{2,1}(t)\}$, $M_k = \max\{f_1(t,u(t),v(t)) \mid t \in [\frac{1}{k},\frac{k-1}{k}],\ u \in \bar{K}_R\setminus K_r\}$. Since $G_1(t,s)$ is uniformly continuous on $[0,1]\times[0,1]$, $\phi_{1,1}$ and $\phi_{2,1}$ are uniformly continuous on $[0,1]$, then for the above $\epsilon > 0$ and fixed $s \in [\frac{1}{k},\frac{k-1}{k}]$, there exists $\sigma > 0$, for all $t,t' \in [0,1]$, $|t-t'| < \sigma$, such that

$$|G_1(t,s) - G_1(t',s)| \le \left(6M_k \int_0^1 p_1(s)c_1(s)\mathrm{d}s\right)^{-1} \epsilon,$$
$$|\phi_{1,1}(t) - \phi_{1,1}(t')|$$
$$\le \left(6M_k \int_0^1\int_0^1 G_1(s,\tau)p_1(\tau)[k_{3,1}g_1(\tau)+k_{1,1}h_1(\tau)]c_1(s)\mathrm{d}\tau\mathrm{d}s\right)^{-1} k_1\epsilon,$$
$$|\phi_{2,1}(t) - \phi_{2,1}(t')|$$
$$\le \left(6M_k \int_0^1\int_0^1 G_1(s,\tau)p_1(\tau)[k_{4,1}g_1(\tau)+k_{2,1}h_1(\tau)]c_1(s)\mathrm{d}\tau\mathrm{d}s\right)^{-1} k_1\epsilon.$$

Hence, for all $t,t' \in [0,1]$, $|t-t'| < \sigma$ and $u \in B$, we have

$$|A_v u(t) - A_v u(t')| \le 2 \sup_{u\in \bar{K}_R\setminus K_r} \int_{e(k)} \mathscr{H}_1(s)c_1(s)f_1(s,u(s),v(s))\mathrm{d}s$$
$$+ \sup_{u\in \bar{K}_R\setminus K_r} \int_{\frac{1}{k}}^{\frac{k-1}{k}} |G_1(t,s)-G_1(t',s)|p_1(s)c_1(s)f_1(s,u(s),v(s))\mathrm{d}s$$
$$+ \sup_{u\in \bar{K}_R\setminus K_r} \int_{\frac{1}{k}}^{\frac{k-1}{k}} k_1^{-1}|\phi_{1,1}(t)-\phi_{1,1}(t')| \int_0^1 G_1(s,\tau)p_1(\tau)[k_{3,1}g_1(\tau)$$
$$+ k_{1,1}h_1(\tau)]c_1(s)f_1(s,u(s),v(s))\mathrm{d}\tau\mathrm{d}s$$
$$+ \sup_{u\in \bar{K}_R\setminus K_r} \int_{\frac{1}{k}}^{\frac{k-1}{k}} k_1^{-1}|\phi_{2,1}(t)-\phi_{2,1}(t')| \int_0^1 G_1(s,\tau)p_1(\tau)[k_{4,1}g_1(\tau)$$

$$+ k_{2,1}h_1(\tau)]c_1(s)f_1(s,u(s),v(s))\mathrm{d}\tau\mathrm{d}s$$
$$< 2 \times \frac{\epsilon}{4} + \frac{\epsilon}{6} + \frac{\epsilon}{6} + \frac{\epsilon}{6} = \epsilon.$$

Therefore, $A_v(B)$ is equicontinuous. (ii) In the similar way we can get the proof. Then we can get (iii). □

Lemma 2.6. *If* (H_1), (H_2) *and* (H_4) *hold, then*

(i) *for any* $R > r > 0$ *and* $v \in K\backslash\{0\}$, $A_v(\bar{K}_R\backslash K_r) \subset K$;
(ii) *for any* $R > r > 0$ *and* $u \in K\backslash\{0\}$, $B_u(\bar{K}_R\backslash K_r) \subset K$;
(iii) $T((K\backslash\{0\}) \times (K\backslash\{0\})) \subset K \times K$.

Proof. We only prove (i). For any $u \in \bar{K}_R\backslash K_r$, $t \in [0,1]$, by (2.4) we have

$$(A_v u)(t) = \int_0^1 H_1(t,s)c_1(s)f_1(s,u(s),v(s))\mathrm{d}s \leq \int_0^1 \mathscr{H}_1(s)c_1(s)f_1(s,u(s),v(s))\mathrm{d}s,$$

hence

$$\|A_v u\| \leq \int_0^1 \mathscr{H}_1(s)c_1(s)f_1(s,u(s),v(s))\mathrm{d}s.$$

On the other hand, by (2.5) we have

$$(A_v u)(t) = \int_0^1 H_1(t,s)c_1(s)f_1(s,u(s),v(s))\mathrm{d}s \geq \int_0^1 \gamma(t)\mathscr{H}_1(s)c_1(s)f_1(s,u(s),v(s))\mathrm{d}s \geq \gamma(t)\|A_v u\|.$$

Hence, $A_v(\bar{K}_R\backslash K_r) \subset K$. □

Lemma 2.7 [24]. *Let* $T : K \to k$ *be a completely continuous mapping. If there exists* $u_0 \in K\backslash\{0\}$ *such that*

$$u - Tu \neq \mu u_0, \quad u \in \partial K_r,\ \mu > 0,$$

then the fixed-point index $i(T,\ K_r,\ K) = 0$.

Lemma 2.8 [24]. *Let* $T : K \to K$ *be a completely continuous mapping and* $\mu Tu \neq u$ *for* $u \in \partial K_r$ *and* $0 < \mu \leq 1$. *Then* $i(T,\ K_r,\ K) = 1$.

Lemma 2.9 [21]. *Let* X *be a Banach space and let* $P_i \subset X$ *be a closed convex cone in and* W_i *a bounded open subset of* X *with boundary* $\partial W_i\,(i = 1, 2)$. *Suppose that* $A_i : P_i \cap \overline{W_i} \to P_i$ *is a completely continuous mapping and that* $A_i u_i \neq u_i$, $\forall u_i \in P_i \cap \partial W_i$, *then*

$$i(A,\ P_1 \times P_2 \cap (W_1 \times W_2),\ P_1 \times P_2) = i(A_1,\ P_1 \cap W_1,\ P_1) \cdot i(A_2,\ P_2 \cap W_2,\ P_2),$$

where $A(u,v) := (A_1 u,\ A_2 v)$, $\forall (u,v) \in (P_1 \times P_2) \cap \overline{(W_1 \times W_2)}$.

Lemma 2.10 [24]. *Let* P *be a cone in Banach space* X. *For* $r > 0$, *denote* $P_r = \{x \in P : \|x\| < r\}$, $\overline{P}_r = \{x \in P : \|x\| \leq r\}$ *and* $\partial P_r = \{x \in P : \|x\| = r\}$. *Suppose that* $A : \overline{P}_r \to P$ *is a completely continuous operator.*

(i) *If* $\|Au\| \leq \|u\|$ *for* $u \in \partial P_r$, *then the fixed-point index* $i(A, P_r, P) = 1$;
(ii) *If* $\|Au\| \geq \|u\|$ *for* $u \in \partial P_r$, *then the fixed-point index* $i(A, P_r, P) = 0$.

3. Main Results

Lemma 3.1. *If* (H_1), (H_2), (H_4) *and the following condition hold:*

(H_5) *There exist* $p > 0$, $\eta \geq 0$, $\bar{\lambda} \geq 0$ *such that for all* $0 < x \leq p$, $y > 0$ *and* $0 \leq t \leq 1$,

$$f_1(t, x, y) \leq \eta p, \quad \eta \int_0^1 \mathscr{H}_1(s) c_1(s) \mathrm{d}s < 1,$$

and for all $0 < \gamma_\delta p \leq x \leq p$, $y > 0$ *and* $t \in [\delta, 1 - \delta]$,

$$f_2(t, x, y) \geq \bar{\lambda} p, \quad \bar{\lambda} \int_\delta^{1-\delta} H_2\left(\frac{1}{2}, s\right) c_2(s) \mathrm{d}s > 1.$$

Then, for any $u, v \in K \backslash \{0\}$, *we have* $i(A_v, K_p, K) = 1$, $i(B_u, K_p, K) = 0$.

Proof. For any $v \in K\backslash\{0\}$ and $u \in K\backslash\{0\}$ with $\|u\| = p > 0$, we have $0 < u(s) \leq p$, $v(s) > 0$ for any $s \in (0, 1)$. So, by (2.4), (2.6) and (H_5), we have

$$\begin{aligned} \|A_v u\| &\leq \int_0^1 \mathscr{H}_1(s) c_1(s) f_1(s, u(s), v(s)) \mathrm{d}s \\ &\leq p\eta \int_0^1 \mathscr{H}_1(s) c_1(s) \mathrm{d}s < p = \|u\|; \end{aligned}$$

that is $\|A_v u\| < \|u\|$ for any $u \in \partial K_p$, $v \in K\backslash\{0\}$. Therefore, by Lemma 2.10, we obtain $i(A_v,\, K_p,\, K) = 1$.

For any $u \in K\backslash\{0\}$ and $v \in \partial K_p$, we have $\gamma_\delta p \leq v(s) \leq p$, $u(s) > 0$ for any $s \in [\delta, 1 - \delta]$. So, from (H_5) and (2.7), we obtain

$$\begin{aligned} B_u v\left(\frac{1}{2}\right) &= \int_0^1 H_2\left(\frac{1}{2}, s\right) c_2(s) f_2(s, v(s), u(s)) \mathrm{d}s \\ &\geq \int_\delta^{1-\delta} H_2\left(\frac{1}{2}, s\right) c_2(s) f_2(s, v(s), u(s)) \\ &\geq p\bar{\lambda} \int_\delta^{1-\delta} H_2\left(\frac{1}{2}, s\right) c_2(s) \mathrm{d}s \\ &> p = \|v\|; \end{aligned}$$

that is, $\|B_u v\| > \|v\|$ for any $v \in \partial K_p$, $u \in K\backslash\{0\}$. Therefore, by Lemma 2.10, we obtain $i(B_u,\, K_p,\, K) = 0$. □

In the same way, we can get the following three lemmas.

Lemma 3.2. *If* (H_1), (H_2), (H_4) *and the following condition hold:*

(H_5') *There exist* $p>0$ *and* $\eta_1,\eta_2\geq 0$, *such that for all* $0<x\leq p$, $y>0$ *and* $0\leq t\leq 1$,

$$f_1(t,x,y)\leq \eta_1 p,\quad \eta_1\int_0^1 \mathscr{H}_1(s)c_1(s)\mathrm{d}s<1,$$

and

$$f_2(t,x,y)\leq \eta_2 p,\quad \eta_2\int_0^1 \mathscr{H}_2(s)c_2(s)\mathrm{d}s<1.$$

Then, for any $u,v\in K\backslash\{0\}$, $i(A_v,K_p,K)=1$, $i(B_u,K_p,K)=1$.

Lemma 3.3. *If* (H_1), (H_2), (H_4) *and the following conditions hold:*

(H_5^*) *There exist* $p>0$, *and* $\bar{\lambda}_1,\bar{\lambda}_2\geq 0$ *such that for all* $\gamma_\delta p\leq x\leq p$, $y>0$ *and* $t\in[\delta,1-\delta]$,

$$f_1(t,x,y)\geq \bar{\lambda}_1 p,\quad \bar{\lambda}_1\int_\delta^{1-\delta} H_1\left(\frac{1}{2},s\right)c_1(s)\mathrm{d}s>1,$$

and

$$f_2(t,x,y)\geq \bar{\lambda}_2 p,\quad \bar{\lambda}_2\int_\delta^{1-\delta} H_2\left(\frac{1}{2},s\right)c_2(s)\mathrm{d}s>1.$$

Then, for any $u,v\in K\backslash\{0\}$, $i(A_v,K_p,K)=0$, $i(B_u,K_p,K)=0$.

For any $y>0$ and $i=1,2$, we denote

$$f_{i,0}(y)=\liminf_{x\to 0^+}\min_{t\in[0,1]}\frac{f_i(t,x,y)}{x},\quad f_{i,\infty}(y)=\liminf_{x\to+\infty}\min_{t\in[0,1]}\frac{f_i(t,x,y)}{x},$$
$$f_i^0(y)=\limsup_{x\to 0^+}\max_{t\in[0,1]}\frac{f_i(t,x,y)}{x},\quad f_i^\infty(y)=\limsup_{x\to+\infty}\max_{t\in[0,1]}\frac{f_i(t,x,y)}{x}.$$

Lemma 3.4. *Suppose the conditions* $(H_1),(H_2)$ *and* (H_4) *are satisfied.*

(i) *If*

$$\inf_{y\in(0,+\infty)} f_{1,0}(y)>\lambda_{1,1}, \tag{3.1}$$

then for any $v\in K\backslash\{0\}$, *there exists* $r_1>0$ *such that*

$$i(A_v,K_r,K)=0,\quad \forall\, 0<r<r_1.$$

(ii) *If*

$$\inf_{y\in(0,+\infty)} f_{2,0}(y)>\lambda_{1,2}, \tag{3.2}$$

then for any $u\in K\backslash\{0\}$, *there exists* $r_2>0$ *such that*

$$i(B_u,K_r,K)=0,\quad \forall\, 0<r<r_2.$$

Proof. We only prove (i). It follows from (3.1) that for any prescribed $y > 0$ there exists a corresponding $r_1 > 0$ such that $f_1(t, x, y) \geq \lambda_{1,1}x$, $0 < x \leq r_1, y > 0$, and thus for every $u \in \partial K_{r_1}$ and any fixed $v \in K\backslash\{0\}$, we have

$$\begin{aligned}(A_v u)(t) &= \int_0^1 H_1(t,s)c_1(s)f_1(s,u(s),v(s))\mathrm{d}s \\ &\geq \lambda_{1,1}\int_0^1 H_1(t,s)c_1(s)u(s)\mathrm{d}s \\ &= \lambda_{1,1}(T_1 u)(t), \quad t \in [0,1].\end{aligned} \tag{3.3}$$

Let φ^* be the positive eigenfunction of T_1 corresponding to $\lambda_{1,1}$, then $\varphi^* = \lambda_{1,1}T_1\varphi^*$. We may suppose that A_v has no fixed points on ∂K_{r_1} (otherwise, the proof is ended). Now we show that

$$u - A_v u \neq \mu\varphi^*, \quad u \in \partial K_{r_1}, \, \mu \geq 0. \tag{3.4}$$

Assume by contradiction that there exist $u_0 \in \partial K_{r_1}$ and $\mu_0 \geq 0$ such that $u_0 - A_v u_0 = \mu_0\varphi^*$, then $\mu_0 > 0$ and $u_0 = A_v u_0 + \mu_0\varphi^* \geq \mu_0\varphi^*$. Let $\bar{\mu} = \sup\{\mu : u_0 \geq \mu\varphi^*\}$, then $\bar{\mu} \geq \mu_0, u_0 \geq \bar{\mu}\varphi^*, \lambda_{1,1}T_1 u_0 \geq \lambda_{1,1}\bar{\mu}T_1\varphi^* = \bar{\mu}\varphi^*$. Therefore, by (3.3),

$$u_0 = A_v u_0 + \mu_0\varphi^* \geq \lambda_{1,1}T_1 u_0 + \mu_0\varphi^* \geq \bar{\mu}\varphi^* + \mu_0\varphi^* = (\bar{\mu} + \mu_0)\varphi^*,$$

which contradicts the definition of $\bar{\mu}$. So (3.4) is true and by Lemma 2.7 we have

$$i(A_v, K_{r_1}, K) = 0.$$

Then $i(A_v, K_r, K) = 0, \quad \forall\, 0 < r < r_1, \, v \in K\backslash\{0\}$. □

Remark 3.5. If $f_1(t, x, y)$ is singular at $x = 0$ or $f_2(t, x, y)$ is singular at $y = 0$, we have $f_{1,0}(y) = f_{2,0}(x) = +\infty$ for any $x, y > 0$, hence $\inf_{y\in(0,+\infty)} f_{1,0}(y) > \lambda_{1,1}$ or $\inf_{x\in(0,+\infty)} f_{2,0}(x) > \lambda_{1,2}$ holds automatically (see the assumptions (3.1) and (3.2) in Lemma 3.4). But if $f_1(t, x, y)$ is continuous at $x = 0$ and $f_1(t, 0, y) = 0$ for any $t \in [0, 1]$, $y > 0$, the assumption $\inf_{y\in(0,+\infty)} f_{1,0}(y) > \lambda_{1,1}$ may not hold, since the limitation $\liminf_{x\to 0^+} \min_{t\in[0,1]} \frac{f_1(t,x,y)}{x}$ is of $\frac{0}{0}$ type.

Lemma 3.6. *Suppose the conditions* $(H_1), (H_2)$ *and* (H_4) *are satisfied.*

(i) *If*

$$\sup_{y\in(0,+\infty)} f_1^0(y) < \lambda_{1,1}, \tag{3.5}$$

then for any $v \in K\backslash\{0\}$, *there exists* $r_3 > 0$ *such that*

$$i(A_v, K_r, K) = 1, \quad \forall\, 0 < r < r_3.$$

(ii) *If*

$$\sup_{y\in(0,+\infty)} f_2^0(y) < \lambda_{1,2}, \tag{3.6}$$

then for any $u \in K\backslash\{0\}$, *there exists* $r_4 > 0$ *such that*

$$i(B_u, K_r, K) = 1, \quad \forall\, 0 < r < r_4.$$

Proof. We only prove (i). It follows from (3.5) that for any prescribed $y > 0$ there exists a corresponding $r_3 > 0$ such that for all $0 < r < r_3$,

$$f_1(t,x,y) \le \lambda_{1,1} x, \quad 0 < x \le r_3,\ y > 0. \tag{3.7}$$

For any $u \in \partial K_r$, and any fixed $v \in K\backslash\{0\}$, it follows from (3.7) that

$$\begin{aligned}(A_v u)(t) &= \int_0^1 H_1(t,s)c_1(s)f_1(s,u(s),v(s))\mathrm{d}s \\ &\le \lambda_{1,1}\int_0^1 H_1(t,s)c_1(s)u(s)\mathrm{d}s \\ &= \lambda_{1,1}(T_1u)(t), \quad t\in[0,1],\end{aligned}$$

and hence $A_v u \le \lambda_{1,1}T_1u$, $u \in \partial K_r\, v \in K\backslash\{0\}$. We may suppose that A_v has no fixed point on ∂K_r (otherwise, the proof is finished). Now we show that $A_v u \ne \mu u$ for any $u \in \partial K_r$, $\mu \ge 1$. Assume, by contradiction, that there exist $\varphi \in \partial K_r$ and $\mu_2 \ge 1$ satisfying $A_v\varphi = \mu_2\varphi$. Then, $\mu_2 > 1$ and $\mu_2\varphi = A_v\varphi \le \lambda_{1,1}T_1\varphi$. By induction, we have $\mu_2^n\varphi \le \lambda_{1,1}^n T_1^n\varphi$ $(n = 1, 2, \ldots)$. Thus,

$$\|T_1^n\| \ge \frac{\|T_1^n\varphi\|}{\|\varphi\|} \ge \frac{\mu_2^n\|\varphi\|}{\lambda_{1,1}^n\|\varphi\|} = \frac{\mu_2^n}{\lambda_{1,1}^n}.$$

By the Gelfand's formula we have

$$r(T_1) = \lim_{n\to\infty}\sqrt[n]{\|T_1^n\|} \ge \frac{\mu_2}{\lambda_{1,1}} > \frac{1}{\lambda_{1,1}},$$

which is contradiction with $r(T_1) = \lambda_{1,1}^{-1}$. So $A_v u \ne \mu u$ for any $u \in \partial K_r$, $\mu \ge 1$. By Lemma 2.8, we have $i(A_v, K_r, K) = 1, \quad \forall\, 0 < r < r_3,\ v \in K\backslash\{0\}$. □

Remark 3.7. If $f_1(t,x,y)$ is continuous at $x = 0$ and $f_1(t,0,y) = 0$ for any $t \in [0,1]$, $y > 0$, the assumptions $\sup_{y\in(0,+\infty)} f_1^0(y) < \lambda_{1,1}$ may hold, since the limitation $\limsup_{x\to0^+}\max_{t\in[0,1]}\frac{f_1(t,x,y)}{x}$ is of $\frac{0}{0}$ type.

Lemma 3.8. *Suppose the conditions* $(H_1), (H_2)$ *and* (H_4) *are satisfied.*

(i) *If*

$$\inf_{y\in(0,+\infty)} f_{1,\infty}(y) > \tilde{\lambda}_{1,1}, \tag{3.8}$$

then for any $v \in K\backslash\{0\}$, *there exists* $R_1 > 0$ *such that*

$$i(A_v, K_R, K) = 0, \quad \forall\, R > R_1.$$

(ii) *If*

$$\inf_{y\in(0,+\infty)} f_{2,\infty}(y) > \tilde{\lambda}_{1,2}, \tag{3.9}$$

then for any $u \in K\backslash\{0\}$, *there exists* $R_2 > 0$ *such that*

$$i(B_u, K_R, K) = 0, \quad \forall\, R > R_2.$$

Proof. We only prove (i). By (3.8) and Lemma 2.4, it is easy to see that for any prescribed $y > 0$ there exists a corresponding sufficiently small $\tau > 0$ and $R_1 > 0$ such that for all $R > R_1$,

$$f_1(t,x,y) \ge \lambda_{\tau,1}x, \quad x \ge \gamma_\tau R,\ y > 0,$$

where $\lambda_{\tau,1}$ is the first eigenvalue of $T_{\tau,1}$ defined by (2.10) and $\gamma_\tau = \min_{t\in[\tau,1-\tau]}\{\phi_{1,1}(t), \phi_{2,1}(t)\}$.

Let φ_τ be the positive eigenfunction of $T_{\tau,1}$ corresponding to $\lambda_{\tau,1}$, then $\varphi_\tau = \lambda_{\tau,1}T_{\tau,1}\varphi_\tau$. For every $u \in \partial K_R$, $t \in [\tau, 1-\tau]$ and any prescribed $v \in K\backslash\{0\}$, we have $u(t) \geq \gamma_\tau\|u\| = \gamma_\tau R$, so

$$\begin{aligned}(A_v u)(t) &= \int_0^1 H_1(t,s)c_1(s)f_1(s,u(s),v(s))\mathrm{d}s \\ &\geq \int_\tau^{1-\tau} H_1(t,s)c_1(s)f_1(s,u(s),v(s))\mathrm{d}s \\ &\geq \lambda_{\tau,1}\int_\tau^{1-\tau} H_1(t,s)c_1(s)u(s)\mathrm{d}s \\ &= \lambda_{\tau,1}(T_{\tau,1}u)(t), \quad t\in[0,1].\end{aligned}$$

We may suppose that A_v has no fixed points on ∂K_R (otherwise, the proof is ended). Following the procedure used in Lemma 3.4, we have

$$u - A_v u \neq \mu\varphi_\tau, \quad u \in \partial K_R,\ \mu \geq 0.$$

By Lemma 2.7, we have $i(A_v, K_R, K) = 0, \quad \forall R > R_3,\ v \in K\backslash\{0\}$. □

Lemma 3.9. *Suppose the conditions* $(H_1), (H_2)$ *and* (H_4) *are satisfied.*

(i) *If*

$$\sup_{y\in(0,+\infty)} f_1^\infty(y) < \lambda_{1,1}, \tag{3.10}$$

then for any $v \in K\backslash\{0\}$*, there exists* $R_3 > 0$ *such that*

$$i(A_v, K_R, K) = 1, \quad \forall R > R_3.$$

(ii) *If*

$$\sup_{y\in(0,+\infty)} f_2^\infty(y) < \lambda_{1,2}, \tag{3.11}$$

then for any $u \in K\backslash\{0\}$*, there exists* $R_4 > 0$ *such that*

$$i(B_u, K_R, K) = 1, \quad \forall R > R_4.$$

Proof. We only prove (i). By (3.5), for any prescribed $y > 0$ there exists a corresponding $R_0 > 0$ and $0 < \sigma < 1$ such that $f_1(t,x,y) \leq \sigma\lambda_{1,1}x$ for $x \geq R_0$. Let $T_{1,1}u = \sigma\lambda_{1,1}T_1 u$, then $T_{1,1} : E \to E$ is a bounded linear operator and $T_{1,1}(K) \subset K$. Since $\lambda_{1,1}$ is the first eigenvalue of T_1 and $0 < \sigma < 1$, we have

$$(r(T_{1,1}))^{-1} = (\sigma\lambda_{1,1})^{-1}(r(T_1))^{-1} = \sigma^{-1} > 1. \tag{3.12}$$

Let $\epsilon_0 = \frac{1}{2}(1 - r(T_{1,1}))$, then by the Gelfand's formula, we know that there exists a natural number $N \geq 1$ such that $n \geq N$ implies that $\|T_{1,1}^n\| \leq [r(T_{1,1}) + \epsilon_0]^n$. For any $u \in E$, define

$$\|u\|^* = \sum_{i=1}^N [r(T_{1,1}) + \epsilon_0]^{N-i}\|T_{1,1}^{i-1}u\|,$$

where $T^0_{1,1} = I$ is the identity operator. It is easy to verify that $\|u\|^*$ is a new norm in E. Let

$$M_0 = \sup_{u\in\partial K_{R_0}} \int_0^1 \mathscr{H}_1(s)c_1(s)f_1(s,u(s),v(s))\mathrm{d}s,$$

by (2.11) we know that $M_0 < +\infty$.

Select $R'_0 > \max\{R_0, 2M^*_0\epsilon_0^{-1}\}$, where $M^*_0 = \|M_0\|^*$. Since $\|u\|^* > [r(T_{1,1}) + \epsilon_0]^{N-1}\|u\|$, we may choose $R_3 > R'_0$ large enough such that $\|u\| \geq R_3$ implies $\|u\|^* > R'_0$.

Next, we prove

$$A_v u \neq \mu u, \quad u \in \partial K_{R_3},\ \mu \geq 1. \tag{3.13}$$

If otherwise, there exist $u_1 \in \partial K_{R_3}$ and $\mu_1 \geq 1$ such that $A_v u = \mu_1 u_1$. Let $\tilde{u}(t) = \min\{u_1(t), R_0\}$, $D(u_1) = \{t \in [0,1] \,|\, u_1(t) > R_0\}$, then $\tilde{u} \in \partial K_{R_0}$,

$$\begin{aligned}
\mu_1 u_1(t) = (Au_1)(t) &= \int_0^1 H_1(t,s)c_1(s)f_1(s,u_1(s),v(s))\mathrm{d}s\\
&\leq \int_{D(u_1)} H_1(t,s)c_1(s)f_1(s,u_1(s),v(s))\mathrm{d}s\\
&\quad + \int_{[0,1]\setminus D(u_1)} \mathscr{H}_1(s)c_1(s)f_1(s,u_1(s),v(s))\mathrm{d}s\\
&\leq \sigma\lambda_{1,1}\int_0^1 H_1(t,s)c_1(s)u_1(s)\mathrm{d}s + \int_0^1 \mathscr{H}_1(s)c_1(s)f_1(s,\tilde{u}(s),v(s))\mathrm{d}s\\
&\leq (T_{1,1}u_1)(t) + M_0, \quad t \in [0,1].
\end{aligned}$$

Since $T_{1,1}(K) \subset K$, we have $0 \leq (T^j_{1,1}(A_v u_1)(t)) \leq (T^j_{1,1}(T_{1,1}u_1 + M_0)(t))$ $(j = 0,1,2,\ldots,N-1)$, and consequently

$$\|T^j_{1,1}(A_v u_1)\| \leq \|T^j_{1,1}(T_{1,1}u_1 + M_0)\|, \quad j = 0,1,2,\ldots,N-1.$$

Hence,

$$\begin{aligned}
\|A_v u_1\|^* &= \sum_{i=1}^N [r(T_{1,1}) + \epsilon_0]^{N-i}\|T^{i-1}_{1,1}(A_v u_1)\|\\
&\leq \sum_{i=1}^N \left[r(T_{1,1}) + \epsilon_0\right]^{N-i}\|T^{i-1}_{1,1}(T_{1,1}u_1 + M_0)\| = \|T_{1,1}u_1 + M_0\|^*.
\end{aligned}$$

From the selection of R'_0, we obtain $M^*_0 \leq \frac{\epsilon_0}{2}R'_0$. Since $\|u_1\| = R_3$ implies $\|u_1\|^* > R'_0$, we have

$$\begin{aligned}
\mu_1\|u_1\|^* &= \|A_v u_1\|^* \le \|T_{1,1}u_1\|^* + M_0^* = \sum_{i=1}^{N}[r(T_{1,1})+\epsilon_0]^{N-i}\|T_{1,1}^i u_1\| + M_0^*\\
&= [r(T_{1,1})+\epsilon_0]\sum_{i=1}^{N-1}[r(T_{1,1})+\epsilon_0]^{N-i-1}\|T_{1,1}^i u_1\| + \|T_{1,1}^N u_1\| + M_0^*\\
&\le [r(T_{1,1})+\epsilon_0]\sum_{i=1}^{N-1}[r(T_{1,1})+\epsilon_0]^{N-i-1}\|T_{1,1}^i u_1\|\\
&\quad + [r(T_{1,1})+\epsilon_0]^N\|u_1\| + M_0^*\\
&= [r(T_{1,1})+\epsilon_0]\sum_{i=1}^{N}[r(T_{1,1})+\epsilon_0]^{N-i}\|T_{1,1}^{i-1} u_1\| + M_0^*\\
&= [r(T_{1,1})+\epsilon_0]\|u_1\|^* + M_0^* \le [r(T_{1,1})+\epsilon_0]\|u_1\|^* + \frac{\epsilon_0}{2}R_0'\\
&< [r(T_{1,1})+\epsilon_0]\|u_1\|^* + \frac{\epsilon_0}{2}\|u_1\|^* = \left[\frac{1}{4}r(T_{1,1}) + \frac{3}{4}\right]\|u_1\|^*.
\end{aligned}$$

This together with $\mu_1 \ge 1$ implies that $\frac{1}{4}r(T_{1,1}) + \frac{3}{4} \ge 1$, that is $r(T_{1,1}) \ge 1$, which is a contradiction with (3.12). This implies that (3.13) holds. It follows from Lemma 2.8 that $i(A_v, K_{R_3}, K) = 1$. □

Theorem 3.10. *If* $(H_1), (H_2), (H_4)$ *and* (H_5) *are satisfied, and the following conditions hold:*

(i)

$$\inf_{y\in(0,+\infty)} f_{1,0}(y) > \lambda_{1,1}, \qquad \inf_{y\in(0,+\infty)} f_1^{\infty}(y) > \tilde{\lambda}_{1,1};$$

(ii)

$$\sup_{y\in(0,+\infty)} f_2^{0}(y) < \lambda_{1,2}, \qquad \sup_{y\in(0,+\infty)} f_2^{\infty}(y) < \lambda_{1,2}.$$

Then, system (1.1) *has at least two positive solutions* (u_1, v_1) *and* (u_2, v_2) *with* $0 < \|(u_1, v_1)\| < p < \|(u_2, v_2)\|$.

Proof. For any $u \in K\backslash\{0\}$, $v \in K\backslash\{0\}$, it follows from Lemma 3.1 that $i(A_v, K_p, K) = 1$, $i(B_u, K_p, K) = 0$. Next, according to Lemmas 3.4 and 3.9, and then additivity of fixed-point index, we can find r_1, r_2, R_1, R_2, which satisfy $0 < r_1 < p < R_1$, $0 < r_2 < p < R_2$, such that

$$i(A_v, K_p\backslash\bar{K}_{r_1}, K) = 1, \quad i(A_v, K_{R_1}\backslash\bar{K}_p, K) = -1;$$

$$i(B_u, K_p\backslash\bar{K}_{r_2}, K) = -1, \quad i(B_u, K_{R_2}\backslash\bar{K}_p, K) = 1.$$

Since $A_v : K\backslash\{0\} \to K$, $B_u : K\backslash\{0\} \to K$, $T : K\backslash\{0\}\times K\backslash\{0\} \to K\times K$ are completely continuous, from Theorem 2.9, we get

$$\begin{aligned}
i(T,\, K_p\backslash\overline{K}_{r_1} \times K_p\backslash\overline{K}_{r_2},\, K\times K) &= i(A_v,\, K_p\backslash\overline{K}_{r_1},\, K)\\
&\quad \times i(B_u,\, K_p\backslash\overline{K}_{r_2},\, K) = -1,\\
i(T,\, K_{R_1}\backslash\overline{K}_p \times K_{R_2}\backslash\overline{K}_p,\, K\times K) &= i(A_v,\, K_{R_1}\backslash\overline{K}_p,\, K)\\
&\quad \times i(B_u,\, K_{R_2}\backslash\overline{K}_p,\, K) = -1.
\end{aligned}$$

So, system (1.1) has at least two positive solutions (u_1, v_1) and (u_2, v_2) with $0 < \|(u_1, v_1)\| < p < \|(u_2, v_2)\|$. □

In the same way, we can prove the following theorems.

Theorem 3.11. *If* $(H_1), (H_2), (H_4)$ *and* (H_5^*) *are satisfied, and the following conditions hold:*

(i)
$$\sup_{y\in(0,+\infty)} f_1^0(y) < \lambda_{1,1}, \qquad \sup_{y\in(0,+\infty)} f_1^\infty(y) < \lambda_{1,1};$$
(ii)
$$\sup_{y\in(0,+\infty)} f_2^0(y) < \lambda_{1,2}, \qquad \sup_{y\in(0,+\infty)} f_2^\infty(y) < \lambda_{1,2}.$$
Then, system (1.1) *has at least two positive solutions* (u_1, v_1) *and* (u_2, v_2) *with* $0 < \|(u_1, v_1)\| < p < \|(u_2, v_2)\|$.

Theorem 3.12. *If* $(H_1), (H_2), (H_4)$ *and* (H_5') *are satisfied, and the following conditions hold:*

(i)
$$\inf_{y\in(0,+\infty)} f_{1,0}(y) > \lambda_{1,1}, \qquad \inf_{y\in(0,+\infty)} f_{1,\infty}(y) > \tilde{\lambda}_{1,1};$$
(ii)
$$\inf_{y\in(0,+\infty))} f_{2,0}(y) > \lambda_{1,2}, \qquad \inf_{y\in(0,+\infty)} f_{2,\infty}(y) > \tilde{\lambda}_{1,2}.$$
Then, system (1.1) *has at least two positive solutions* (u_1, v_1) *and* (u_2, v_2) *with* $0 < \|(u_1, v_1)\| < p < \|(u_2, v_2)\|$.

By Lemma 2.9, we can also prove the following theorem.

Theorem 3.13. *If* $(H_1), (H_2)$ *and* (H_4) *are satisfied, and one of the following conditions holds:*

(i)
$$\sup_{y\in(0,+\infty)} f_1^0(y) < \lambda_{1,1}, \qquad \inf_{y\in(0,+\infty)} f_{1,\infty}(y) > \tilde{\lambda}_{1,1}$$
and
$$\inf_{y\in(0,+\infty)} f_{2,0}(y) > \lambda_{1,2}, \qquad \sup_{y\in(0,+\infty)} f_2^\infty(y) < \lambda_{1,2};$$
(ii)
$$\sup_{y\in(0,+\infty)} f_1^0(y) < \lambda_{1,1}, \qquad \inf_{y\in(0,+\infty)} f_{1,\infty}(y) > \tilde{\lambda}_{1,1}$$
and
$$\sup_{y\in(0,+\infty)} f_2^0(y) < \lambda_{1,2}, \qquad \inf_{y\in(0,+\infty)} f_{2,\infty}(y) > \tilde{\lambda}_{1,2};$$
(iii)
$$\inf_{y\in(0,+\infty)} f_{1,0}(y) > \lambda_{1,1}, \qquad \sup_{y\in(0,+\infty)} f_1^\infty(y) < \lambda_{1,1}$$
and
$$\inf_{y\in(0,+\infty)} f_{2,0}(y) > \lambda_{1,2}, \qquad \sup_{y\in(0,+\infty)} f_2^\infty(y) < \lambda_{1,2}.$$
Then, system (1.1) *has at least one positive solution.*

4. Examples

Consider the following integral boundary value system

$$\begin{cases} u''(t) - u(t) + \frac{1}{t} y_1(t) = 0, & t \in (0,1), \\ v''(t) - v(t) + \frac{1}{t} y_2(t) = 0, & t \in (0,1), \\ u(0) = \int_0^1 u(s)\mathrm{d}s, & u(1) = \int_0^1 u(s)\mathrm{d}s, \\ v(0) = \int_0^1 v(s)\mathrm{d}s, & v(1) = \int_0^1 v(s)\mathrm{d}s, \end{cases} \tag{4.1}$$

where $y_1, y_2 \in C((0,1)) \cap L^1((0,1))$. System (4.1) is a special case of the (1.1), where $a_1(t) = a_2(t) \equiv 0$, $b_1(t) = b_2(t) \equiv -1$, $c_1(t) = c_2(t) = \frac{1}{t}$, $h_1(t) = h_2(t) = g_1(t) = g_2(t) \equiv 1$. Obviously $c_1(t)$, $c_2(t)$ are singular at $t = 0$.

Based on Lemma 2.1, let $\varphi_{1,i}$ and $\varphi_{2,i}$ be the unique solutions of the following two boundary value problems, respectively

$$\begin{cases} \varphi''_{1,i}(t) - \varphi_{1,i}(t) = 0, & t \in (0,1), \\ \varphi_{1,i}(0) = 0, & \varphi_{1,i}(1) = 1, \ i = 1,2, \end{cases}$$

$$\begin{cases} \varphi''_{2,i}(t) - \varphi_{2,i}(t) = 0, & t \in (0,1), \\ \varphi_{2,i}(0) = 1, & \varphi_{2,i}(1) = 0, \ i = 1,2. \end{cases}$$

Then it is easy to verify that

$$\varphi_{1,i}(t) = \tfrac{e}{e^2-1}(e^t - e^{-t}), \quad \varphi_{2,i}(t) = \tfrac{1}{e^2-1}(e^{2-t} - e^t),$$

$$k_{1,i} = k_{4,i} = \tfrac{2}{e+1}, \quad k_{2,i} = k_{3,i} = \tfrac{e-1}{e+1}, \quad k_i = \tfrac{4-(e-1)^2}{(e+1)^2},$$

$$\rho_i = \varphi'_{1,i}(0) = \tfrac{2e}{e^2-1}, \quad p_i(t) = 1,$$

$$G_i(t,s) = \tfrac{1}{2(e^2-1)} \begin{cases} (e^t - e^{-t})(e^{2-s} - e^s), & 0 \le t \le s \le 1, \\ (e^s - e^{-s})(e^{2-t} - e^t), & 0 \le s \le t \le 1. \end{cases}$$

By computation, we know that $0 \le G_i(t,s) \le 2s, t, s \in [0,1]$ and

$$\mathscr{H}_i(s) = G_i(s,s) + \frac{1}{k}\int_0^1 G_i(s,\tau)\mathrm{d}\tau + \frac{1}{k}\int_0^1 G_i(s,\tau)\sigma\tau \le 2s + \frac{4s}{k} \le 80s, \quad s \in [0,1],$$

then $0 < \int_0^1 \mathscr{H}_i(s)c_i(s)\mathrm{d}s < +\infty$ for $i = 1,2$.

Example 4.1. *Let*

$$f_1(t,x,y) = \left(x^{\frac{1}{3}} + x^3\right)\frac{2 + \sin(\ln y)}{A},$$

$$f_2(t,x,y) = \left(x^{\frac{1}{2}} + x^2\right)\frac{2 + \cos(\ln y)}{B},$$

where $A, B > 0$ are arbitrary real positive numbers. Assume that $p = 1$, then for all $0 < x \le 1$, $y > 0$ and $0 \le t \le 1$, we have

$$f_1(t,x,y) \le \frac{3}{A}, \quad f_2(t,x,y) \le \frac{3}{B}.$$

Since $0 < \int_0^1 \mathscr{H}_i(s)c_i(s)\mathrm{d}s < +\infty$ *for* $i = 1, 2$*, we can choose* A, B *large enough such that*

$$\frac{3}{A} < \frac{1}{\int_0^1 \mathscr{H}_1(s)c_2(s)\mathrm{d}s}, \quad \frac{3}{B} < \frac{1}{\int_0^1 \mathscr{H}_2(s)c_2(s)\mathrm{d}s}.$$

Choose $\eta_1 \in \left(\frac{3}{A}, \frac{1}{\int_0^1 \mathscr{H}_1(s)c_2(s)\mathrm{d}s}\right)$ *and* $\eta_2 \in \left(\frac{3}{B}, \frac{1}{\int_0^1 \mathscr{H}_2(s)c_2(s)\mathrm{d}s}\right)$*, then the assumption* (H_5') *in Lemma 3.2 is satisfied. It is easy to verify that* f_1, f_2 *satisfy* (H_4)*. Since* $\inf_{y\in(0,+\infty)} f_{1,0}(y) = \inf_{y\in(0,+\infty)} f_{1,\infty}(y) = \inf_{y\in(0,+\infty)} f_{2,0}(y) = \inf_{y\in(0,+\infty)} f_{2,0}(y) = +\infty$*. So by Theorem 3.12, we conclude that the following singular integral boundary value system*

$$\begin{cases} u''(t) - u(t) + \frac{1}{t}\left(u^{\frac{1}{3}}(t) + u^3(t)\right) \frac{2+\sin(\ln v(t))}{A} = 0, & t \in (0,1), \\ v''(t) - v(t) + \frac{1}{t}\left(v^{\frac{1}{2}}(t) + v^2(t)\right) \frac{2+\cos(\ln u(t))}{B} = 0, & t \in (0,1), \\ u(0) = \int_0^1 u(s)\mathrm{d}s, & u(1) = \int_0^1 u(s)\mathrm{d}s, \\ v(0) = \int_0^1 v(s)\mathrm{d}s, & v(1) = \int_0^1 v(s)\mathrm{d}s, \end{cases}$$

has at least two positive solutions (u_1, v_1) *and* (u_2, v_2) *with* $0 < \|(u_1, v_1)\| < 1 < \|(u_2, v_2)\|$*.*

Example 4.2. *Let*

$$f_1(t, x, y) = (1 + |\ln x|)\,(2 + \sin(\ln y))\,,$$

$$f_2(t, x, y) = \left(1 + \sin^2 \frac{1}{x}\right)(2 + \cos(\ln y))\,.$$

It is easy to verify that f_1, f_2 *satisfy the assumption* (H_4) *and*

$$\inf_{y\in(0,+\infty)} f_{1,0}(y) = \inf_{y\in(0,+\infty)} f_{2,0}(y) = +\infty,$$

$$\inf_{y\in(0,+\infty)} f_1^{\infty}(y) = \inf_{y\in(0,+\infty)} f_2^{\infty}(y) = 0.$$

So by Theorem 3.13, we conclude that the following singular integral boundary value system

$$\begin{cases} u''(t) - u(t) + \frac{1}{t}\,(1 + |\ln u(t)|)\,(2 + \sin(\ln v(t))) = 0, & t \in (0,1), \\ v''(t) - v(t) + \frac{1}{t}\left(1 + \sin^2 \frac{1}{u(t)}\right)(2 + \cos(\ln u(t))) = 0, & t \in (0,1), \\ u(0) = \int_0^1 u(s)\mathrm{d}s, & u(1) = \int_0^1 u(s)\mathrm{d}s, \\ v(0) = \int_0^1 v(s)\mathrm{d}s, & v(1) = \int_0^1 v(s)\mathrm{d}s, \end{cases}$$

has at least one positive solution.

Acknowledgements

The authors are grateful to the referees for their valuable suggestions and comments. The work was supported financially by the National Natural Science Foundation of China (11371221, 11571296).

References

[1] Ma, R.: Positive solutions for a nonlinear three-point boundary-value problem. Electron. J. Differ. Equ. **1999**(34), 1–8 (1999)

[2] Ma, R., Wang, H.: Positive solutions of nonlinear three-point boundary-value problems. Electron. J. Differ. Equ. **279**(1), 216–227 (2003)

[3] Eloe, P. W., Ahmad, B.: Positive solutions of a nonlinear nth order boundary value problem with nonlocal conditions. Appl. Math. Lett. **18**(5), 521–527 (2005)

[4] Li, J., Shen, J.: Multiple positive solutions for a second-order three-point boundary value problem. Appl. Math. Comput. **182**(1), 258–268 (2006)

[5] Liu, B., Liu, L., Wu, Y.: Positive solutions for a singular second-order three-point boundary value problem. Appl. Math. Comput. **196**(2), 532–541 (2008)

[6] Liu, L., Liu, B., Wu, Y.: Nontrivial solutions of m-point boundary value problems for singular second-order differential equations with a sign-changing nonlinear term. J. Comput. Appl. Math. **224**(1), 373–382 (2009)

[7] Liu, B., Liu, L., Wu, Y.: Multiple solutions of singular three-point boundary value problems on $[0, +\infty)$. Nonlinear Anal. Theory Methods Appl. **70**(9), 3348–3357 (2009)

[8] Zhang, X., Feng, M., Ge, W.: Symmetric positive solutions for p-Laplacian fourth-order differential equations with integral boundary conditions. J. Comput. Appl. Math. **222**(2), 561–573 (2008)

[9] Webb, J.R.L.: Nonlocal conjugate type boundary value problems of higher order. Nonlinear Anal. **71**(5-6), 1933–1940 (2009)

[10] Kong, L.: Second order singular boundary value problems with integral boundary conditions. Nonlinear Anal. **72**(5), 2628–2638 (2010)

[11] Jankowski, T.: Differential equations with integral boundary conditions. J. Comput. Appl. Math. **147**(1), 1–8 (2002)

[12] Jankowski, T.: Positive solutions for second order impulsive differential equations involving Stieltjes integral conditions. Nonlinear Anal. **74**(11), 3775–3785 (2011)

[13] Hao, X., Liu, L., Wu, Y.: Positive solutions for second order impulsive differential equations with integral boundary conditions. Commun. Nonlinear Sci. Numer. Simul. **16**(1), 101–111 (2011)

[14] Wu, J., Zhang, X., Liu, L., Wu, Y.: Positive solutions of higher-order nonlinear fractional differential equations with changing-sign measure. Adv. Differ. Equ. **2012**(1), 1–14 (2012)

[15] Jiang, J., Liu, L., Wu, Y.: Positive solutions for second order impulsive differential equations with Stieltjes integral boundary conditions. Adv. Differ. Equ. **2012**(1), 1–18 (2012)

[16] Jiang, J., Liu, L., Wu, Y.: Positive solutions for p-Laplacian fourth-order differential system with integral boundary conditions. Discrete Dyn. Nat. Soc. (2012)

[17] Liu, L., Hao, X., Wu, Y.: Positive solutions for singular second order differential equations with integral boundary conditions. Math. Comput. Model. **57**(3), 836–847 (2013)

[18] Liu, B., Li, J., Liu, L.: Nontrivial solutions for a boundary value problem with integral boundary conditions. Bound. Value Probl. **2014**(1), 1–8 (2014)

[19] Wang, Y., Liu, L., Wu, Y.: Extremal solutions for p-Laplacian fractional integro-differential equation with integral conditions on infinite intervals via iterative computation. Adv. Differ. Equ. **2015**(1), 1–14 (2015)

[20] Wang, Y., Liu, L., Zhang, X., Wu, Y.: Positive solutions of an abstract fractional semipositone differential system model for bioprocesses of HIV infection. Appl. Math. Comput. **258**, 312–324 (2015)

[21] Cheng, X., Zhong, C.: Existence of positive solutions for a second-order ordinary differential system. J. Math. Anal. Appl. **312**(1), 14–23 (2005)

[22] Cheng, X.: Existence of positive solutions for a class of second-order ordinary differential systems. Nonlinear Anal. Theory Methods Appl. **69**(9), 3042–3049 (2008)

[23] Liu, L., Hu, L., Wu, Y.: Positive solutions of two-point boundary value problems for systems of nonlinear second-order singular and impulsive differential equations. Nonlinear Anal. Theory Methods Appl. **69**(11), 3774–3789 (2008)

[24] Guo, D., Lakshmikantham, V.: Nonlinear Problems in Abstract Cones. Academic Press, Inc., New York (1988)

Fenglong Sun and Lishan Liu
School of Mathematical Sciences
Qufu Normal University
Qufu 273165, Shandong
People's Republic of China
e-mail: sfenglong@sina.com

Lishan Liu
e-mail: mathlls@163.com

Lishan Liu, Xinguang Zhang and Yonghong Wu
Department of Mathematics and Statistics
Curtin University
Perth
WA 6845
Australia
e-mail: yhwu@maths.curtin.edu.au

Xinguang Zhang
Department of Mathematics and Informational Science
Yantai University
Yantai 264005
Shandong
People's Republic of China
e-mail: zxg123242@163.com

Received: February 23, 2016.
Revised: June 25, 2016.
Accepted: July 6, 2016.