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Numerical Solutions for Systems of Nonlinear Fractional Ordinary Differential Equations Using the FNDM

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Abstract. A new technique has been developed for analytical solutions of fractional order nonlinear ODE system. We propose a reliable method called the fractional natural decomposition method (FNDM). The FNDM is based on the natural transform method (NTM) and the Adomian decomposition method. We use the FNDM to construct new analytical approximate and exact solutions to systems of nonlinear fractional ordinary differential equation (NLFODEs). The fractional derivatives are described in the Caputo sense.

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Keywords. Fractional natural decomposition method, System of fractional differential equations, Caputo fractional derivative.

1. Introduction

Differential equations with fractional order have recently proved to be valuable tools to the modeling of many physical phenomena and started to attract much more attention of Physicists and Mathematicians [4–6,8–10,18,20]. These equations are represented by linear and nonlinear ODEs and solving such fractional differential equations (FDEs) is very important. So it is very important to find efficient methods for solving FDEs. Most of the fractional differential equations do not have exact analytical solutions; hence considerable effort has been focused on approximate and numerical solutions of these equations.

Recently, various researchers have introduced new methods in the literature. These methods include fractional Sumudu Transform [12,16], fractional matrix method [6], fractional Adomian decomposition method (FADM) [7,19,27], the fractional reduced differential transform method (FRDTM) [25,26],

fractional Laplace decomposition method (FLDM) [30], the fractional homotopy analysis method (FHAM) [11,31] and the fractional homotopy perturbation method (FHPM) [28,29].
In this paper, we introduce a new method, called the fractional natural decomposition method (FNDM). The suggested FNDM provides the solution in a rapid convergent series which may lead us to the solution in a closed form. This method combines two powerful methods, the Natural transform method (NTM) [3,13] and the Adomian decomposition method (ADM) [1,2], for obtaining approximate solutions for systems of fractional partial differential equations. It is worth mentioning that the FNDM is applied without any discretization or restrictive assumptions or transformations and it is free from round-off errors. Also this method provides an analytical solution by using the initial conditions only, unlike the variables separation method, which requires initial and boundary conditions. The boundary conditions can be used to justify the obtained results. The natural decomposition method (NDM) was first introduced by Rawashdeh and Maitama in 2014 [21,23,24], to solve linear and nonlinear ODEs and PDEs that appears in many mathematical physics and engineering applications.
In this paper, we give analytical approximate solutions for $0 < \alpha, \beta, \gamma < 1$ and exact solutions in the case when $\alpha = \beta = \gamma = 1$ to two nonlinear systems of fractional ordinary differential equations.
The rest of this paper is organized as follows: in Sect. 2, we give some preliminaries and definitions of fractional calculus. In Sects. 3 and 4, the natural transform method is introduced. Section 4 is devoted to apply the method to two test problems and presents graphs to show the effectiveness of the FNDM for some values of x and t. In Sect. 5, we present tables for different values of α, β, γ and t. Section 6 is for discussion and conclusion of this paper.

2. Preliminaries of Fractional Calculus

In this section, we give some of the main definitions and facts that we will use in our study. Some of these basic definitions are due to Liouville which are given as follows [4,5,9,15]:

Definition 2.1. A real function $f(x),\ x > 0$ is said to be in the space $C_\mu,\ \mu \in \mathbb{R}$ if there exists a real number $q(> \mu)$, such that $f(x) = x^q g(x)$, where $g(x) \in C\,[0, \infty)$, and it is said to be in the space C_μ^m if $f^{(m)} \in C_\mu,\ m \in \mathbb{N}$.

Definition 2.2. For an integrable function $f \in C_\mu$, the Riemann–Liouville fractional integral operator of order $\alpha \geq 0$ is defined as

$$\begin{cases} J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\,\mathrm{d}t, & \text{when} \quad \alpha > 0, \quad x > 0 \\ J^0 f(x) = f(x). \end{cases}$$

Caputo and Mainardi [5] presented a modified fractional differentiation operator D^α in their work on the theory of viscoelasticity to overcome the disadvantages of the Riemann–Liouville derivative when someone tries to model real-world problems.

Definition 2.3. The fractional derivative of $f \in C_{-1}^m$ in the Caputo sense can be defined as

$$\begin{aligned} D^\alpha f(x) &= J^{m-\alpha} D^m f(x) \\ &= \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t)\mathrm{d}t, \\ &\quad \text{where } m-1 < \alpha \leq m, \quad m \in \mathbb{N}, \quad x > 0 \end{aligned}$$

Lemma 2.1 [14]. *If $m-1 < \alpha \leq m,\ m \in \mathbb{N}$ and $f \in C_\mu^m, \mu \geq -1$, then*

$$\begin{cases} D^\alpha J^\alpha f(x) = f(x), & \text{if } x > 0 \\ J^\alpha D^\alpha f(x) = f(x) - \sum\limits_{k=0}^{m-1} f^{(k)}(0^+)\frac{x^k}{k!}, & \text{if } m-1 < \alpha < m. \end{cases}$$

We would like to mention here the Caputo fractional derivative is used because it allows traditional initial and boundary conditions to be included in the formulation of our problem.

3. Definitions and Properties of the N–Transform

In this section, we present some background about the nature of the natural transform method (NTM). Given a function $f(t)$, $t \in \mathbb{R}$, the general integral transform is defined by [3,13]:

$$\Im\left[f(t)\right](s) = \int_{-\infty}^{\infty} K(s,t)\, f(t)\, \mathrm{d}t, \tag{3.1}$$

where $K(s,t)$ represent the kernel of the transform and s is the real (complex) number which is independent of t. Note that when $K(s,t)$ is e^{-st}, $t\, J_n(st)$ and $t^{s-1}(st)$, Eq. (3.1) gives, respectively, Laplace transform, Hankel transform and Mellin transform. Now, for $f(t),\ t \in (-\infty, \infty)$ consider the integral transforms defined by

$$\Im\left[f(t)\right](u) = \int_{-\infty}^{\infty} K(t)\, f(ut)\, \mathrm{d}t, \tag{3.2}$$

and

$$\Im\left[f(t)\right](s,u) = \int_{-\infty}^{\infty} K(s,t)\, f(ut)\, \mathrm{d}t. \tag{3.3}$$

It is worth mentioning that when $K(t) = e^{-t}$, Eq. (3.2) gives the integral Sumudu transform, where the parameter s is replaced by u. Moreover, for any value of n the generalized Laplace and Sumudu transform are, respectively, defined by [3,13]:

$$\ell\left[f(t)\right] = F(s) = s^n \int_0^{\infty} e^{-s^{n+1}t}\, f(s^n t)\, \mathrm{d}t, \tag{3.4}$$

and

$$\mathbb{S}\left[f(t)\right] = G(u) = u^n \int_0^{\infty} e^{-u^n t}\, f(tu^{n+1})\, \mathrm{d}t. \tag{3.5}$$

Note that when $n = 0$, Eqs. (3.4) and (3.5) are the Laplace and Sumudu transform, respectively. The natural transform of the function $f(t)$ for $t \in \mathbb{R}$ is defined by [8,10]:

$$\mathbb{N}\left[f(t)\right] = R(s,u) = \int_{-\infty}^{\infty} e^{-st} f(ut)\,\mathrm{d}t; \quad s,\, u \in (-\infty, \infty)\,, \tag{3.6}$$

where $\mathbb{N}\left[f(t)\right]$ is the natural transformation of the time function $f(t)$ and the variables s and u are the natural transform variables. Note that Eq. (3.6) can be written in the following form [3,13]:

$$\mathbb{N}\left[f(t)\right] = R^{-}(s,u) + R^{+}(s,u).$$

It is worth mentioning here that if the function $f(t)H(t)$ is defined on the positive real axis, where $H(.)$ is the Heaviside function, $t \in (0, \infty)$, and suppose that $A = \left\{ f(t) \;:\; \exists\, M, \tau_1, \tau_2 > 0,\ with\ |f(t)| < Me^{\frac{|t|}{\tau_j}},\ for\ t \in (-1)^j \times [0,\infty)\,,\ j \in \mathbb{Z}^+ \right\}$.

Then, we define the Natural transform (N-Transform) as

$$\mathbb{N}\left[f(t)H(t)\right] = \mathbb{N}^{+}\left[f(t)\right] = R^{+}(s,u) = \int_{0}^{\infty} e^{-st} f(ut)\,\mathrm{d}t; \quad s,\, u \in (0, \infty)\,. \tag{3.7}$$

Note if $u = 1$ Eq. (3.7) can be reduced to the Laplace transform and if $s = 1$ Eq. (3.7) can be reduced to the Sumudu transform.

Important properties: Some basic properties of the N-Transforms are given as follows [3,13]:

1. $\mathbb{N}^{+}\left[1\right] = \frac{1}{s}$.
2. $\mathbb{N}^{+}\left[t^{\alpha}\right] = \frac{\Gamma(\alpha+1)\,u^{\alpha}}{s^{\alpha+1}}$, where $\alpha > -1$.

4. Analysis of the Fractional Natural Decomposition Method

In this section, we present some theorems of the fractional natural transform method. Such results are in [22]. Also, in [17] the authors used different approach to prove Theorems 4.1 and 4.3.

Theorem 4.1. *If $R(s,u)$ is the Natural transform of $f(t)$, then the Natural transform of the Riemann–Liouville fractional integral for $f(t)$ of order α denoted by $J^{\alpha} f(t)$ is given by*

$$\mathbb{N}^{+}\left[J^{\alpha} f(t)\right] = \frac{u^{\alpha}}{s^{\alpha}} R(s,u).$$

Theorem 4.2. *If n is any positive integer, where $n - 1 \le \alpha < n$ and $R(s,u)$ is the Natural transform of the function $f(t)$, then the Natural transform, $R_{\alpha}(s,u)$ of the Riemann–Liouville fractional derivative of the function $f(t)$ of order α denoted by $D^{\alpha} f(t)$ is given by*

$$\mathbb{N}^{+}\left[D^{\alpha} f(t)\right] = R_{\alpha}(s,u) = \frac{s^{\alpha}}{u^{\alpha}} R(s,u) - \sum_{k=0}^{n-1} \frac{s^{k}}{u^{k+1}} \left(D^{\alpha-k-1} f(t)\right)_{t=0}.$$

Theorem 4.3. *If n is any positive integer, where $n-1 \leq \alpha < n$ and $R(s,u)$ is the Natural transform of the function $f(t)$, then the Natural transform, $R_{\alpha}^{c}(s,u)$ of the Caputo fractional derivative of the function $f(t)$ of order α denoted by ${}^{c}D^{\alpha}f(t)$ is given by*

$$\mathbb{N}^{+}\left[{}^{c}D^{\alpha}f(t)\right] = R_{\alpha}^{c}(s,u) = \frac{s^{\alpha}}{u^{\alpha}}R(s,u) - \sum_{k=0}^{n-1}\frac{s^{\alpha-(k+1)}}{u^{\alpha-k}}\left[D^{k}f(t)\right]_{t=0}.$$

Methodology of the FNDM: We illustrate the FNDM by considering the general fractional nonlinear ODEs system of the form

$$\begin{aligned} D_t^{\alpha}x(t) + R\,x(t) + F\,x(t) &= g(t) \\ D_t^{\beta}y(t) + R\,y(t) + F\,y(t) &= h(t), \end{aligned} \tag{4.1}$$

where $0 < \alpha, \beta \leq 1$,
subject to the initial conditions

$$x(0) = g(t); \quad y(0) = h(t). \tag{4.2}$$

Note that $D_t^{\alpha}\,x(t)$, $D_t^{\beta}\,y(t)$ are the Caputo fractional derivative of the functions $x(t)$, $y(t)$, respectively, R is the linear differential operator, F represents the general nonlinear differential operator and $g(t)$, $h(t)$ are the source terms. We apply the N-Transform and Theorem 4.3 to Eq. (4.1) to get

$$\begin{aligned} X(s,u) &= \frac{u^{\alpha}}{s^{\alpha}}\sum_{k=0}^{n-1}\frac{s^{\alpha-(k+1)}}{u^{\alpha-k}}\left[D^{k}x(t)\right]_{t=0} + \frac{u^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\left[g(t)\right] \\ &\quad -\frac{u^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\left[R\,x(t) + F\,x(t)\right] \\ Y(s,u) &= \frac{u^{\beta}}{s^{\beta}}\sum_{k=0}^{n-1}\frac{s^{\beta-(k+1)}}{u^{\beta-k}}\left[D^{k}y(t)\right]_{t=0} + \frac{u^{\beta}}{s^{\beta}}\mathbb{N}^{+}\left[h(t)\right] \\ &\quad -\frac{u^{\beta}}{s^{\beta}}\mathbb{N}^{+}\left[R\,y(t) + F\,y(t)\right]. \end{aligned} \tag{4.3}$$

Using Eq. (4.2), Eq. (4.3) becomes

$$\begin{aligned} X(s,u) &= g(t) + \frac{u^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\left[g(t)\right] - \frac{u^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\left[R\,x(t) + F\,x(t)\right] \\ Y(s,u) &= h(t) + \frac{u^{\beta}}{s^{\beta}}\mathbb{N}^{+}\left[h(t)\right] - \frac{u^{\beta}}{s^{\beta}}\mathbb{N}^{+}\left[Ry(t) + Fy(t)\right]. \end{aligned} \tag{4.4}$$

Now we apply the inverse Natural transform of Eq. (4.4) to obtain

$$\begin{aligned} x(t) &= G(t) - \mathbb{N}^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\left[R\;x(t) + F\;x(t)\right]\right] \\ y(t) &= H(t) - \mathbb{N}^{-1}\left[\frac{u^{\beta}}{s^{\beta}}\mathbb{N}^{+}\left[R\,y(t) + F\,y(t)\right]\right]. \end{aligned} \tag{4.5}$$

Note $G(t)$ and $H(t)$ are arising from the nonhomogeneous term and the prescribed initial conditions.
Now we assume an infinite series solutions form:

$$x(t) = \sum_{n=0}^{\infty} x_n(t), \quad y(t) = \sum_{n=0}^{\infty} y_n(t). \tag{4.6}$$

Using Eq. (4.6) we can re-write Eq. (4.5) as follows:

$$\sum_{n=0}^{\infty} x_n(t) = G(t) - \mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbb{N}^+\left[R\sum_{n=0}^{\infty} x_n(t)\right] + \sum_{n=0}^{\infty} A_n\right]$$
$$\sum_{n=0}^{\infty} y_n(t) = H(t) - \mathbb{N}^{-1}\left[\frac{u^\beta}{s^\beta}\mathbb{N}^+\left[R\sum_{n=0}^{\infty} y_n(t)\right] + \sum_{n=0}^{\infty} B_n\right], \quad (4.7)$$

where the A_n, B_n are the polynomials representing the nonlinear term $F\,x(t)$, $F\,y(t)$, respectively.
By comparing both sides of Eq. (4.7) we conclude

$$\begin{array}{ll} x_0(t) = G(t), & y_0(t) = H(t) \\ x_1(t) = -\mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbb{N}^+\left[R\,x_0(t)\right] + A_0\right], & y_1(t) = -\mathbb{N}^{-1}\left[\frac{u^\beta}{s^\beta}\mathbb{N}^+\left[R\,y_0(t)\right] + B_0\right] \\ x_2(t) = -\mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbb{N}^+\left[R\,x_1(t)\right] + A_1\right], & y_2(t) = -\mathbb{N}^{-1}\left[\frac{u^\beta}{s^\beta}\mathbb{N}^+\left[R\,y_1(t)\right] + B_1\right]. \end{array}$$

We continue in this manner to get the general recursive relation given by

$$\begin{array}{l} x_{n+1}(t) = -\mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbb{N}^+\left[R\,x_n(t)\right] + A_n\right], \; n \geq 1 \\ y_{n+1}(t) = -\mathbb{N}^{-1}\left[\frac{u^\beta}{s^\beta}\mathbb{N}^+\left[R\,y_n(t)\right] + B_n\right], \; n \geq 1. \end{array} \quad (4.8)$$

5. Applications

To demonstrate the effectiveness of the FNDM, two examples of nonlinear systems will be studied. We choose two nonlinear systems to show the features of FNDM and the convergence of the FNDM solution.

Example 5.1. Consider the nonlinear systems of fractional ordinary differential equations of the form

$$\begin{array}{l} D^\alpha x(t) = \frac{1}{2}x(t) \\ D^\beta y(t) = y(t) + x^2(t) \end{array}, \quad 0 < \alpha, \quad \beta \leq 1 \quad (5.1)$$

subject to the initial conditions

$$x(0) = 1, \quad y(0) = 0. \quad (5.2)$$

The exact solutions in the case, $\alpha = \beta = 1$, are $x(t) = e^{\frac{t}{2}}$ and $y(t) = t\,e^t$.
First, we apply the N-transform to Eq. (5.1) to get

$$\begin{array}{l} \mathbb{N}^+\left[D^\alpha x(t)\right] = \frac{1}{2}\mathbb{N}^+\left[x(t)\right] \\ \mathbb{N}^+\left[D^\beta y(t)\right] = \mathbb{N}^+\left[y(t)\right] + \mathbb{N}^+\left[x^2(t)\right]. \end{array} \quad (5.3)$$

Apply Theorem 4.3 to Eq. (5.3) to obtain

$$\begin{array}{l} \frac{s^\alpha}{u^\alpha}\mathbb{N}^+\left[x(t)\right] - \sum_{k=0}^{n-1}\frac{u^{\alpha-(k+1)}}{s^{\alpha-k}}\left[D^k x(t)\right]_{t=0} = \frac{1}{2}\mathbb{N}^+\left[x(t)\right] \\ \frac{s^\beta}{u^\beta}\mathbb{N}^+\left[y(t)\right] - \sum_{k=0}^{n-1}\frac{u^{\beta-(k+1)}}{s^{\beta-k}}\left[D^k y(t)\right]_{t=0} = \mathbb{N}^+\left[y(t)\right] + \mathbb{N}^+\left[x^2(t)\right]. \end{array} \quad (5.4)$$

Substitute Eq. (5.2) into Eq. (5.4) to get

$$\begin{array}{l} \mathbb{N}^+\left[x(t)\right] = \frac{1}{s} + \frac{1}{2}\frac{u^\alpha}{s^\alpha}\mathbb{N}^+\left[x(t)\right] \\ \mathbb{N}^+\left[y(t)\right] = \frac{u^\beta}{s^\beta}\mathbb{N}^+\left[y(t)\right] + \frac{u^\beta}{s^\beta}\mathbb{N}^+\left[x^2(t)\right]. \end{array} \quad (5.5)$$

Take the inverse N-Transform of Eq. (5.5) to get

$$\begin{aligned} x(t) &= 1 + \tfrac{1}{2}\mathbb{N}^{-1}\left[\tfrac{u^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\left[x(t)\right]\right] \\ y(t) &= \mathbb{N}^{-1}\left[\tfrac{u^{\beta}}{s^{\beta}}\mathbb{N}^{+}\left[y(t)\right]\right] + \mathbb{N}^{-1}\left[\tfrac{u^{\beta}}{s^{\beta}}\mathbb{N}^{+}\left[x^{2}(t)\right]\right]. \end{aligned} \tag{5.6}$$

Now from Eq. (5.6) we conclude

$$\begin{aligned} \sum_{n=0}^{\infty} x_n(t) &= 1 + \tfrac{1}{2}\mathbb{N}^{-1}\left[\tfrac{u^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\left[\sum_{n=0}^{\infty} x_n(t)\right]\right] \\ \sum_{n=0}^{\infty} y_n(t) &= \mathbb{N}^{-1}\left[\tfrac{u^{\beta}}{s^{\beta}}\mathbb{N}^{+}\left[\sum_{n=0}^{\infty} y_n(t) + \sum_{n=0}^{\infty} A_n\right]\right]. \end{aligned} \tag{5.7}$$

Also note that $x^2(t) = \sum_{n=0}^{\infty} A_n$.
Thus

$$\begin{aligned} A_0 &= x_0^2 \\ A_1 &= 2x_0x_1 \\ A_2 &= 2x_0x_2 + x_1^2 \\ A_3 &= 2x_0x_3 + 2x_1x_2. \end{aligned}$$

Since $x_0(t) = 1$ and $y_0(t) = 0$, using Eq. (5.7) we can find the following components:

$$\begin{aligned} x_1(t) &= \frac{1}{2}\mathbb{N}^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\left[x_0\right]\right] \\ &= \frac{1}{2}\frac{t^{\alpha}}{\Gamma(\alpha+1)}. \end{aligned}$$

And

$$\begin{aligned} y_1(t) &= \mathbb{N}^{-1}\left[\frac{u^{\beta}}{s^{\beta}}\mathbb{N}^{+}\left[y_0 + A_0\right]\right] \\ &= \mathbb{N}^{-1}\left[\frac{u^{\beta}}{s^{\beta}}\frac{1}{s}\right] \\ &= \frac{t^{\beta}}{\Gamma(\beta+1)}. \end{aligned}$$

Similarly,

$$\begin{aligned} x_2(t) &= \frac{1}{2}\mathbb{N}^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}\frac{1}{2}\frac{u^{\alpha}}{s^{\alpha+1}}\right] \\ &= \frac{1}{4}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}. \end{aligned}$$

And

$$\begin{aligned} y_2(t) &= \mathbb{N}^{-1}\left[\frac{u^{\beta}}{s^{\beta}}\mathbb{N}^{+}\left[y_1 + A_1\right]\right] \\ &= \mathbb{N}^{-1}\left[\frac{u^{\beta}}{s^{\beta}}\mathbb{N}^{+}\left[\frac{t^{\beta}}{\Gamma(\beta+1)} + \frac{t^{\alpha}}{\Gamma(\alpha+1)}\right]\right] \\ &= \frac{t^{2\beta}}{\Gamma(2\beta+1)} + \frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}. \end{aligned}$$

We continue in this manner to get

$$x_3(t) = \tfrac{1}{8}\tfrac{t^{3\alpha}}{\Gamma(3\alpha+1)}$$
$$y_3(t) = \tfrac{t^{3\beta}}{\Gamma(3\beta+1)} + \tfrac{t^{\alpha+2\beta}}{\Gamma(\alpha+2\beta+1)} + \tfrac{1}{2}\tfrac{t^{2\alpha+\beta}}{\Gamma(2\alpha+\beta+1)} + \tfrac{t^{3\beta}}{\Gamma(3\beta+1)}\tfrac{\Gamma(2\beta+1)}{4(\Gamma(\beta+1))^2}.$$

Finally, the approximate solutions are given by

$$x(t) = 1 + \frac{1}{2}\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{1}{4}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{1}{8}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{1}{16}\frac{t^{4\alpha}}{\Gamma(4\alpha+1)} + \cdots$$
$$y(t) = 0 + \frac{t^{\beta}}{\Gamma(\beta+1)} + \frac{t^{2\beta}}{\Gamma(2\beta+1)} + \frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{t^{3\beta}}{\Gamma(3\beta+1)}$$
$$+ \frac{t^{\alpha+2\beta}}{\Gamma(\alpha+2\beta+1)} + \frac{1}{2}\frac{t^{2\alpha+\beta}}{\Gamma(2\alpha+\beta+1)} + \frac{t^{3\beta}}{\Gamma(3\beta+1)}\frac{\Gamma(2\beta+1)}{4(\Gamma(\beta+1))^2} + \cdots$$

Now when $\alpha = \beta = 1$, we get

$$x(t) = e^{\frac{t}{2}}, \quad y(t) = t\,e^{t}.$$

This is the exact solution of Eq. (5.1).

Remark 5.1. Clearly, from Figs. 1 and 2 below, the FNDM approximation and the exact solution are in excellent agreement for different values of α, β.

Example 5.2. Consider the nonlinear systems of fractional ordinary differential equations of the form

$$\begin{aligned} D^{\alpha}x(t) &= x(t) \\ D^{\beta}y(t) &= 2\,x^2(t), \qquad 0<\alpha, \quad \beta, \gamma \le 1 \\ D^{\gamma}z(t) &= 3\,x(t)\,y(t) \end{aligned} \tag{5.8}$$

subject to the initial conditions

$$x(0) = 1, \quad y(0) = 1, \quad z(0) = 0. \tag{5.9}$$

The exact solutions in the case when $\alpha = \beta = \gamma = 1$ are $x(t) = e^t$, $y(t) = e^{2t}$ and $z(t) = e^{2t} - 1$.

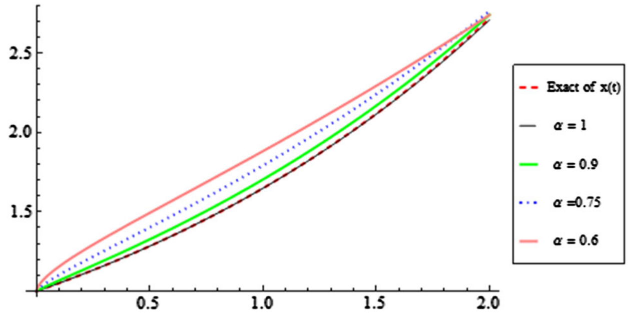

Figure 1. The approximate and exact solutions of $x(t)$ for Example 5.1 for different values of α when $0 < x < 2$

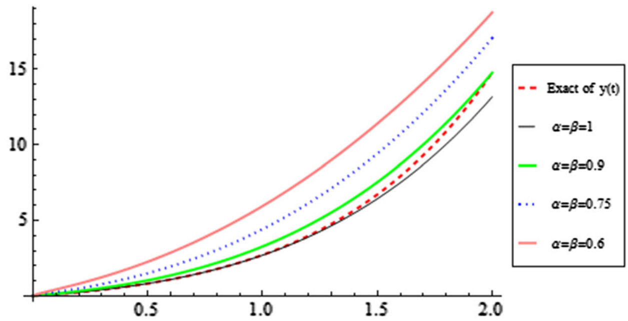

Figure 2. The approximate and exact solutions of $y(t)$ for Example 5.1 for different values of α, β when $0 < x < 2$

First, we apply the N-transform to Eq. (5.8) to get

$$\begin{aligned}\mathbb{N}^{+}\left[D^{\alpha}x(t)\right] &= \mathbb{N}^{+}\left[x(t)\right]\\ \mathbb{N}^{+}\left[D^{\beta}y(t)\right] &= 2\,\mathbb{N}^{+}\left[x^{2}(t)\right]\\ \mathbb{N}^{+}\left[D^{\gamma}z(t)\right] &= 3\,\mathbb{N}^{+}\left[x(t)y(t)\right].\end{aligned} \tag{5.10}$$

Apply Theorem 4.3 to Eq. (5.10) to get

$$\begin{aligned}\frac{s^{\alpha}}{u^{\alpha}}\mathbb{N}^{+}\left[x(t)\right]-\sum_{k=0}^{n-1}\frac{u^{\alpha-(k+1)}}{s^{\alpha-k}}\left[D^{k}x(t)\right]_{t=0} &= \mathbb{N}^{+}\left[x(t)\right]\\ \frac{s^{\beta}}{u^{\beta}}\mathbb{N}^{+}\left[y(t)\right]-\sum_{k=0}^{n-1}\frac{u^{\beta-(k+1)}}{s^{\beta-k}}\left[D^{k}y(t)\right]_{t=0} &= 2\,\mathbb{N}^{+}\left[x^{2}(t)\right]\\ \frac{s^{\gamma}}{u^{\gamma}}\mathbb{N}^{+}\left[z(t)\right]-\sum_{k=0}^{n-1}\frac{u^{\gamma-(k+1)}}{s^{\gamma-k}}\left[D^{k}z(t)\right]_{t=0} &= 3\,\mathbb{N}^{+}\left[x(t)y(t)\right].\end{aligned} \tag{5.11}$$

Substitute Eq. (5.9) into Eq. (5.11) to get

$$\begin{aligned}\mathbb{N}^{+}\left[x(t)\right] &= \frac{1}{s}+\frac{u^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\left[x(t)\right]\\ \mathbb{N}^{+}\left[y(t)\right] &= \frac{1}{s}+2\frac{u^{\beta}}{s^{\beta}}\mathbb{N}^{+}\left[x^{2}(t)\right]\\ \mathbb{N}^{+}\left[z(t)\right] &= 3\,\frac{u^{\gamma}}{s^{\gamma}}\mathbb{N}^{+}\left[x(t)y(t)\right].\end{aligned} \tag{5.12}$$

Take the inverse N-Transform of Eq. (5.12) to get

$$\begin{aligned}x(t) &= 1+\frac{1}{2}\mathbb{N}^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\left[x(t)\right]\right]\\ y(t) &= 1+2\mathbb{N}^{-1}\left[\frac{u^{\beta}}{s^{\beta}}\mathbb{N}^{+}\left[x^{2}(t)\right]\right]\\ z(t) &= 3\,\mathbb{N}^{-1}\left[\frac{u^{\gamma}}{s^{\gamma}}\mathbb{N}^{+}\left[x(t)y(t)\right]\right].\end{aligned} \tag{5.13}$$

Assume our approximate solutions are given by

$$x(t) = \sum_{n=0}^{\infty} x_n(t)$$
$$y(t) = \sum_{n=0}^{\infty} y_n(t)$$
$$z(t) = \sum_{n=0}^{\infty} z_n(t). \tag{5.14}$$

Note that the Adomian Polynomials are

$$x^2(t) = \sum_{n=0}^{\infty} A_n, \quad x(t)y(t) = \sum_{n=0}^{\infty} B_n.$$

Thus,

$$\begin{aligned}
&A_0 = x_0^2 B_0 = x_0 y_0\\
&A_1 = 2x_0x_1 B_1 = x_0y_1 + y_0x_1\\
&A_2 = 2x_0x_2 + x_1^2 B_2 = y_0x_2 + y_1x_1 + y_2x_0\\
&A_3 = 2x_0x_3 + 2x_1x_2 B_3 = y_0x_3 + y_1x_2 + y_2x_1 + y_3x_0.
\end{aligned}$$

Now from Eqs. (5.12) and (5.14) we conclude

$$\sum_{n=0}^{\infty} x_n(t) = 1 + \mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbb{N}^+\left[\sum_{n=0}^{\infty} x_n(t)\right]\right]$$
$$\sum_{n=0}^{\infty} y_n(t) = 1 + 2\mathbb{N}^{-1}\left[\frac{u^\beta}{s^\beta}\mathbb{N}^+\left[\sum_{n=0}^{\infty} A_n\right]\right]$$
$$\sum_{n=0}^{\infty} z_n(t) = 3\ \mathbb{N}^{-1}\left[\frac{u^\gamma}{s^\gamma}\mathbb{N}^+\left[\sum_{n=0}^{\infty} B_n\right]\right]. \tag{5.15}$$

Note that

$$A_0 = x_0^2$$
$$A_1 = 2x_0x_1$$
$$A_2 = 2x_0x_2 + x_1^2$$
$$A_3 = 2x_0x_3 + 2x_1x_2,$$

where $x_0(t) = 1$, $y_0(t) = 1$ and $z_0(t) = 0$.
Using Eq. (5.15) we can find the following components:

$$x_1(t) = \mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbb{N}^+[x_0]\right] = \frac{t^\alpha}{\Gamma(\alpha+1)}$$
$$y_1(t) = 2\mathbb{N}^{-1}\left[\frac{u^\beta}{s^\beta}\mathbb{N}^+[A_0(t)]\right] = 2\mathbb{N}^{-1}\left[\frac{u^\beta}{s^\beta}\frac{1}{s}\right] = 2\frac{t^\beta}{\Gamma(\beta+1)}$$
$$z_1(t) = 3\mathbb{N}^{-1}\left[\frac{u^\gamma}{s^\gamma}\mathbb{N}^+[B_0(t)]\right] = 3\mathbb{N}^{-1}\left[\frac{u^\gamma}{s^\gamma}\frac{1}{s}\right] = 3\frac{t^\gamma}{\Gamma(\gamma+1)}.$$

Similarly,

$$\begin{aligned} x_2(t) &= \mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbb{N}^+\left[x_1\right]\right] \\ &= \mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\frac{u^\alpha}{s^{\alpha+1}}\right] \\ &= \frac{t^{2\alpha}}{\Gamma\left(2\alpha+1\right)}. \end{aligned}$$

And

$$\begin{aligned} y_2(t) &= 2\mathbb{N}^{-1}\left[\frac{u^\beta}{s^\beta}\mathbb{N}^+\left[A_1(t)\right]\right] \\ &= 4\mathbb{N}^{-1}\left[\frac{u^\beta}{s^\beta}\frac{u^\alpha}{s^{\alpha+1}}\right] \\ &= 4\frac{t^{\beta+\alpha}}{\Gamma\left(\alpha+\beta+1\right)}. \\ z_2(t) &= 3\mathbb{N}^{-1}\left[\frac{u^\gamma}{s^\gamma}\mathbb{N}^+\left[B_1(t)\right]\right] \\ &= 3\mathbb{N}^{-1}\left[\frac{u^\gamma}{s^\gamma}\left[2\frac{u^\beta}{s^{\beta+1}}+\frac{u^\alpha}{s^{\alpha+1}}\right]\right] \\ &= 6\frac{t^{\gamma+\alpha_2}}{\Gamma\left(\gamma+\beta+1\right)}+3\frac{t^{\gamma+\alpha_1}}{\Gamma\left(\gamma+\alpha_1+1\right)}. \end{aligned}$$

We continue in this manner to get

$$\begin{aligned} x_3(t) &= \mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbb{N}^+\left[x_1(t)\right]\right] \\ &= \mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\frac{u^{2\alpha}}{s^{2\alpha+1}}\right] \\ &= \frac{t^{3\alpha}}{\Gamma\left(3\alpha+1\right)}. \\ y_3(t) &= 2\mathbb{N}^{-1}\left[\frac{u^\beta}{s^\beta}\mathbb{N}^+\left[A_1(t)\right]\right] \\ &= 4\mathbb{N}^{-1}\left[\frac{u^\beta}{s^\beta}\frac{u^{2\alpha_1}}{s^{2\alpha_1+1}}\right]+2\mathbb{N}^{-1}\left[\frac{u^{\beta+2\alpha}}{s^{\beta+2\alpha+1}}\frac{\Gamma\left(2\alpha+1\right)}{\left(\Gamma\left(\alpha+1\right)\right)^2}\right] \\ &= 4\frac{t^{\beta+2\alpha}}{\Gamma\left(\beta+2\alpha+1\right)}+2\frac{\Gamma\left(2\alpha+1\right)}{\left(\Gamma\left(\alpha+1\right)\right)^2}\frac{t^{\beta+2\alpha}}{\Gamma\left(\beta+2\alpha+1\right)}. \\ z_3(t) &= 3\mathbb{N}^{-1}\left[\frac{u^\gamma}{s^\gamma}\mathbb{N}^+\left[\frac{t^{2\alpha}}{\Gamma\left(2\alpha+1\right)}+\frac{2t^{\alpha+\beta}}{\Gamma\left(\alpha+1\right)\Gamma\left(\beta+1\right)}+4\frac{t^{\alpha+\beta}}{\Gamma\left(\alpha+\beta+1\right)}\right]\right] \\ &= 3\frac{t^{2\alpha+\gamma}}{\Gamma\left(2\alpha+\gamma+1\right)}+6\frac{\Gamma\left(\alpha+\beta+1\right)t^{\alpha+\beta+\gamma}}{\Gamma\left(\alpha+1\right)\Gamma\left(\beta+1\right)\Gamma\left(\gamma+\beta+1\right)} \\ &\quad+12\frac{t^{\alpha+\beta+\gamma}}{\Gamma\left(\alpha+\beta+\gamma+1\right)}. \end{aligned}$$

Finally, the approximate solution is given by

$$x(t) = 1 + \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{t^{4\alpha}}{\Gamma(4\alpha+1)} + \cdots$$

$$y(t) = 1 + \frac{2t^{\beta}}{\Gamma(\beta+1)} + \frac{4t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{4t^{2\alpha+\beta}}{\Gamma(2\alpha+\beta+1)} + \frac{2\Gamma(2\beta+1)\,t^{\alpha+\beta}}{(\Gamma(\alpha+1))^2\,\Gamma(2\alpha+\beta+1)} + \frac{4t^{3\alpha+\beta}}{\Gamma(3\alpha+\beta+1)} + \frac{4\Gamma(3\alpha+1)\,t^{3\alpha+\beta}}{\Gamma(\alpha+1)\,\Gamma(2\alpha+1)\,\Gamma(3\alpha+\beta+1)} + \frac{t^{\alpha+3\beta}}{\Gamma(\alpha+3\beta+1)} + \cdots z$$

$$z(t) = \frac{3t^{\gamma}}{\Gamma(\alpha_3+1)} + \frac{6t^{\beta+\gamma}}{\Gamma(\alpha_3+\beta+1)} + \frac{3t^{\alpha+\gamma}}{\Gamma(\alpha_3+\alpha_1+1)} + \frac{3t^{2\alpha+\gamma}}{\Gamma(2\alpha+\gamma+1)} + \frac{6\Gamma(\alpha+\beta+1)\,t^{\alpha+\beta+\gamma}}{\Gamma(\alpha+1)\,\Gamma(\beta+1)\,\Gamma(\alpha+\beta+\gamma+1)} + \frac{12t^{\alpha+\beta+\gamma}}{\Gamma(\alpha+\beta+\gamma+1)} + \cdots$$

Now when $\alpha = \beta = \gamma = 1$, we get

$$x(t) = e^{t}, \quad y(t) = e^{2t}, \quad z(t) = e^{3t} - 1.$$

This is the exact solution of Eq. (5.8).

Remark 5.2. Clearly, from Figs. 3, 4 and 5 below, the FNDM approximation and the exact solution are in excellent agreement for different values of α, β. Finally, from Fig. 5 below, the FNDM approximation and the exact solution are in excellent agreement for different values of α, β, γ.

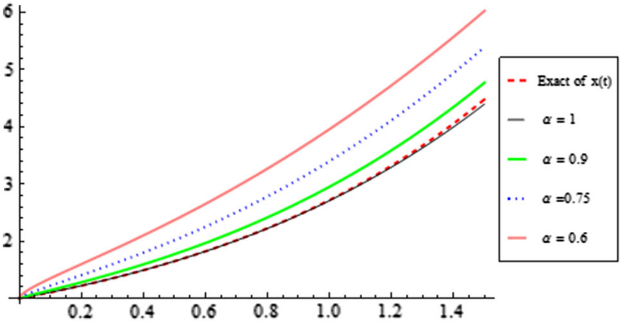

Figure 3. The approximate and exact solutions of $x(t)$ for Example 5.2 for different values of α when $0 < x < 2$

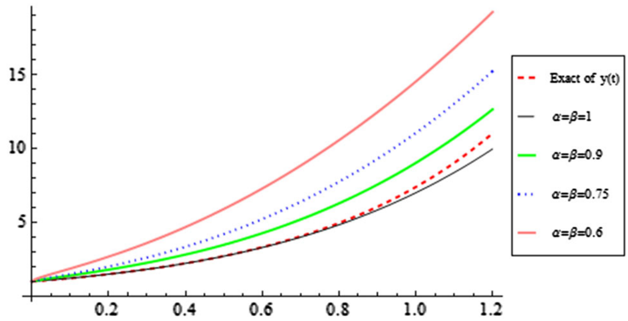

Figure 4. The approximate and exact solutions of $y(t)$ for Example 5.2 for different values of α, β when $0 < x < 2$

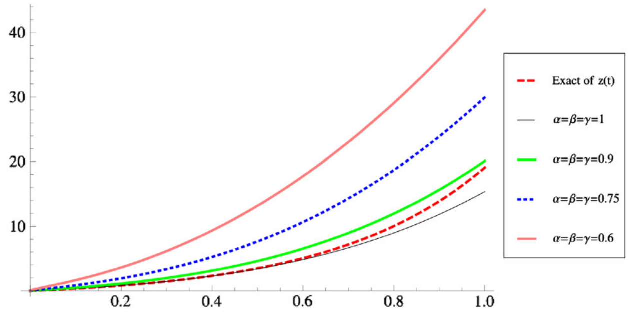

Figure 5. The approximate and exact solutions of $z(t)$ for Example 5.2 for different values of α, β, γ when $0 < x < 2$

6. Numerical Tables

In this section, we shall illustrate the accuracy and efficiency of the FNDM by comparing the approximate and exact solutions. In Tables 1, 2 we consider the same values of t for $x(t)$ and $y(t)$, specifically, $t = \{0, 0.5, 1, 1.5, 2\}$. Also, In Tables 3, 4 and 5 we consider the same values of t for $x(t)$, $y(t)$ and $z(t)$, specifically, $t = \{0.2, 0.4, 0.6, 0.8, 1\}$.

Table 1. The approximate and exact solution of $x(t)$ for Example 5.1 with $n = 6$ for different values of α

t	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 1$	
			Approximate	Exact
0	1	1	1	1
0.5	1.565	1.40158	1.28402	1.28403
1	1.93947	1.79155	1.64844	1.64872
1.5	2.30905	2.24013	2.11475	2.117
2	2.68885	2.76345	2.70833	2.71828

Table 2. The approximate and exact solution of $y(t)$ for Example 5.1 with $n = 6$ for values of α and β

t	$\alpha = \beta = 0.5$	$\alpha = \beta = 0.75$	$\alpha = \beta = 1$	
			Approximate	Exact
0	0	0	0	0
0.5	3.02474	1.50472	0.82487	0.824361
1	7.15762	4.42537	2.69792	2.71828
1.5	12.5724	9.41998	6.43945	6.72253
2	19.2289	17.0667	13.1667	14.7781

Table 3. The approximate and exact solutions of $x(t)$ for Example 5.2 with $n = 6$ for different values of α

t	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 1$	
			Approximate	Exact
0.2	1.79191	1.40452	1.2214	1.2214
0.4	2.38396	1.79815	1.49173	1.49182
0.6	3.00365	2.25167	1.8214	1.82212
0.8	3.66752	2.78142	2.2224	2.22554
1	4.38063	3.39926	2.70833	2.71828

Table 4. The approximate and exact solutions of $y(t)$ for Example 5.2 with $n = 6$ for values of α and β

t	$\alpha = \beta = 0.5$	$\alpha = \beta = 0.75$	$\alpha = \beta = 1$	
			Approximate	Exact
0.2	3.44972	2.01141	1.49173	1.49182
0.4	6.07314	3.34422	2.2224	2.22554
0.6	9.23683	5.21482	3.2944	3.32012
0.8	12.9423	7.74447	4.83573	4.95303
1	17.1814	11.0466	7	7.38906

Table 5. The approximate and exact solutions of $z(t)$ for Example 5.2 with $n = 6$ for values of α, β, γ

t	$\alpha = \beta = \gamma = 0.5$	$\alpha = \beta = \gamma = 0.75$	$\alpha = \beta = \gamma = 1$	
			Approximate	Exact
0.2	5.82993	1.92092	0.8214	0.822119
0.4	14.0205	5.23594	2.2944	2.32012
0.6	24.8723	10.6418	4.8294	5.04965
0.8	38.2985	18.6998	8.9664	10.0232
1	54.2354	29.9561	15.375	19.0855

7. Conclusion

In this work, the FNDM has been successfully applied to construct approximate solutions for nonlinear fractional systems of ordinary differential equations. The FNDM provides the solution in terms of convergent series with easily computable components. We successfully found exact solutions to both example 1, in the case when $\alpha = \beta = 1$ and example 2 in the case when $\alpha = \beta = \gamma = 1$. The FNDM is effective and simple to solve fractional nonlinear systems of ODES. Our goal in the future is to apply the FNDM to other systems of fractional differential equations that arise in other areas of science.

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