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Fredholm Spectra and Weyl Type Theorems for Drazin Invertible Operators

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Abstract. In this paper we investigate the relationship between some spectra originating from Fredholm theory of a Drazin invertible operator and its Drazin inverse, if this does exist. Moreover, we study the transmission of Weyl type theorems from a Drazin invertible operator R, to its Drazin inverse S.

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1. Introduction

Let $L(X)$ be the Banach algebra of all bounded linear operators on an infinite-dimensional complex Banach space X. If $T \in L(X)$, we denote by $\sigma(T)$ the spectrum of T, and we set $\alpha(T) := \dim \ker T$ and $\beta(T) := \operatorname{codim} T(X)$. A simple consequence of the spectral mapping theorem for the spectrum shows that if $T \in L(X)$ is invertible then the points of the spectrum of its inverse T^{-1} are the reciprocals of the spectrum $\sigma(T)$. If T is invertible, also some other spectra that originated from Fredholm theory satisfy this relationship of reciprocity, see next Corollary 2.6.

In literature, the concept of invertibility for an operator $T \in L(X)$ admits several generalizations and has some interest to investigate the relationships between the spectral properties of T and the spectral properties of a "generalized inverse" of T, if this exists. For instance, the relationship of "reciprocity" mentioned above between the nonzero parts of the spectrum has been also observed in the case that the "generalized inverse" is given in the sense of *Drazin invertibility*, while this relation of reciprocity between the nonzero points of spectrum of T and the nonzero points of spectrum of any of its "pseudo-inverses" may fail if we consider the concept of *relatively regular* operators, see [13, p. 53] for definition and details. The concept of Drazin invertibility has been introduced by Drazin in the more abstract setting of

Birkhäuser

Banach algebras [14]. In the case of the Banach algebra $L(X)$, an operator $R \in L(X)$ is said to be *Drazin invertible* (with a finite index) if there exists an operator $S \in L(X)$ and $n \in \mathbb{N}$ such that

$$RS = SR, \quad SRS = S, \quad R^n SR = R^n, \tag{1}$$

see [17, Chap. 3, Theorem 10]. The operator S is called *Drazin inverse* of R. The smallest nonnegative integer ν such that (1) holds is called the index $i(R)$ of R. Recall that the *ascent* of an operator $T \in L(X)$ is defined as the smallest non-negative integer $p := p(T)$ such that $\ker T^p = \ker T^{p+1}$. If such integer does not exist we set $p(T) = \infty$. Analogously, the *descent* of T is defined as the smallest non-negative integer $q := q(T)$ such that $T^q(X) = T^{q+1}(X)$, and if such integer does not exist we set $q(T) = \infty$. A classical result establishes that if $p(T)$ and $q(T)$ are both finite, then $p(T) = q(T)$, see [1, Theorem 3.3]. An operator $R \in L(X)$ is Drazin invertible if and only if $p(R) = q(R) < \infty$, see [17, Chap. 3, Theorem 10]. Evidently, an invertible operator R is Drazin invertible with Drazin inverse $S := R^{-1}$, while if $0 \in \sigma(R)$ then R is Drazin invertible if and only if 0 is a pole of the resolvent of R. In this case $i(R)$ is the order of the pole 0, i.e. $i(R) = p(R) = q(R)$, see [13, § 5.2]. From [17, Chap. 3, Theorem 10] we also know that if $R \in L(X)$ is Drazin invertible if and only if there exist two closed invariant subspaces Y and Z such that $X = Y \oplus Z$ and, with respect to this decomposition,

$$R = R_1 \oplus R_2, \quad \text{with} \quad R_1 := R|Y \text{ nilpotent} \quad \text{and} \quad R_2 := R|Z \text{ invertible}. \tag{2}$$

Note that the Drazin inverse S of an operator, if it exists, is uniquely determined ([13]), and with respect to the decomposition $X = Y \oplus Z$, the Drazin inverse S may be represented as the directed sum

$$S := 0 \oplus S_2 \quad \text{with} \quad S_2 := {R_2}^{-1}. \tag{3}$$

The decompositions (2) and (3) are very useful to study the spectral properties of a Drazin invertible operator; in particular, the decomposition (3) shows that the Drazin inverse S is itself Drazin invertible, since is the direct sum of the nilpotent operator 0 and the invertible operator S_2. It should be noted that if $0 \in \sigma(R)$, then $0 \in \sigma(S)$ and 0 is a pole of the first order of the resolvent of S, see [18]. Furthermore, the following relationship of reciprocity holds for the spectra of S and R:

$$\sigma(S)\setminus\{0\} = \left\{\frac{1}{\lambda} : \lambda \in \sigma(R)\setminus\{0\}\right\}, \tag{4}$$

see [8].

The structure of the spectrum of the Drazin inverse S of a Drazin invertible operator R is strongly related to the structure of the spectrum of R. In [8] has been studied the transmission of several local spectral properties from a Drazin invertible operator R to its Drazin inverse S. In this paper we show that the nonzero points of some other spectra of R and S, that originated from Fredholm theory, satisfy a relationship of reciprocity. This relationship will be proved in the more general context of the spectrum generated by a regularity. Furthermore, we shall show that if a Drazin invertible operator R

is algebraic then also its Drazin inverse S is algebraic and analogously, if R is a Riesz operator then also S is Riesz. In the last section, we also study the transmission of Browder type theorems and Weyl type theorems from a Drazin invertible operator R to its Drazin inverse S.

2. Preliminary Results

Given a bounded linear operator $T \in L(X)$, the *local resolvent set* $\rho_T(x)$ of T at a point $x \in X$ is defined as the union of all open subsets $\mathcal{U}$ of $\mathbf{C}$ such that there exists an analytic function $f : \mathcal{U} \to X$ satisfying

$$(\lambda I - T)f(\lambda) = x \quad \text{for all} \quad \lambda \in \mathcal{U}. \tag{5}$$

The local spectrum $\sigma_T(x)$ of T at x is the set defined by $\sigma_T(x) := \mathbf{C} \setminus \rho_T(x)$. Obviously, $\sigma_T(x) \subseteq \sigma(T)$.

An operator $T \in L(X)$ is said to have *the single valued extension property* at $\lambda_0 \in \mathbf{C}$ (abbreviated SVEP at λ_0), if for every open disc $\mathbf{D}_{\lambda_0}$ centered at λ_0 the only analytic function $f : \mathbf{D}_{\lambda_0} \to X$ which satisfies the equation

$$(\lambda I - T)f(\lambda) = 0 \tag{6}$$

is the function $f \equiv 0$. Evidently, every operator T has SVEP at the isolated points of $\sigma(T)$. An operator $T \in L(X)$ is said to have the SVEP if T has the SVEP at every point $\lambda \in \mathbf{C}$. It is easily seen from definition that the SVEP is inherited by restrictions to invariant closed subspaces. The SVEP for T is equivalent to saying that $\sigma_T(x) = \emptyset$ if and only if $x = 0$, see [15, Proposition 1.2.16]. Note that

$$p(\lambda I - T) < \infty \Rightarrow T \quad \text{has SVEP at } \lambda,$$

and, by duality,

$$q(\lambda I - T) < \infty \Rightarrow T^* \quad \text{has SVEP at } \lambda,$$

see [1, Chap. 3].

The localized SVEP satisfies a spectral mapping theorem, see [1, Theorem 2.39]:

Theorem 2.1. *Let $T \in L(X)$ and suppose that f is an analytic function on the open neighborhood U of $\sigma(T)$ such that f is non-constant on each of the connected components of U. Then $f(T)$ has the SVEP at $\lambda \in \mathbb{C}$ if and only if T has the SVEP at every point $\mu \in \sigma(T)$ for which $f(\mu) = \lambda$.*

Corollary 2.2. *Let $\lambda_0 \neq 0$ and suppose that T is invertible. Then T^{-1} has SVEP at $1/\lambda_0$.*

Proof. Let $g(\lambda) = \frac{1}{\lambda}$. Since $0 \notin \sigma(T)$, there is an open neighborhood U containing the spectrum such that $0 \notin U$ and obviously g is analytic on D. Since T has SVEP at λ_0, by Theorem 2.1, $g(T) = T^{-1}$ has SVEP at $\frac{1}{\lambda_0}$. □

In [8] it has been proved that the SVEP for a Drazin invertible operator R is transmitted to its Drazin inverse S. We give now a localized version of this result.

Theorem 2.3. *Suppose that $R \in L(X)$ is Drazin invertible with Drazin inverse S. Then R has SVEP at $\lambda_0 \neq 0$ if and only if S has SVEP at $\frac{1}{\lambda_0}$.*

Proof. Let $X = Y \oplus Z$, $R = R_1 \oplus R_2$ and $S = 0 \oplus S_2$, where $S_2 = {R_2}^{-1}$. Suppose that R has SVEP at λ_0. Then $R_2 = R|Z$ has SVEP at λ_0, since the localized SVEP is inherited by the restriction on invariant closed subspaces. By Corollary 2.2, S_2 has SVEP at $\frac{1}{\lambda_0}$. Since the null operator has SVEP at every point, by [1, Theorem 2.9], S has SVEP at $\frac{1}{\lambda_0}$. The reverse is proved similarly, since every nilpotent operator has SVEP. Let $\mathcal{A}$ be an unital Banach algebra with unit u. □

Definition 2.4. *A non-empty subset $\mathcal{K}$ of $\mathcal{A}$ is said to be a* regularity *if the following conditions are satisfied:*

(i) $a \in \mathcal{K} \Leftrightarrow a^n \in \mathcal{K}$ *for all* $n \in \mathbb{N}$.

(ii) *If a, b, c, d are mutually commuting elements of $\mathcal{A}$ and $ac + bd = u$ then*

$$ab \in \mathcal{K} \Leftrightarrow a \in \mathcal{K} \quad \text{and} \quad b \in \mathcal{K}.$$

Let us now consider the Banach algebra $\mathcal{A} = L(X)$ and let $\mathcal{H}(\sigma(T))$ denote the space of all analytic functions defined on an open neighborhood U of $\sigma(T)$ which are non-constant on each component of its domain of definition. Denote by

$$\sigma_{\mathcal{K}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin \mathcal{K}\},$$

the *spectrum corresponding to the regularity* $\mathcal{K}$ in $L(X)$. It should be noted that $\sigma_{\mathcal{K}}(T) \subseteq \sigma(T)$ for every regularity $\mathcal{K}$ and every $T \in L(X)$. The proof of the following theorem may be found in [17]:

Theorem 2.5. *Let $\mathcal{K}$ be a regularity in $L(X)$. Then*

$$\sigma_{\mathcal{K}}(f(T)) = f(\sigma_{\mathcal{K}}(T))$$

for every $T \in L(X)$ and every $f \in \mathcal{H}(\sigma(T))$.

The axioms of regularity are usually rather easy to verify and there are many classes of operators in Fredholm theory which satisfy them. An excellent survey concerning the regularity of various classes of bounded linear operators in Banach spaces may be found in [16] and [17].

In the sequel we always assume the nontrivial case $\mathcal{K} \neq L(X)$. It should be noted that $\sigma_{\mathcal{K}}(T)$ may be empty. For instance, if T is *algebraic* (i.e., there exists a non-trivial polynomial h such that $h(T) = 0$) and $\mathcal{K}$ is the class of Drazin invertible operators (this class is a regularity, see [10]), then the Drazin spectrum is empty, since $\sigma(T)$ is a finite set of poles, see [1, Theorem 3.83]. In particular, $\sigma_{\mathcal{K}}(N) = \emptyset$ for every nilpotent operator N, since N is algebraic.

Corollary 2.6. *If $T \in L(X)$ is invertible and $\mathcal{K}$ is a regularity, then*

$$\sigma_{\mathcal{K}}(T^{-1}) = \left\{\frac{1}{\lambda} : \lambda \in \sigma_{\mathcal{K}}(T)\right\}, \tag{7}$$

Proof. Suppose that $T \in L(X)$ is invertible. Then $0 \notin \sigma(T)$. Consider the function $f(\lambda) = \frac{1}{\lambda}$ defined on an open neighborhood U of $\sigma(T)$ which does not contain 0. Then $f(T) = T^{-1}$, so, by Theorem 2.5, the statement follows. □

Let $\mathcal{K}$ be a regularity in $L(X)$ and suppose that $X = X_1 \oplus X_2$ where, $X_1 \neq \{0\}$ and $X_2 \neq \{0\}$. If $T \in L(X)$ and X_i are invariant under T, define $T_1 := T|X_1$, $T_2 := T|X_2$. Write

$$\mathcal{K}_1 := \{T_1 \in L(X_1) : T_1 \oplus I \in \mathcal{K}\},$$

and analogously,

$$\mathcal{K}_2 := \{T_2 \in L(X_2) : I \oplus T_2 \in \mathcal{K}\}.$$

Then $\mathcal{K}_1$ and $\mathcal{K}_2$ are regularities in $L(X_1)$ and $L(X_2)$, respectively. Further, assume that $\mathcal{K}$ satisfies the following condition: (I) $\sigma_{\mathcal{K}_1}(T_1) \neq \emptyset$ for all $T_1 \in L(X_1)$ and $\mathcal{K}_1 \neq L(X_1)$. If $\mathcal{K}$ satisfies the condition (I) we have $\sigma_{\mathcal{K}}(T) = \sigma_{\mathcal{K}_1}(T_1) \cup \sigma_{\mathcal{K}_2}(T_2)$, see [17, p. 53].

Theorem 2.7. *Let $\mathcal{K}$ be a regularity in $L(X)$ which satisfies the condition (I). Let $R \in L(X)$ be a Drazin invertible operator with Drazin inverse S; then*

$$\sigma_{\mathcal{K}}(S)\backslash 0 = \{1/\lambda : \lambda \in \sigma_{\mathcal{K}}(R)\backslash\{0\}\}. \tag{8}$$

Proof. Observe first that if R is either invertible or nilpotent, then the equality (8) holds for every regularity. Indeed, if R is invertible then $S = R^{-1}$, so (8) follows from Corollary 2.6. If R is nilpotent then $S = 0$ and $\sigma_{\mathcal{K}}(R)$, as well as $\sigma_{\mathcal{K}}(S)$, are subsets of $\{0\}$, so the right-hand side and the left-hand side in (8) are both empty.

Suppose that $0 \in \sigma(R)$ (and hence $0 \in \sigma(S)$) and that R is not nilpotent. Then in the decomposition $X = Y \oplus Z$, $R_1 = R|Y$, $R_2 = R|Z$, with R_1 nilpotent and R_2 invertible, we have $Y \neq \{0\}$ and $Z \neq \{0\}$. If $\mathcal{K}$ is a regularity in $L(X)$, let $\mathcal{K}_1$ and $\mathcal{K}_2$ be as above. Since R_1 is nilpotent and, by assumption, $\sigma_{\mathcal{K}_1}(R_1) \neq \emptyset$, $\sigma_{\mathcal{K}_1}(R_1) = \{0\}$, while $0 \notin \sigma_{\mathcal{K}_2}(R_2)$, since $0 \notin \sigma(R_2)$. Therefore,

$$\sigma_{\mathcal{K}}(R) = \sigma_{\mathcal{K}_1}(R_1) \cup \sigma_{\mathcal{K}_2}(R_2) = \{0\} \cup \sigma_{\mathcal{K}_2}(R_2),$$

and hence $\sigma_{\mathcal{K}}(R)\backslash\{0\} = \sigma_{\mathcal{K}_2}(R_2)$. Analogously, $\sigma_{\mathcal{K}}(S)\backslash\{0\} = \sigma_{\mathcal{K}_2}(S_2)$. In view of the equality (7), we then have

$$\sigma_{\mathcal{K}}(S)\backslash\{0\} = \sigma_{\mathcal{K}_2}(S_2) = \left\{\frac{1}{\lambda} : \lambda \in \sigma_{\mathcal{K}_2}(R_2)\right\}$$
$$= \left\{\frac{1}{\lambda} : \lambda \in \sigma_{\mathcal{K}}(R)\backslash\{0\}\right\},$$

as desired. □

The following theorem shows that the nonzero poles of the resolvent of a Drazin invertible S operator are the reciprocals of the nonzero poles of its Drazin inverse. Note that the equality (iii) in [18] has been observed only for the closure of the ranges.

Theorem 2.8. *Let $\lambda \neq 0$.*

(i) *If* $T \in L(X)$ *is invertible, then*

$$(\lambda I - T)^k(X) = \left(\frac{1}{\lambda}I - T^{-1}\right)^k (X) \quad \textit{for all} \quad k \in \mathbb{N}.$$

(ii) *If* R *is Drazin invertible with Drazin inverse* S*, then*

$$\ker(\lambda I - S)^k = \ker\left(\frac{1}{\lambda}I - R\right)^k \quad \textit{for all} \quad k \in \mathbb{N}.$$

(iii) *If* R *is Drazin invertible with Drazin inverse* S*, then*

$$(\lambda I - S)^k(X) = \left(\frac{1}{\lambda}I - R\right)^k (X) \quad \textit{for all} \quad k \in \mathbb{N}.$$

(iv) *If* R *is Drazin invertible with Drazin inverse* S*, then* λ *is a pole of the resolvent of* R *if and only if* $\frac{1}{\lambda}$ *is a pole of the resolvent of* S*.*

Proof.

(i) Let $y = (\lambda I - T)^k x$. Then

$$\left(\frac{1}{\lambda}I - T^{-1}\right)^k T^k x = \left(\frac{1}{\lambda}T^k - I\right)^k x = \left(-\frac{1}{\lambda}\right)^k y,$$

so $(\lambda I - T)^k(X) \subseteq (\frac{1}{\lambda}I - T^{-1})^k(X)$. The reverse inclusion follows by symmetry.

(ii) See [8].

(iii) Let $X = Y \oplus Z$, $R = R_1 \oplus R_2$ and $S = 0 \oplus S_2$ with $S_2 = R_2^{-1}$. Since R_1 is nilpotent then $\frac{1}{\lambda}I - R_1$ is invertible, and hence $(\frac{1}{\lambda}I - R_1)^k(Y) = Y$. Hence

$$\left(\frac{1}{\lambda}I - R\right)^k (X) = \left(\frac{1}{\lambda}I - R_1\right)^k (Y) \oplus \left(\frac{1}{\lambda}I - R_2\right)^k (Z) = Y \oplus \left(\frac{1}{\lambda}I - R_2\right)^k (Z),$$

and analogously

$$(\lambda I - S)^k(X) = Y \oplus (\lambda I - S_2)^k(Z).$$

From part (i) we have $(\lambda I - S_2)^k(Z) = (\frac{1}{\lambda}I - R_2)^k(Z)$, so

$$\left(\frac{1}{\lambda}I - R\right)^k (X) = Y \oplus (\lambda I - S_2)^k(Z) = (\lambda I - S)^k(X).$$

(iv) From part (ii) and part (iii) we have $p(\lambda I - R) = p(\frac{1}{\lambda}I - S)$ and $q(\lambda I - R) = q(\frac{1}{\lambda}I - S)$.

□

Recall that an operator T is algebraic if and only if the spectrum of T is a finite set of poles of the resolvent [1, Theorem 3.83]. Obviously, every algebraic operator T is Drazin invertible.

Corollary 2.9. *If* $T \in L(X)$ *is algebraic, then its Drazin inverse is also algebraic.*

Proof. Let S be the Drazin inverse of T. Since $\sigma(T)$ is a finite set, from (4) it then follows that also $\sigma(S)$ is a finite set. We show that every point of $\sigma(S)$ is a pole of the resolvent. If $0 \in \sigma(S)$ then, since S is Drazin invertible, 0 is a pole (of the first order) of the resolvent of S. Let $0 \neq \lambda \in \sigma(S)$. Then $\frac{1}{\lambda} \in \sigma(T)$ and hence $\frac{1}{\lambda}$ is a pole of the resolvent of T. From part (iv) of Theorem 2.8 it then follows that λ is a pole of the resolvent of S. □

3. Weyl and Browder Spectra

Let $T \in L(X)$ be a bounded linear operator defined on an infinite-dimensional complex Banach space X. Let

$$\Phi_+(X) := \{T \in L(X) : \alpha(T) < \infty \quad \text{and} \quad T(X) \text{ is closed}\}$$

denote the class of all *upper semi-Fredholm* operators, and let

$$\Phi_-(X) := \{T \in L(X) : \beta(T) < \infty\}$$

denote the class of all *lower semi-Fredholm* operators. If $T \in \Phi_\pm(X) := \Phi_+(X) \cup \Phi_-(X)$, the *index* of T is defined by ind $(T) := \alpha(T) - \beta(T)$. If $\Phi(X) := \Phi_+(X) \cap \Phi_-(X)$ denotes the set of all *Fredholm* operators, the set of *Weyl operators* is defined by

$$W(X) := \{T \in \Phi(X) : \operatorname{ind} T = 0\},$$

the class of *upper semi-Weyl operators* is defined by

$$W(X) := \{T \in \Phi(X) : \operatorname{ind} T = 0\},$$

$$W_+(X) := \{T \in \Phi_+(X) : \operatorname{ind} T \leq 0\},$$

and class of *lower semi-Weyl operators* is defined by

$$W_-(X) := \{T \in \Phi_-(X) : \operatorname{ind} T \geq 0\}.$$

Clearly, $W(X) = W_+(X) \cap W_-(X)$. The classes of operators above defined generate the following spectra: the *Weyl spectrum*, defined by

$$\sigma_{\mathrm{w}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin W(X)\};$$

and the *upper semi-Weyl spectrum*, defined by

$$\sigma_{\mathrm{uw}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin W_+(X)\}.$$

The class of all *Browder operators* is defined

$$B(X) := \{T \in \Phi(X) : p(T), q(T) < \infty\};$$

while the class of all *upper semi-Browder operators* is defined

$$B_+(X) := \{T \in \Phi_+(X) : p(T) < \infty\}.$$

The *Browder spectrum* is denoted by $\sigma_{\mathrm{b}}(T)$, while the *upper semi-Browder spectrum* is denoted by $\sigma_{\mathrm{ub}}(T)$. Obviously, $B(X) \subseteq W(X)$ and $B_+(X) \subseteq W_+(X)$, see [1, Theorem 3.4], so $\sigma_{\mathrm{w}}(T) \subseteq \sigma_{\mathrm{b}}(T)$ and $\sigma_{\mathrm{uw}}(T) \subseteq \sigma_{\mathrm{ub}}(T)$.

In the sequel we denote by $\sigma_{\rm a}(T)$ the *approximate point spectrum*, defined by

$$\sigma_{\rm a}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \quad \text{is not bounded below}\},$$

where $T \in L(X)$ is said to be *bounded below* if it is injective and has closed range.

Remark 3.1. It should be noted that, by Theorem 3.4 of [1], if R is Drazin invertible then

$$R \text{ is upper semi-Weyl} \Leftrightarrow R \text{ is Weyl } \Leftrightarrow R \text{ is Browder.}$$

Theorem 3.2. *Suppose that $R \in L(X)$ is Drazin invertible with Drazin inverse S. Then we have*

(i) *R is Browder if and only if S is Browder.*
(ii) $\sigma_{\rm b}(S)\setminus\{0\} = \{\frac{1}{\lambda} : \lambda \in \sigma_{\rm b}(R)\setminus\{0\}\}$.
(iii) $\sigma_{\rm ub}(S)\setminus\{0\} = \{\frac{1}{\lambda} : \lambda \in \sigma_{\rm ub}(R)\setminus\{0\}\}$.

Proof.

(i) If $0 \notin \sigma(R)$, then R is invertible and the Drazin inverse is $S = R^{-1}$ so the assertion is trivial in this case. Suppose that $0 \in \sigma(R)$ and that R is Browder. Then 0 is a pole of the resolvent of R and is also a pole (of the first order) of the resolvent of S. Let $X = Y \oplus Z$ such that $R = R_1 \oplus R_2$, $R_1 = R|Y$ nilpotent and $R_2 = R|Z$ invertible. Observe that

$$\ker R = \ker R_1 \oplus \ker R_2 = \ker R_1 \oplus \{0\}, \tag{9}$$

and, analogously, since $S = 0 \oplus S_2$ with $S_2 = {R_2}^{-1}$, we have

$$\ker S = \ker 0 \oplus \ker S_2 = Y \oplus \{0\}. \tag{10}$$

Since R is Browder we have $\alpha(R) = \dim \ker R < \infty$, and from the inclusion $\ker R_1 \subseteq \ker R$ it then follows that $\alpha(R_1) < \infty$. Consequently, $\alpha(R_1^n) < \infty$ for all $n \in \mathbb{N}$. Let $R_1^\nu = 0$. Since $Y = \ker R_1^\nu$ we then conclude that the subspace Y is finite-dimensional and hence $\ker S = Y \oplus \{0\}$ is finite-dimensional, i.e. $\alpha(S) < \infty$. Now, S is Drazin invertible, so $p(S) = q(S) < \infty$ and hence, by [1, Theorem 3.4], $\alpha(S) = \beta(S) < \infty$. Hence S is Browder.

Conversely, suppose that S is Browder. Then $\alpha(S) < \infty$ and hence by (10) the subspace Y is finite-dimensional, from which it follows that also $\ker R_1 = \ker R|Y$ is finite-dimensional. From (9) we then have that $\alpha(R) < \infty$ and since $p(R) = q(R) < \infty$ we then conclude that $\alpha(R) = \beta(R)$, again by [1, Theorem 3.4]. Therefore, R is a Browder operator.

(ii) The class of Browder operators is a regularity and the spectrum $\sigma_{\rm b}(T)$ is non-empty for all $T \in L(X)$. Hence, from Theorem 2.7, the equality (ii) holds.

(iii) Also the class of upper semi-Browder operators is a regularity and the spectrum $\sigma_{\rm ub}(T)$ is non-empty for all $T \in L(X)$. Again, from Theorem 2.7, the equality (iii) holds. □

Recall that $T \in L(X)$ is said to be a *Riesz operator* if $\lambda I - T \in \Phi(X)$ for all $\lambda \neq 0$, or equivalently if $\lambda I - T$ is Browder for all $\lambda \neq 0$, see [1, Theorem 3.111]. A Drazin invertible operator R which is also Riesz is evidently algebraic, (since 0 is a pole, and hence an isolated point of the spectrum, so $\sigma(R)$ is a finite set of poles), but the converse is not true.

Corollary 3.3. *If a Drazin invertible operator $R \in L(X)$ is a Riesz operator, then its Drazin inverse is also Riesz.*

Proof. Since X is infinite dimensional and R is Riesz, then $\sigma_{\rm b}(R) = \{0\}$. Suppose that the Drazin inverse S is not Riesz. Then there exists $0 \neq \lambda$ such that $\lambda \in \sigma_{\rm b}(S)$. From part (ii) of Theorem 3.2 then $0 \neq \frac{1}{\lambda} \in \sigma_{\rm b}(R)$, a contradiction. □

The spectral theorem may fail for the Weyl spectrum $\sigma_{\rm w}(T)$, see [1, Example 3.64]. However we have, by [1, Theorem 3.63],

$$\sigma_{\rm w}(f(T)) \subseteq f(\sigma_{\rm w}(T)) \quad \text{for all} \quad T \in L(X). \tag{11}$$

Note that if $T \in L(X)$ is invertible, then $0 \notin \sigma_{\rm w}(T)$ and $0 \notin \sigma_{\rm w}(T^{-1})$. Although the spectral mapping theorem does not hold for $\sigma_{\rm w}(T)$ we show that for a Drazin invertible operator R, the relationship of reciprocity between the nonzero parts of the $\sigma_{\rm w}(R)$ and the Weyl spectrum of its Drazin inverse $\sigma_{\rm w}(S)$ is still true. We need first the following Lemma:

Lemma 3.4. *Suppose that $T \in L(X)$ is invertible. Then*

$$\sigma_{\rm w}(T^{-1}) = \left\{ \frac{1}{\lambda} : \lambda \in \sigma_{\rm w}(T) \right\}.$$

Proof. Consider the analytic function $f(\lambda) := \frac{1}{\lambda}$ defined on a open neighborhood U containing the spectrum of T and such that $0 \notin U$. Then $T^{-1} = f(T)$, so, from the inclusion (11) we have

$$\sigma_{\rm w}(T^{-1}) = \sigma_{\rm w}(f(T)) \subseteq f(\sigma_{\rm w}(T)) = \left\{ \frac{1}{\lambda} : \lambda \in \sigma_{\rm w}(T) \right\}.$$

To show the opposite inclusion $\{\frac{1}{\lambda} : \lambda \in \sigma_{\rm w}(T)\} \subseteq \sigma_{\rm w}(T^{-1})$, consider again the function $f(\lambda) := \frac{1}{\lambda}$. Then $f(T^{-1}) = T$, so, always from (11), we have

$$\sigma_{\rm w}(T) = \sigma_{\rm w}(f(T^{-1})) \subseteq f(\sigma_{\rm w}(T^{-1})) = \left\{ \frac{1}{\lambda} : \lambda \in \sigma_{\rm w}(T^{-1}) \right\}. \tag{12}$$

Let $\lambda_0 := \frac{1}{\mu_0}$ with $\mu_0 \in \sigma_{\rm w}(T)$. From the inclusion (12) we have $\mu_0 = \frac{1}{\lambda_0} \in \{\frac{1}{\lambda} : \lambda \in \sigma_{\rm w}(T^{-1})\}$, so $\lambda_0 \in \sigma_{\rm w}(T^{-1})$.

Therefore, the points of $\sigma_{\rm w}(T^{-1})$ are the reciprocals of the spectrum $\sigma_{\rm w}(T)$. □

Theorem 3.5. *Let $R \in L(X)$ be Drazin invertible with Drazin inverse S. Then we have*

$$\sigma_{\rm w}(S)\setminus\{0\} = \left\{ \frac{1}{\lambda} : \lambda \in \sigma_{\rm w}(R)\setminus\{0\} \right\},$$

and

$$\sigma_{\mathrm{uw}}(S)\setminus\{0\} = \{\frac{1}{\lambda} : \lambda \in \sigma_{\mathrm{uw}}(R)\setminus\{0\}\}.$$

Proof. With respect to the decomposition $R = R_1 \oplus R_2$ and $S = 0 \oplus S_2$, with $S_2 = {R_2}^{-1}$, we have

$$\sigma_{\mathrm{w}}(R) = \sigma_{\mathrm{w}}(R_1) \cup \sigma_{\mathrm{w}}(R_2) = \{0\} \cup \sigma_{\mathrm{w}}(R_2)$$

and

$$\sigma_{\mathrm{w}}(S) = \sigma_{\mathrm{w}}(0) \cup \sigma_{\mathrm{w}}(R_2) = \{0\} \cup \sigma_{\mathrm{w}}(S_2).$$

Observe that R_2 and S_2 are invertible, so $0 \notin \sigma_{\mathrm{w}}(R_2)$ and $0 \notin \sigma_{\mathrm{w}}(S_2)$. Hence, $\sigma_{\mathrm{w}}(R)\setminus\{0\} = \sigma_{\mathrm{w}}(R_2)$ and $\sigma_{\mathrm{w}}(S)\setminus\{0\} = \sigma_{\mathrm{w}}(S_2)$. By the previous lemma, the points of $\sigma_{\mathrm{w}}(R_2)$ and $\sigma_{\mathrm{w}}(S_2)$ are reciprocal; hence

$$\sigma_{\mathrm{w}}(S)\setminus\{0\} = \sigma_{\mathrm{w}}(S_2) = \left\{\frac{1}{\lambda} : \lambda \in \sigma_{\mathrm{w}}(R_2)\right\} = \left\{\frac{1}{\lambda} : \lambda \in \sigma_{\mathrm{w}}(R)\setminus\{0\}\right\},$$

so the first equality is proved.

Let $0 \neq \lambda$ and suppose that $\frac{1}{\lambda} \notin \sigma_{\mathrm{uw}}(R)$, i.e., $\frac{1}{\lambda}I - R$ is upper semi-Weyl. Then $\frac{1}{\lambda}I - R \in \Phi_+(X)$ and $\mathrm{ind}\,(\frac{1}{\lambda}I - R) \leq 0$. By Theorem 2.8 we have $\ker\,(\frac{1}{\lambda}I - R) = \ker\,(\lambda I - S)$, so $\alpha(\lambda I - S) < \infty$. Moreover,

$$(\lambda I - S)(X) = (\lambda I - 0)(Y) \oplus (\lambda I - S_2)(Z) = Y \oplus (\lambda I - S_2)(Z).$$

Now, R_1 is nilpotent so $\frac{1}{\lambda}I - R_1$ is invertible, and hence $(\frac{1}{\lambda}I - R_1)(Y) = Y$, while $(\frac{1}{\lambda}I - R_2)(Z) = (\lambda I - S_2)(Z)$, by part (i) of Theorem 2.8. Therefore,

$$\begin{aligned}\left(\frac{1}{\lambda}I - R\right)(X) &= \left(\frac{1}{\lambda}I - R_1\right)(Y) \oplus \left(\frac{1}{\lambda}I - R_2\right)(Z) = Y \oplus (\lambda I - S_2)(Z)\\ &= (\lambda I - S)((X),\end{aligned}$$

so $(\lambda I - S)((X)$ is closed, because $(\frac{1}{\lambda}I - R)(X)$ is closed by assumption, and this shows that $\lambda I - S \in \Phi_+(X)$. It remains only to prove that $\mathrm{ind}\,(\lambda I - S) \leq 0$. Clearly, $\beta(\frac{1}{\lambda}I - R) = \beta(\lambda I - S)$ and $\alpha(\frac{1}{\lambda}I - R) = \alpha(\lambda I - S)$, by Theorem 2.8, so $\mathrm{ind}\,(\lambda I - S) = \mathrm{ind}\,(\frac{1}{\lambda}I - R) \leq 0$. Therefore, $\lambda I - S$ is upper semi-Weyl and hence $\lambda \notin \sigma_{\mathrm{uw}}(S)$.

Conversely, suppose that $\lambda \notin \sigma_{\mathrm{uw}}(S)$, i.e. $\lambda I - S$ is upper semi-Weyl. From the equalities $\ker\,(\frac{1}{\lambda}I - R) = \ker\,(\lambda I - S)$ and $(\frac{1}{\lambda} - R)(X) = (\lambda I - S)((X)$ we then obtain that $\frac{1}{\lambda}I - R \in \Phi_+(X)$. As above, $\mathrm{ind}\,(\frac{1}{\lambda}I - R) = \mathrm{ind}\,(\lambda I - S) \leq 0$, so $\frac{1}{\lambda}I - R$ is upper semi-Weyl, and hence $\frac{1}{\lambda} \notin \sigma_{\mathrm{uw}}(R)$. □

4. Browder and Weyl Type Theorems

Browder type theorems and Weyl type theorems concern the structure of the spectrum of some classes of operators see [2,5]. In this section, by using the results of the previous sections, we show that Browder and Weyl type theorems are transferred from a Drazin invertible operator R to its Drazin inverse S.

An operator $T \in L(X)$ is said to satisfy *Browder's theorem* if $\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{b}}(T)$, or, equivalently, T has SVEP at every $\lambda \notin \sigma_{\mathrm{w}}(T)$ see [3]. An operator

$T \in L(X)$ is said to satisfy *a-Browder's theorem* if $\sigma_{\mathrm{uw}}(T) = \sigma_{\mathrm{ub}}(T)$, or equivalently T has SVEP at every $\lambda \notin \sigma_{\mathrm{uw}}(T)$, see [4].

Theorem 4.1. *Suppose that $R \in L(X)$ is Drazin invertible with Drazin inverse S. Then*

(i) *R satisfies Browder's theorem if and only if S satisfies Browder's theorem.*
(ii) *R satisfies a-Browder's theorem if and only if S satisfies a-Browder's theorem.*

Proof.

(i) Suppose that R satisfies Browder's theorem and let $X = Y \oplus Z$, $R = R_1 \oplus R_2$ and $S = 0 \oplus S_2$, where $S_2 = {R_2}^{-1}$. Let $\lambda \notin \sigma_{\mathrm{w}}(S)$ be arbitrary given. To prove that Browder's theorem holds for S it suffices to show that S has SVEP at λ. If $\lambda = 0$, then S has SVEP at 0, since $p(S) < \infty$ (recall that the Drazin inverse S is itself Drazin invertible). If $\lambda \neq 0$, then $\frac{1}{\lambda} \notin \sigma_{\mathrm{w}}(R)$, and since R satisfies Browder's theorem then R has SVEP at $1/\lambda$. By Theorem 2.3 it then follows that S has SVEP at λ. Hence S satisfies Browder's theorem. The converse may be proved by using similar arguments.
(ii) Suppose that R satisfies a-Browder's theorem and let $\lambda \notin \sigma_{\mathrm{uw}}(S)$. If $\lambda = 0$, since S is Drazin invertible we have $p(S) < \infty$, hence S has SVEP at 0. If $\lambda \neq 0$ then $\frac{1}{\lambda} \notin \sigma_{\mathrm{uw}}(R)$, and since R satisfies a-Browder's theorem then R has SVEP at $1/\lambda$. By Theorem 2.3 then S has SVEP at λ, and hence S satisfies a-Browder's theorem.

□

For every operator $T \in L(X)$ let $p_{00}(T) := \sigma(T) \setminus \sigma_{\mathrm{b}}(T)$, and set

$$\pi_{00}(T) := \{\lambda \in \operatorname{iso} \sigma(T) : 0 < \alpha(\lambda I - T) < \infty.\}$$

Clearly, $p_{00}(T) \subseteq \pi_{00}(T)$ for every $T \in L(X)$. Define $p_{00}^a(T) := \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{ub}}(T)$ and

$$\pi_{00}^a(T) := \{\lambda \in \operatorname{iso} \sigma_{\mathrm{a}}(T) : 0 < \alpha(\lambda I - T) < \infty.\}$$

We also have $p_{00}^a(T) \subseteq \pi_{00}^a(T)$ for every $T \in L(X)$.
Set

$$\Delta(T) := \sigma(T) \setminus \sigma_{\mathrm{w}}(T) \quad \text{and} \quad \Delta_a(T) := \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{uw}}(T).$$

It should be noted that the class of all bounded below operators is a regularity and $\sigma_{\mathrm{a}}(T)$ is nonempty for all $T \in L(X)$; thus, by Theorem 2.7,

$$\sigma_{\mathrm{a}}(S) \setminus \{0\} = \left\{ \frac{1}{\lambda} : \lambda \in \sigma_{\mathrm{a}}(R) \setminus \{0\} \right\}. \tag{13}$$

The following properties have been introduced in [11] and [12], see also [6].

Definition 4.2. *Let $T \in L(X)$.*

(i) *T is said to satisfy* property (b) *if $\Delta_a(T) = p_{00}(T)$.*
(ii) *T is said to satisfy* property (ab) *if $\Delta(T) = p_{00}^a(T)$.*

Property (b) for T entails a-Browder's theorem for T, while property (ab) for T entails Browder's theorem for T. Moreover, property (b) for T implies property (ab) for T. Browder type theorems recently have been investigated in [19]. Set $\Sigma_a(T) := \Delta(T) \cup p_{00}^a(T)$. The following results have been proved in [9, Theorem 3.3] and [6, Theorem 2.4].

Theorem 4.3. *Let $T \in L(X)$. Then we have the following:*

(i) *T satisfies property (ab) if and only if $\Sigma_a(T) \subseteq iso\,\sigma(T)$.*
(ii) *T satisfies property (b) if and only if $\Delta_a(T) \subseteq iso\,\sigma(T)$.*

Both properties (b) and (ab) are transmitted from a Drazin invertible operator to its Drazin inverse. To show this we need a preliminary result.

Lemma 4.4. *Let $R \in L(X)$ be Drazin invertible with Drazin inverse S. We have:*

(i) *$0 \in p_{00}(R) \Leftrightarrow 0 \in p_{00}(S)$. If $\lambda \neq 0$, then $\lambda \in p_{00}(R) \Leftrightarrow \frac{1}{\lambda} \in p_{00}(S)$.*
(ii) *$0 \in p_{00}^a(R) \Leftrightarrow 0 \in p_{00}^a(S)$. If $\lambda \neq 0$, then $\lambda \in p_{00}^a(R) \Leftrightarrow \frac{1}{\lambda} \in p_{00}^a(S)$.*

Proof.

(i) Since $0 \in \sigma(R)$ if and only if $0 \in \sigma(S)$, then the first assertion follows from part (i) of Theorem 3.2. The second assertion is clear from part (ii) of Theorem 3.2.
(ii) The proof is similar to part (i).

□

Theorem 4.5. *Suppose that $R \in L(X)$ is Drazin invertible with Drazin inverse S. Then*

(i) *R satisfies property (ab) and only if S satisfies property (ab).*
(ii) *R satisfies property (b) and only if S satisfies property (ab).*

Proof.

(i) Suppose that R satisfies property (ab). Then R satisfies Browder's theorem and hence also S satisfies Browder's theorem, by Theorem 4.1. Therefore, $\sigma_{\mathrm{b}}(S) = \sigma_{\mathrm{w}}(S)$. Let $\lambda \in \Sigma_a(S)$. By Theorem 4.3 it suffices to show that $\lambda \in \operatorname{iso}\sigma(S)$. We distinguish the two cases $\lambda = 0$ and $\lambda \neq 0$.

    If $\lambda = 0$ then $0 \in \operatorname{iso}\sigma(S)$, since S is Drazin invertible. Suppose that $\lambda \neq 0$. Then either $\lambda \in \Delta(S)$ or $\lambda \in p_{00}(S)$. If $\lambda \in \Delta(S) = \sigma(S)\setminus\sigma_{\mathrm{w}}(S) = \sigma(S)\setminus\sigma_{\mathrm{b}}(S) = p_{00}(S)$, then $\lambda I - S$ is Browder, so $\lambda \in \operatorname{iso}\sigma(S)$. If $\lambda \in p_{00}^a(S)$ then, by Lemma 4.4, $\frac{1}{\lambda} \in p_{00}^a(R)$. Property (ab) for R entails, by Theorem 4.3, $\frac{1}{\lambda} \in \operatorname{iso}\sigma(R)$. Consequently, $\lambda \in \operatorname{iso}\sigma(S)$.

    Therefore, S has property (ab). The converse may be proved by using similar arguments.
(ii) Suppose that R satisfies property (b), or equivalently $\Delta_a(R) \subseteq \operatorname{iso}\sigma(R)$. Then R satisfies a-Browder's theorem and hence also S satisfies a-Browder's theorem, by Theorem 4.1, so that $\sigma_{\mathrm{ub}}(R) = \sigma_{\mathrm{uw}}(SR)$ and $\sigma_{\mathrm{ub}}(S) = \sigma_{\mathrm{uw}}(S)$. Consequently,

$$\Delta_a(R) = p_{00}^a(R) \quad \text{and} \quad \Delta_a(S) = p_{00}^a(S).$$

To show property (b) for T it suffices to prove, by Theorem 4.3, the inclusion $\Delta_a(S) \subseteq \text{iso}\,\sigma(S)$. Let $\lambda \in \Delta_a(S)$. If $\lambda = 0$ then 0 is an isolated point of $\sigma(S)$, since S is Drazin invertible. Suppose that $\lambda \neq 0$. Since $\lambda \in \Delta_a(S) = p_{00}^a(S)$ then, by Lemma 4.4 and Theorem 4.3, $\frac{1}{\lambda} \in p_{00}^a(R) = \Delta_a(R) \subseteq \text{iso}\,\sigma(R)$. Consequently, $\lambda \in \text{iso}\,\sigma(S)$. Thus, S has property (b). The converse may be proved in a similar way. □

Lemma 4.6. *Let $R \in L(X)$ be Drazin invertible with Drazin inverse S. Then we have*

(i) $0 \in \pi_{00}(R) \Leftrightarrow 0 \in \pi_{00}(S)$. *If* $\lambda \neq 0$, *then* $\lambda \in \pi_{00}(R) \Leftrightarrow \frac{1}{\lambda} \in \pi_{00}(S)$.
(ii) $0 \in \pi_{00}^a(R) \Leftrightarrow 0 \in \pi_{00}^a(S)$. *If* $\lambda \neq 0$, *then* $\lambda \in \pi_{00}^a(R) \Leftrightarrow \frac{1}{\lambda} \in \pi_{00}^a(S)$.

Proof.

(i) Suppose first that $0 \in \pi_{00}(R)$. Then $0 \in \text{iso}\,\sigma(R)$ and $0 < \alpha(R) < \infty$. Obviously, $0 \in \text{iso}\,\sigma(S)$ and arguing as in the proof of part (i) of Theorem 3.2 we have $\alpha(S) < \infty$. With respect to the usual decomposition $X = Y \oplus Z$, $R = R_1 \oplus R_2$, we have $0 < \alpha(R) = \alpha(R_1) + \alpha(R_2) = \alpha(R_1)$, and hence $\alpha(R_1^n) > 0$ for all $n \in \mathbb{N}$. If $R_1^\nu = 0$, then $Y = \ker R_1^\nu$ has dimension greater than 0, and from $\ker S = Y \oplus \{0\}$ we conclude that $\alpha(S) > 0$. Hence $0 \in \pi_{00}(S)$. Analogous arguments show the reverse implication. The second assertion easily follows from the equality $\ker(\lambda I - R) = \ker(\frac{1}{\lambda} I - S)$ for all $\lambda \neq 0$.
(ii) If $0 \in \pi_{00}^a(R)$, then $0 \in \text{iso}\,\sigma_{\text{a}}(R)$ and $0 < \alpha(R) < \infty$. Obviously, from $0 \in \text{iso}\,\sigma_{\text{a}}(S)$. To show that $0 < \alpha(S) < \infty$, proceed as in part (i). An analogous reasoning shows that if $0 \in \pi_{00}^a(S)$, then $0 \in \pi_{00}^a(R)$. The second assertion follows from the equality $\ker(\lambda I - R) = \ker(\frac{1}{\lambda} I - S)$ for all $\lambda \neq 0$. □

An operator $T \in L(X)$ satisfies *Weyl's theorem* if $\Delta(T) = \pi_{00}(T)$. Weyl's theorem for T is equivalent to saying that T satisfies Browder's theorem and $p_{00}(T) = \pi_{00}(T)$, see [3, Theorem 3.3]. An operator $T \in L(X)$ satisfies *a-Weyl's theorem* if $\Delta_a(T) = \pi_{00}^a(T)$. a-Weyl's theorem for T is equivalent to saying that T satisfies a-Browder's theorem and $p_{00}^a(T) = \pi_{00}^a(T)$, see [4, Theorem 2.14]. An operator $T \in L(X)$ is said to verify *property* (w) if $\Delta_a(T) = \pi_{00}(T)$. Property (w) for T is equivalent to saying that T satisfies a-Browder's theorem and $\pi_{00}^a(T) = \pi_{00}(T)$, see [7]. Property (w) or a-Weyl's theorem for T entails Weyl's theorem for T, but in general property (w) and a-Weyl's theorem are independent.

Theorem 4.7. *Suppose that $R \in L(X)$ is Drazin invertible with Drazin inverse S. Then*

(i) *R satisfies Weyl's theorem if and only if S satisfies Weyl's theorem.*
(ii) *R satisfies a-Weyl's theorem if and only if S satisfies aWeyl's theorem.*
(iii) *R satisfies property (w) if and only if S satisfies property (w).*

Proof.

(i) Suppose that R satisfies Weyl's theorem. Then R satisfies Browder's theorem and $\pi_{00}(R) = p_{00}(R)$; hence, from part (i) of Theorem 4.1, Browder's theorem holds for S.

Let $\lambda \in \pi_{00}(S)$. If $\lambda = 0$, then, by Lemma 4.6, $0 \in \pi_{00}(R) = p_{00}(R)$ and hence, by Lemma 4.4, $0 \in p_{00}(S)$. If $\lambda \neq 0$, then $\frac{1}{\lambda} \in \pi_{00}(R) = p_{00}(R)$, so $\lambda \in p_{00}(S)$, again by Lemma 4.4. Therefore, $\pi_{00}(S) \subseteq p_{00}(S)$, and since the opposite inclusion holds for every operator, we then conclude that $\pi_{00}(S) = p_{00}(S)$; thus S satisfies Weyl's theorem. In a similar way Weyl's theorem for S implies Weyl's theorem for R.

(ii) If R satisfies a-Weyl's theorem, then R satisfies a-Browder's theorem and $p^a_{00}(R) = \pi^a_{00}(R)$. By part (ii) of Theorem 4.1, a-Browder's theorem holds for S. To show that S satisfies aWeyl's, it suffices to prove that $\pi^a_{00}(S) = p^a_{00}(S)$.

Let $\lambda \in \pi^a_{00}(S)$. If $\lambda = 0$, then, by Lemma 4.6, $0 \in \pi^a_{00}(R) = p^a_{00}(R)$. Hence, by Lemma 4.4, $0 \in p^a_{00}(S)$. If $\lambda \neq 0$, then, by Lemma 4.6, $\frac{1}{\lambda} \in \pi^a_{00}(R) = p^a_{00}(R)$, and hence $\lambda \in p^a_{00}(S)$, by Lemma 4.4. Therefore, $\pi^a_{00}(S) \subseteq p^a_{00}(S)$. The opposite inclusion holds for every operator; hence $\pi^a_{00}(S) = p^a_{00}(S)$; thus S satisfies a-Weyl's theorem. In a similar way, property a-Weyl's theorem for S implies property a-Weyl's theorem for R.

(iii) If R satisfies property (w), then R satisfies a-Browder's theorem and $p^a_{00}(R) = \pi_{00}(R)$. From part (ii) of Theorem 4.1, a- Browder's theorem holds for S, so it suffices to prove that $\pi_{00}(S) = p^a_{00}(S)$.

Let $\lambda \in \pi_{00}(S)$. If $\lambda = 0$, then, by Lemma 4.4, $0 \in \pi_{00}(R) = p^a_{00}(R)$. Hence, always by Lemma 4.4, $0 \in p^a_{00}(S)$. Suppose $\lambda \neq 0$, then $\frac{1}{\lambda} \in \pi_{00}(R) = p^a_{00}(R)$, and hence $\lambda \in p^a_{00}(S)$, by Lemma 4.4. Therefore, $\pi_{00}(S) \subseteq p^a_{00}(S)$.

It remains to prove that $p^a_{00}(S) \subseteq \pi_{00}(S)$. Let $\lambda \in p^a_{00}(S)$. If $\lambda = 0$, then S is upper semi-Browder and hence is Browder, see Remark 3.1, so 0 is an isolated point of $\sigma(S)$. Clearly, $0 < \alpha(S)$, since $0 \in \sigma_{\mathrm{a}}(S)$ and $S(X)$ is closed. Moreover, $\alpha(S) < \infty$, so $0 \in \pi_{00}(S)$. If $\lambda \neq 0$, then always by Lemma 4.4, $\frac{1}{\lambda} \in p^a_{00}(R) = \pi_{00}(R)$, since R has property (w). By Lemma 4.6, then $\lambda \in \pi_{00}(S)$. Therefore, $\pi_{00}(S) = p^a_{00}(S)$; thus S satisfies property (w). In a similar way property (w) for S implies property (w) for R.

□

Let $U := R^2S = RSR$. It is easily seen from the equalities (1) that U commutes with S and $S^2U = S$, while $USU = U$, so U is the Drazin inverse of S with index $i(S) = 1$, or equivalently 0 is a simple pole of the resolvent.

Theorem 4.8. *Suppose that $R \in L(X)$ is Drazin invertible with Drazin inverse S. Then*

$$\sigma_{\mathrm{b}}(R^2S)\setminus\{0\} = \sigma_{\mathrm{b}}(R)\setminus\{0\},$$

and a similar relationship holds for the Weyl spectrum, the semi-Browder spectra and the semi-Weyl spectra. Moreover, if R satisfies Browder's theorem, (respectively, a-Browder's theorem, property (b), property (ab), Weyl's theorem, a-Weyl's theorem, property (w)), then R^2S satisfies Browder's theorem, (respectively, a-Browder's theorem, property (b), property (ab), Weyl's theorem, a-Weyl's theorem, property (w)).

Proof. R^2S is the Drazin inverse of S, so

$$\sigma_{\mathrm{b}}(R^2S)\setminus\{0\} = \left\{\frac{1}{\lambda} : \lambda \in \sigma_{\mathrm{b}}(S)\setminus\{0\}\right\} = \sigma_{\mathrm{b}}(R)\setminus\{0\}.$$

The same argument shows the assertion for the other spectra. The last assertion follows from Theorems 4.1 and 4.7: S^2R satisfies Browder's theorem since S satisfies Browder's theorem. The assertions concerning the other properties follow similarly from Theorems 4.5 and 4.7. □

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