Mediterr. J. Math. 13 (2016), 4333–4352
DOI 10.1007/s00009-016-0748-y
1660-5446/16/064333-20
published online June 18, 2016

Mediterranean Journal of Mathematics

CrossMark

L^p-Boundedness of the Littlewood–Paley g-Function Associated with the Spherical Mean Operator for $1 < p < +\infty$

A. Hammami and S. Omri

Abstract. We prove the L^p-boundedness of the Littlewood–Paley g-function associated with the spherical mean operator for $p \in]1, +\infty[$.

Mathematics Subject Classification. Primary : 43A32, Secondary : 42B10.

Keywords. spherical mean operator, poisson semigroup, poisson integral, Littlewood–Paley g-function.

1. Introduction

According to Stein [10], the Littlewood–Paley g-function is defined in the Euclidean case by

$$\forall x \in \mathbb{R}^n,\ g(f)(x) = \left(\int_0^{+\infty} \left|\nabla\left(\mathscr{U}(f)\right)(x,t)\right|^2 t\mathrm{d}t\right)^{\frac{1}{2}},$$

where $\mathscr{U}(f)$ is the Poisson integral of f defined on $\mathbb{R}^n \times]0, +\infty[$, by

$$\mathscr{U}(f)(x,t) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \int_{\mathbb{R}^n} \frac{tf(y)}{(t^2 + |x-y|^2)^{\frac{n+1}{2}}} \mathrm{d}y,$$

and ∇ is the standard gradient on $\mathbb{R}^{n+1}$. According to Stein [10], it is well known that the Littlewood–Paley g-function is bounded from the Lebesgue space L^p, $p \in]1, +\infty[$ into it self. The Littlewood–Paley theory constitutes one of the most important ways to study many function spaces as the Hardy spaces H^p, or the various forms of Lipshitz and BMO spaces, and remains closely related to the theory of Fourier multipliers in harmonic analysis. For more details, we refer the reader to Stein [10]. In the literature, many authors notably A. Achour, A.Fitouhi, and K. Stempak [1,2,11] generalized the Littlewood–Paley g-function to several other hypergroups and integral transforms, and showed similarly its L^p-boundedness.

The spherical mean operator $\mathscr{R}$ is defined by [7]

$$\mathscr{R}(f)(r,x)=\int_{S^n} f(r\eta, x+r\xi)d\sigma_n(\eta,\xi), \quad (r,x)\in\mathbb{R}\times\mathbb{R}^n,$$

where S^n is the unit sphere of $\mathbb{R}^{n+1}$ and $d\sigma_n$ is the surface measure on S^n normalized to have total measure one. The Fourier transform associated with the spherical mean operator is defined by [7]

$$\mathscr{F}(f)(\mu,\lambda)=\int_0^{+\infty}\int_{\mathbb{R}^n}\mathscr{R}\left(\cos(\mu.)e^{-i\langle\lambda|.\rangle}\right)(r,x)d\nu_{n+1}(r,x),$$

where $d\nu_{n+1}$ is a measure that will be defined later. Many harmonic analysis results related to spherical mean operator and its the Fourier transform $\mathscr{F}$ have already been proved by Rachdi and Trimèche [7,9] or also by Hleili and Omri [3,5,8]. Hleili and Omri [3] defined the Littlewood–Paley g-function associated with the spherical mean operator by

$$\in [0,+\infty[\times\mathbb{R}^n,\ g(f)(r,x)=\left(\int_0^{+\infty}\left|\nabla\left(\mathscr{U}(f)\right)(r,x,t)\right|^2 t\mathrm{d}t\right)^{1/2},$$

where $\mathscr{U}(f)$ is the Poisson integral associated with the spherical mean operator (see [3]). The authors showed that for every $p\in]1,2]$ and for every $f\in L^p(d\nu_{n+1})$ the function $g(f)$ belongs to the space $L^p(d\nu_{n+1})$ and satisfies

$$\|g(f)\|_{p,\nu_{n+1}}\leqslant\frac{2^{\frac{2+p}{2p}}}{\sqrt{p}(p-1)^{\frac{1}{p}}}\|f\|_{p,\nu_{n+1}}.$$

The aim of this work is to extend this result to every $p\in]1,+\infty[$.

2. The Spherical Mean Operator

2.1. Eigenfunction Associated with the Spherical Mean Operator

In [7], Nessibi, Rachdi and Trimèche showed that for every $(\mu,\lambda)\in\mathbb{C}\times\mathbb{C}^n$, the function $\varphi_{(\mu,\lambda)}$ defined on $\mathbb{R}\times\mathbb{R}^n$ by

$$(r,x)=\mathscr{R}\left(\cos(\mu.)e^{-i\langle\lambda|.\rangle}\right)(r,x), \tag{2.1}$$

is the unique infinitely differentiable function on $\mathbb{R}\times\mathbb{R}^n$, even with respect to the first variable, satisfying the following system:

$$\begin{cases}\dfrac{\partial u}{\partial x_j}(r,x_1,...,x_n)=-i\lambda_j u(r,x_1,...,x_n), & 1\leqslant j\leqslant n,\\ \ell_{\frac{n-1}{2}}u(r,x_1,...,x_n)-\Delta u(r,x_1,...,x_n)=-\mu^2 u(r,x_1,...,x_n), & \\ u(0,...,0)=1, \quad \dfrac{\partial u}{\partial r}(0,x_1,...,x_n)=0, \quad (x_1,\dots,x_n)\in\mathbb{R}^n. & \end{cases}$$

where $\ell_{\frac{n-1}{2}}$ is the Bessel operator defined by $\ell_{\frac{n-1}{2}}=\dfrac{\partial^2}{\partial r^2}+\dfrac{n}{r}\dfrac{\partial}{\partial r}$, and $\Delta=\sum_{j=1}^n\dfrac{\partial^2}{\partial x_j^2}$ denotes the Laplacian operator. Then, according to Nessibi, Rachdi and Trimèche [7], it is known that for every $r,\mu\in\mathbb{R}_+$

$$\ell_{\frac{n-1}{2}}\left(j_{\frac{n-1}{2}}(r.)\right)(\mu)=-r^2 j_{\frac{n-1}{2}}(r\mu) \tag{2.2}$$

where $j_{\frac{n-1}{2}}$ is the modified Bessel function [6]. In [7], the authors proved also that the eigenfunction $\varphi_{(\mu,\lambda)}$ defined by relation (2.1) is explicitly given by

$$\forall (r,x)\in\mathbb{R}\times\mathbb{R}^n,\quad \varphi_{(\mu,\lambda)}(r,x)=j_{\frac{n-1}{2}}(r\sqrt{\mu^2+|\lambda|^2})e^{-i\langle\lambda|x\rangle}, \tag{2.3}$$

From the properties of the modified Bessel function $j_{\frac{n-1}{2}}$, we deduce that the eigenfunction $\varphi_{(\mu,\lambda)}$ is bounded on $\times\mathbb{R}^n$ if, and only if, $(\mu,\lambda)\in\Upsilon$, where

$$\Upsilon=\mathbb{R}\times\mathbb{R}^n\cup\big\{(ir,x),\ (r,x)\in\mathbb{R}\times\mathbb{R}^n,\ |r|\leqslant|x|\big\}, \tag{2.4}$$

and in this case

$$\sup_{(r,x)\in\mathbb{R}\times\mathbb{R}^n}\left|\varphi_{(\mu,\lambda)}(r,x)\right|=1. \tag{2.5}$$

In the following, we denote by

- $\mathrm{d}\nu_{n+1}$ is the measure defined on $[0,+\infty[\times\mathbb{R}^n$ by $d\nu_{n+1}(r,x)=\dfrac{r^n\mathrm{d}r\,\mathrm{d}x}{2^{n-\frac12}\pi^{\frac n2}\Gamma(\frac{n+1}{2})}$.
- $\mathscr{C}_e\left(\mathbb{R}\times\mathbb{R}^n\right)$ the space of continuous functions on $\mathbb{R}\times\mathbb{R}^n$, even with respect to the first variable.
- $\mathscr{C}_{0,e}\left(\mathbb{R}\times\mathbb{R}^n\right)$ the space of continuous functions on $\mathbb{R}\times\mathbb{R}^n$, even with respect to the first variable such that $\lim_{r^2+|x|^2\to+\infty}f(r,x)=0$.
- $\mathscr{C}_e^k\left(\mathbb{R}\times\mathbb{R}^n\right)$ the space of functions of class C^k on $\mathbb{R}\times\mathbb{R}^n$, even with respect to the first variable.
- $\mathscr{C}_e^\infty\left(\mathbb{R}\times\mathbb{R}^n\right)$ the space of infinitely differentiable functions on $\mathbb{R}\times\mathbb{R}^n$, even with respect to the first variable.
- $S_e\left(\mathbb{R}\times\mathbb{R}^n\right)$ the space of infinitely differentiable functions, rapidly decreasing together with all their derivatives, even with respect to the first variable.
- $\mathscr{D}_e\left(\mathbb{R}\times\mathbb{R}^n\right)$ the space of smooth functions on $\mathbb{R}\times\mathbb{R}^n$ with compact support, even with respect to the first variable.
- $\Upsilon_+=[0,+\infty[\times\mathbb{R}^n\cup\big\{(is,y)\ ;\ (s,y)\in[0,+\infty[\times\mathbb{R}^n;\ s\leqslant|y|\big\}$.
- $\mathscr{B}_{\Upsilon_+}$ the σ-algebra defined on Υ_+ by $\mathscr{B}_{\Upsilon_+}=\theta^{-1}\left(\mathscr{B}_{Bor}([0,+\infty[\times\mathbb{R}^n)\right)$ where θ is the bijective function defined on the set Υ_+ by $\theta(s,y)=(\sqrt{s^2+|y|^2},y)$.
- γ_{n+1} the measure defined on $\mathscr{B}_{\Upsilon_+}$ by $\gamma_{n+1}(B)=\nu_{n+1}(\theta(B))$.

2.2. Generalized Translation Operator and Convolution Product

According to Nessibi, Rachdi and Trimèche [7], for every $(r,x)\in[0,+\infty[\times\mathbb{R}^n$, the generalized translation operator $\mathcal{T}_{(r,x)}$ associated with the spherical mean operator is defined by

$$\mathcal{T}_{(r,x)}(f)(s,y)=\frac{\Gamma(\frac{n+1}{2})}{\sqrt{\pi}\Gamma(\frac n2)}\int_0^\pi f(\sqrt{r^2+s^2+2rs\cos\theta},x+y)(\sin\theta)^{n-1}\mathrm{d}\theta, \tag{2.6}$$

whenever the integral in the right hand side is well defined. The convolution product of two measurable functions f and g is defined on $[0,+\infty[\times\mathbb{R}^n$ by

$$f*g(r,x)=\int_0^{+\infty}\int_{\mathbb{R}^n}\mathcal{T}_{(r,-x)}(\check{f})(s,y)g(s,y)d\nu_{n+1}(s,y), \tag{2.7}$$

whenever the integral of the right-hand side is well defined, where $\check{f}(s,y)=f(s,-y)$. Then, it is well know that for every $p\in[1,+\infty]$, $\mathcal{T}_{(r,x)}$ is bounded form $L^p(d\nu_{n+1})$ into itself and satisfies

$$|||\mathcal{T}_{(r,x)}|||\leqslant 1. \tag{2.8}$$

Moreover, for every $f\in L^p(d\nu_{n+1}), p\in[1,+\infty[$, we have

$$\lim_{(r,x)\longrightarrow(0,0)}\|\mathcal{T}_{(r,x)}(f)-f\|_{p,\nu_{n+1}}=0. \tag{2.9}$$

We have also the following Young inequality [7], that is for every $p,q,r\in[1,+\infty]$ such that $\dfrac{1}{p}+\dfrac{1}{q}=1+\dfrac{1}{r}$ and for every $f\in L^p(d\nu_{n+1})$ and $g\in L^q(d\nu_{n+1})$, the function $f*g$ belongs to the space $L^r(d\nu_{n+1})$, and we have

$$||f*g||_{r,\nu_{n+1}}\leqslant||f||_{p,\nu_{n+1}}||g||_{q,\nu_{n+1}}. \tag{2.10}$$

2.3. The Fourier Transform Associated with the Spherical Mean Operator

The Fourier transform $\mathscr{F}$ associated with the spherical mean operator is defined on $L^1(d\nu_{n+1})$ by [7]

$$\forall(\mu,\lambda)\in\Upsilon\ ,\ \mathscr{F}(f)(\mu,\lambda)=\int_0^{+\infty}\int_{\mathbb{R}^n}f(r,x)\varphi_{(\mu,\lambda)}(r,x)\,d\nu_{n+1}(r,x),$$

where $\varphi_{(\mu,\lambda)}$ is the eigenfunction given by relation (2.3), and Υ is the set defined by relation (2.4). Then, according to [7], it is known that for every $f\in L^1(d\nu_{n+1})$,

$$\mathscr{F}(f)=\widetilde{\mathscr{F}}(f)\circ\theta, \tag{2.11}$$

where $\widetilde{\mathscr{F}}$ is the integral transform defined on $L^1(d\nu_{n+1})$, by

$$\forall(s,y)\in\mathbb{R}\times\mathbb{R}^n,\widetilde{\mathscr{F}}(f)(s,y)=\int_0^{+\infty}\int_{\mathbb{R}^n}f(r,x)j_{\frac{n-1}{2}}(rs)e^{-i\langle y|x\rangle}d\nu_{n+1}(r,x). \tag{2.12}$$

We know also that for every $f,g\in L^1(d\nu_{n+1})$, we have

$$\widetilde{\mathscr{F}}(f*g)=\widetilde{\mathscr{F}}(f)\widetilde{\mathscr{F}}(g). \tag{2.13}$$

Moreover, relation (2.5) implies that the Fourier transform $\mathscr{F}$ is a bounded linear operator from $L^1(d\nu_{n+1})$ into $L^\infty(d\gamma_{n+1})$, and that for every $f\in L^1(d\nu_{n+1})$

$$\|\mathscr{F}(f)\|_{\infty,\gamma_{n+1}}\leqslant\|f\|_{1,\nu_{n+1}}. \tag{2.14}$$

Theorem 2.1 *[Inversion formula]. Let $f\in L^1(d\nu_{n+1})$ such that $\mathscr{F}(f)\in L^1(d\gamma_{n+1})$, then for almost every $(r,x)\in\mathbb{R}\times\mathbb{R}^n$, we have*

$$f(r,x)=\iint_{\Upsilon_+}\mathscr{F}(f)(\mu,\lambda)\overline{\varphi_{(\mu,\lambda)}(r,x)}\,d\gamma_{n+1}(\mu,\lambda)$$

Theorem 2.2 (Plancherel theorem). *The Fourier transform $\mathscr{F}$ can be extended to an isometric isomorphism from $L^2(d\nu_{n+1})$ onto $L^2(d\gamma_{n+1})$. In particular, we have the following Parseval equality that is, for every $f, g \in L^2(d\nu_{n+1})$*

$$\int_0^{+\infty}\int_{\mathbb{R}^n} f(r,x)\overline{g(r,x)}\, d\nu_{n+1}(r,x) = \iint_{\Upsilon_+} \mathscr{F}(f)(\mu,\lambda)\overline{\mathscr{F}(g)(\mu,\lambda)} d\gamma_{n+1}(\mu,\lambda). \tag{2.15}$$

3. The Generalized Poisson Integral Associated with the Spherical Mean Operator

3.1. The Generalized Poisson Integral Associated with the Spherical Mean Operator

In [3], Hleili and Omri introduced the Poisson kernel associated with the spherical mean operator, by

$$\begin{aligned}\forall (r,x) \in \mathbb{R}\times\mathbb{R}^n,\ p_t(r,x) &= \iint_{\Upsilon_+} e^{-t|\theta(s,y)|}\, \overline{\varphi_{(s,y)}(r,x)} d\gamma_{n+1}(s,y),\ t > 0\\ &= \frac{2^{n+\frac{1}{2}} n! t}{\sqrt{\pi}(t^2 + r^2 + |x|^2)^{n+1}}.\end{aligned} \tag{3.1}$$

According to [3], the generalized Poisson integral associated with the spherical mean operator is defined for every $f \in L^1(d\nu_{n+1})$ by

$$\forall (r,x,t) \in \mathbb{R}\times\mathbb{R}^n \times]0,+\infty[, \mathscr{U}(f)(r,x,t) = \mathscr{P}^t(f)(r,x),$$

where $\mathscr{P}^t$ is the convolution operator defined on $L^1(d\nu_{n+1})$, by $\mathscr{P}^t(f) = p_t * f$. Then, by inversion formula, we deduce that for every $f \in L^1(d\nu_{n+1})$,

$$\begin{aligned}\mathscr{U}(f)(r,x,t) &= \iint_{\Upsilon_+} e^{-t|\theta(\mu,\lambda)|}\mathscr{F}(f)(\mu,\lambda)\overline{\varphi_{(\mu,\lambda)}(r,x)}\, d\gamma_{n+1}(\mu,\lambda),\ a.e\\ &= \int_0^{+\infty}\int_{\mathbb{R}^n} e^{-t\sqrt{s^2+|y|^2}}\widetilde{\mathscr{F}}(f)(s,y) j_{\frac{n-1}{2}}(rs) e^{i\langle y|x\rangle} d\nu_{n+1}(s,y),\ a.e.\end{aligned} \tag{3.2}$$

Lemma 3.1. *Let $f \in L^1(d\nu_{n+1}) \cap \mathscr{C}_{0,e}(\mathbb{R}\times\mathbb{R}^n)$, then the function $\mathscr{U}(f)$ belongs to $\mathscr{C}(\mathbb{R}\times\mathbb{R}^n\times[0,+\infty[)$, and satisfies*

$$\forall (r,x) \in \mathbb{R}\times\mathbb{R}^n,\ \mathscr{U}(f)(r,x,0) = f(r,x). \tag{3.3}$$

In the following for every nonnegative real number η, we denote by $B_\eta = \{(r,x) \in \mathbb{R}\times\mathbb{R}^n \mid r^2 + |x|^2 \leqslant \eta^2\}$ and $B_\eta^+ = B_\eta \cap (\mathbb{R}_+ \times \mathbb{R}^n)$. For every measurable function f on $\mathbb{R}\times\mathbb{R}^n$, we denote by $supp(f)$ the support of f.

Proposition 3.2. *Let η be a positive real number, and $f \in \mathscr{D}_e(\mathbb{R}\times\mathbb{R}^n)$. If $supp(f) \subset B_\eta$, then*

(i) *For every $(r,x,t) \in B_{2\eta}^c \times]0,+\infty[$, we have*

$$\left|\frac{\partial \mathscr{U}(f)}{\partial t}(r,x,t)\right| \leqslant \frac{2^{4n+3}(2n+3)\ (n!)^2\ \eta^{2n+1}}{\pi\ (2n+1)!}\frac{\|f\|_{\infty,\nu_{n+1}}}{(t^2+r^2+|x|^2)^{n+1}}. \tag{3.4}$$

(ii) *For every* $(r,x,t) \in \mathbb{R} \times \mathbb{R}^n \times]0,+\infty[$, *we have*

$$\left|\frac{\partial \mathscr{U}(f)}{\partial r}(r,x,t)\right| \leqslant \frac{\sqrt{\pi}(2n+1)!}{2^{n-\frac{3}{2}}\Gamma(\frac{n+1}{2})^2} \ \frac{\|f\|_{1,\nu_{n+1}}}{t^{2n+2}}. \tag{3.5}$$

(iii) *For every* $1 \leqslant j \leqslant n$ *and for every* $(r,x,t) \in \mathbb{R} \times \mathbb{R}^n \times]0,+\infty[$, *we have*

$$\left|\frac{\partial \mathscr{U}(f)}{\partial x_j}(r,x,t)\right| \leqslant \frac{\sqrt{\pi}(2n+1)!}{2^{n-\frac{3}{2}}\Gamma(\frac{n+1}{2})^2} \ \frac{\|f\|_{1,\nu_{n+1}}}{t^{2n+2}}. \tag{3.6}$$

Proof. (i) According to relations (2.6) and (3.1), we have

$$|\frac{\partial \mathcal{T}_{(r,-x)}p_t}{\partial t}(s,y)| \leqslant \frac{2^{n+\frac{1}{2}}n!}{\sqrt{\pi}} \frac{2n+3}{(t^2+(r-s)^2+|x-y|^2)^{n+1}} \tag{3.7}$$

Hence by relations (2.7) and (3.7), we have

$$\left|\frac{\partial \mathscr{U}(f)}{\partial t}(r,x,t)\right| \leqslant \frac{2^{n+\frac{1}{2}}n!(2n+3)\|f\|_{\infty,\nu_{n+1}}}{\sqrt{\pi}}$$
$$\int\int_{B_\eta^+} \frac{d\nu_{n+1}(s,y)}{(t^2+(r-s)^2+|x-y|^2)^{n+1}},$$

therefore, for every $(r,x,t) \in B_{2\eta}^c \times]0,+\infty[$, we get

$$\left|\frac{\partial \mathscr{U}(f)}{\partial t}(r,x,t)\right| \leqslant \frac{2^{3n+\frac{5}{2}}n!(2n+3)}{\sqrt{\pi}} \frac{\|f\|_{\infty,\nu_{n+1}}\nu_{n+1}(B_\eta^+)}{(t^2+r^2+|x|^2)^{n+1}},$$

however, a standard calculus leads to $\nu_{n+1}(B_\eta^+) = \dfrac{\eta^{2n+1}n!2^{n+\frac{1}{2}}}{\sqrt{\pi}(2n+1)!}$. □

Lemma 3.3. *Let* W *be the mapping defined on* $\mathscr{D}_e(\mathbb{R} \times \mathbb{R}^n)$ *by*

$$\forall (r,x,t) \in \mathbb{R} \times \mathbb{R}^n \times]0,+\infty[, \ W(f)(r,x,t) = |\nabla \mathscr{U}(f)(r,x,t)|^2, \tag{3.8}$$

and let $f \in \mathscr{D}_e(\mathbb{R} \times \mathbb{R}^n)$. *Then*
(i) $W(f) \in \mathscr{C}_e^\infty(\mathbb{R} \times \mathbb{R}^n \times]0,+\infty[)$.
(ii) $\forall t > 0, \ W(f)(.,.,t) \in L^1(d\nu_{n+1}) \cap \mathscr{C}_{0,e}(\mathbb{R} \times \mathbb{R}^n)$.
(iii) $\lim\limits_{r^2+|x|^2+t^2 \to +\infty} W(f)(r,x,t) = 0$.

Proof. *(ii)* Let $\eta > 0$ such that $supp(f) \subset B_\eta$, then according to Hleili and Omri [3, Lemma 4.2, pp. 900], we know that for every $(r,x,t) \in B_{2\eta}^c \times]0,+\infty[$, the generalized Poisson integral $\mathscr{U}(f)$ satisfies the following relations:

$$\left|\frac{\partial(\mathscr{U}(f))}{\partial r}(r,x,t)\right| \leqslant \frac{C\eta^{2n+1}\|f\|_{\infty,\nu_{n+1}}}{(t^2+r^2+|x|^2)^{n+1}}, \tag{3.9}$$

and for every $1 \leqslant j \leqslant n$

$$\left|\frac{\partial(\mathscr{U}(f))}{\partial x_j}(r,x,t)\right| \leqslant \frac{C\eta^{2n+1}\|f\|_{\infty,\nu_{n+1}}}{(t^2+r^2+|x|^2)^{n+1}}. \tag{3.10}$$

Then, by relations (3.4), (3.9) and (3.10), we get that for every $(r,x,t) \in B_{2\eta}^c \times]0,+\infty[$,

$$W(f)(r,x,t) \leqslant \frac{C\eta^{4n+2}\|f\|^2_{\infty,\nu_{n+1}}}{(t^2+r^2+|x|^2)^{2n+2}}, \tag{3.11}$$

in particular for every $t>0$, $W(f)(.,.,t)$ belongs to $L^1(d\nu_{n+1})$ and

$$\lim_{r^2+|x|^2\to+\infty} W(f)(r,x,t) \leqslant \lim_{r^2+|x|^2\to+\infty} \frac{C\eta^{4n+2}\|f\|^2_{\infty,\nu_{n+1}}}{(t^2+r^2+|x|^2)^{2n+2}} = 0.$$

(iii) • If $|(r,x)| \geqslant 2\eta$, then by relation (3.11), we have

$$\lim_{t^2+r^2+|x|^2\to+\infty} W(f)(r,x,t) \leqslant \lim_{t^2+r^2+|x|^2\to+\infty} \frac{C\eta^{4n+2}\|f\|^2_{\infty,\nu_{n+1}}}{(t^2+r^2+|x|^2)^{2n+2}} = 0.$$

• If $|(r,x)| \leqslant 2\eta$, then according to [3, Lemma 4.1 pp 899], we know that for every $(r,x,t) \in \mathbb{R}\times\mathbb{R}^n\times]0,+\infty[$, we have

$$\left|\frac{\partial \mathscr{U}(f)}{\partial t}(r,x,t)\right| \leqslant \frac{n!(2n+1)!2^{n+\frac{1}{2}}\|f\|_{1,\nu_{n+1}}}{\sqrt{\pi}(2n)!}\frac{1}{t^{2n+2}}. \tag{3.12}$$

The proof is complete by means of relations (3.5), (3.6) and (3.12). □

Lemma 3.4. *Let $f \in \mathscr{D}_e(\mathbb{R}\times\mathbb{R}^n)$, then for every $s>0$, we have*
(i) $\mathscr{U}(W(f)(.,.,s)) \in \mathscr{C}^\infty(\mathbb{R}\times\mathbb{R}^n\times]0,+\infty[)$.
(ii) $\lim_{r^2+|x|^2+t^2\to+\infty}\mathscr{U}(W(f)(.,.,s))(r,x,t) = 0$.

Proof. According to Lemma 3.3, we know that for every $s>0$ the function $W(f)(.,.,s)$ belongs to $L^1(d\nu_{n+1})$ and, therefore, $\mathscr{U}(W(f)(.,.,s))$ is well defined; moreover, by relation (3.2), we have for every $(r,x,t) \in \mathbb{R}\times\mathbb{R}^n\times]0,+\infty[$,

$$\begin{aligned}&\mathscr{U}(W(f)(.,.,s))(r,x,t)\\ &\quad= \int_0^{+\infty}\int_{\mathbb{R}^n} e^{-t\sqrt{\mu^2+|\lambda|^2}}\widetilde{\mathscr{F}}(W(f)(.,.,s))(\mu,\lambda)j_{\frac{n-1}{2}}(r\mu)e^{i\langle\lambda|x\rangle}d\nu_{n+1}(\mu,\lambda),\end{aligned}$$

which implies that $\mathscr{U}(W(f)(.,.,s)) \in \mathscr{C}^\infty(\mathbb{R}\times\mathbb{R}^n\times]0,+\infty[)$.
(ii) Let s be a positive real number, then by relation (2.2),we get

$$\begin{aligned}&\left|r^2\mathscr{U}(W(f)(.,.,s))(r,x,t)\right|\\ &\quad\leqslant\int_0^{+\infty}\int_{\mathbb{R}^n}\left|\ell_{\frac{n-1}{2}}(e^{-t\sqrt{.^2+|\lambda|^2}}\widetilde{\mathscr{F}}(W(f)(.,.,s))(.,\lambda))(\mu)\right|d\nu_{n+1}(\mu,\lambda)\\ &\quad\leqslant C\frac{1+t+t^2}{t^{2n+1}}\end{aligned} \tag{3.13}$$

and, by the same way, we may obtain that there is a nonnegative constant C (not necessarily the same) such that for every $(r,x,t) \in \mathbb{R}\times\mathbb{R}^n\times]0,+\infty[$,

$$\left|x_k^2\mathscr{U}(W(f)(.,.,s))(r,x,t)\right| \leqslant C\frac{1+t+t^2}{t^{2n+1}}, \tag{3.14}$$

and

$$\left|t^2\mathscr{U}(W(f)(.,.,s))(r,x,t)\right| \leqslant C\frac{1+t+t^2}{t^{2n+1}}, \tag{3.15}$$

□

Combining relations (3.13–3.15), we deduce that

$$|\mathscr{U}(W(f)(.,.,s))(r,x,t)| \leqslant \frac{C(1+t+t^2)}{t^{2n+1}(r^2+|x|^2+t^2)}. \tag{3.16}$$

On the other hand by Lemma 3.3, we know that for every $s > 0$, $W(f) \in \mathscr{C}_{0,e}(\mathbb{R} \times \mathbb{R}^n)$, since the family $(p_t)_{t>0}$ is an approximation of identity in $\mathscr{C}_{0,e}(\mathbb{R} \times \mathbb{R}^n)$, then $\lim_{t\to 0^+} \mathscr{U}(W(f)(.,.,s))(.,.,t) = W(f)(.,.,s)$ uniformly. Hence,

- If $t \longrightarrow 0$, then $\lim_{\substack{t\to 0^+\\ r^2+|x|^2\to+\infty}} \mathscr{U}(W(f)(.,.,s))(r,x,t) = 0$.
- If $t \geqslant a$ for some positive constant a, then by relation (3.16) we get

$$\lim_{\substack{t\geqslant a\\ t^2+r^2+|x|^2\to+\infty}} \mathscr{U}(W(f)(.,.,s))(r,x,t) = 0.$$

4. Litellewood–Paley g-Function Associated with the Spherical Mean Operator

The main idea of this section is to prove the L^p-boundedness for $4 \leqslant p < +\infty$ and to use nextly the Marcinkiewicz interpolation theorem. To prove the result for $4 \leqslant p < +\infty$, we are going to apply mainly the Hopf's maximum principle to the uniformly elliptic operator [4]

$$\Delta_{\frac{n-1}{2}} = \frac{\partial^2}{\partial r^2} + \frac{n}{r}\frac{\partial}{\partial r} + \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial t^2}. \tag{4.1}$$

Theorem 4.1 (Strong Hopf's maximum principle). Let

$$L = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^{n} b_j(x)\frac{\partial}{\partial x_j},$$

be an uniformly elliptic operator on a bounded connected domain $\Omega \subset \mathbb{R}^n$, such that the functions a_{ij} and b_j are continuous on $\overline{\Omega}$. Let $u \in \mathscr{C}^2(\Omega) \cap \mathscr{C}(\overline{\Omega})$ such that for every $x \in \Omega$, $Lu(x) \geqslant 0$. If there exists $x_0 \in \Omega$ such that $\sup_{x\in\overline{\Omega}} u(x) = u(x_0)$, then

$$\forall x \in \overline{\Omega},\ u(x) = u(x_0).$$

Proposition 4.2. *Let $a_0, a_1, \ldots, a_n$ and T be positive real numbers and let*

$$\Omega =]-a_0, a_0[\times \prod_{i=1}^{n}]-a_i, a_i[\times]0, T[.$$

Let $u \in \mathscr{C}^2(\Omega) \cap \mathscr{C}(\overline{\Omega})$, be an even function with respect to the first variable, satisfying $\forall (r,x,t) \in \Omega, \quad \Delta_{\frac{n-1}{2}} u(r,x,t) \geqslant 0$. If there is $(r_0,x_0,t_0) \in \Omega$, $r_0 \neq 0$ such that $\sup_{(r,x,t)\in\overline{\Omega}} u(r,x,t) = u(r_0,x_0,t_0)$, then

$$\forall (r,x,t) \in \overline{\Omega},\ u(r,x,t) = u(r_0,x_0,t_0).$$

Proof. Assume that there is $(r_0,x_0,t_0) \in \Omega,\ r_0 \neq 0$ such that

$$\sup_{(r,x,t)\in\overline{\Omega}} u(r,x,t) = u(r_0,x_0,t_0),$$

since the function u is even with respect to the first variable then without loss of generality we can assume that $r_0 > 0$. Let ε be a real number satisfying $0 < \varepsilon < r_0$ and let $\Omega_\varepsilon =]\varepsilon, a_0[\times \prod_{i=1}^{n}]-a_i, a_i[\times]0, T[$, then $(r_0,x_0,t_0) \in \Omega_\varepsilon$

and, therefore, $\sup_{(r,x,t)\in\overline{\Omega_\varepsilon}} u(r,x,t) = \sup_{(r,x,t)\in\overline{\Omega}} u(r,x,t) = u(r_0,x_0,t_0)$. On the other hand, the operator $\Delta_{\frac{n-1}{2}}$ defined by relation (4.1) is uniformly elliptic on the connected bounded domain Ω_ε and satisfies according to the hypothesis $\Delta_{\frac{n-1}{2}} u \geqslant 0$, hence by Theorem 4.1 we deduce that

$$\forall (r,x,t) \in \Omega_\varepsilon,\ u(r,x,t) = u(r_0,x_0,t_0),$$

consequently

$$\forall (r,x,t) \in]0,a_0[\times \left(\prod_{i=1}^{n}]-a_i,a_i[\right) \times]0,T[,\ u(r,x,t) = u(r_0,x_0,t_0),$$

since u is continuous on $\overline{\Omega}$, then

$$\forall (x,t) \in \prod_{i=1}^{n}]-a_i,a_i[\times]0,T[,\ u(0,x,t) = \lim_{r\to 0^+} u(r,x,t) = u(r_0,x_0,t_0),$$

since u is even with respect to the first variable, then

$$\forall (r,x,t) \in \Omega,\ u(r,x,t) = u(r_0,x_0,t_0).$$

However, u is continuous on $\overline{\Omega}$, hence

$$\forall (r,x,t) \in \overline{\Omega},\ u(r,x,t) = u(r_0,x_0,t_0).$$

□

Proposition 4.3. *Let $a_0, a_1, \ldots, a_n$ and T be positive real numbers and let*

$$\Omega =]-a_0,a_0[\times \prod_{i=1}^{n}]-a_i,a_i[\times]0,T[.$$

Let $u \in \mathscr{C}^2(\Omega)\cap\mathscr{C}(\overline{\Omega})$, be a function even with respect to the first variable satisfying

$$\forall (r,x,t) \in \Omega, \quad \Delta_{\frac{n-1}{2}} u(r,x,t) \geqslant 0.$$

If there is $(x_0,t_0) \in \left(\prod_{i=1}^{n}]-a_i,a_i[\right) \times]0,T[$ such that $\sup\limits_{(r,x,t)\in\overline{\Omega}} u(r,x,t) = u(0,x_0,t_0)$, then

$$\forall (r,x,t) \in \overline{\Omega},\ \ u(r,x,t) = u(0,x_0,t_0).$$

Proof. Let $M = u(0,x_0,t_0)$, then by Proposition 4.2, it is sufficient to prove that, there is $(r_1,x_1,t_1) \in \Omega$ such that $r_1 \neq 0$ and $M = u(r_1,x_1,t_1)$. Suppose towards a contradiction that this is not true, then

$$\forall (r,x,t) \in \Omega,\ r \neq 0,\ u(r,x,t) < M. \tag{4.2}$$

Let φ be the function defined on $\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}$, by

$$\varphi(r,x,t) = e^{2r^2-|x-x_0|^2-(t-t_0)^2} - 1,$$

and H be the subset of $\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}$ defined by $H = \varphi^{-1}([0,+\infty[)\cap\Omega$. Since Ω is open and $(0,x_0,t_0) \in \Omega$, then there exists a real $\varepsilon > 0$ such that

$$\overline{B_{(0,x_0,t_0),\varepsilon}} = \{(r,x,t) \in \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}; r^2 + |x-x_0|^2 + (t-t_0)^2 \leqslant \varepsilon^2\} \subset \Omega,$$

and, therefore, $H \cap \partial\overline{B_{(0,x_0,t_0),\varepsilon}} = \varphi^{-1}([0,+\infty[) \cap \overline{B_{(0,x_0,t_0),\varepsilon}}$, in particular the set $K = H \cap \partial\overline{B_{(0,x_0,t_0),\varepsilon}}$ is compact. However, since u is continuous on K, then u attains its maximum on K that is there exists $(r_2, x_2, t_2) \in K$ such that

$$M' = \sup_{(r,x,t)\in K} u(r,x,t) = u(r_2, x_2, t_2).$$

Now, since $(r_2, x_2, t_2) \in K$, then $\varphi(r_2, x_2, t_2) = e^{2r_2^2-|x_2-x_0|^2-(t_2-t_0)^2} - 1 \geqslant 0$, in particular $r_2 \neq 0$ and by assertion (4.2) we get $M' < M$. Now let

$$M'' = \sup_{(r,x,t)\in K} \varphi(r,x,t),$$

since $(\varepsilon, x_0, t_0) \in K$ then $M'' \geqslant \varphi(\varepsilon, x_0, t_0) = e^{2\varepsilon^2} - 1 > 0$. Let $\delta \in]0, \frac{M-M'}{M''}[$, and let ϕ be the function defined on Ω by

$$\phi(r,x,t) = u(r,x,t) + \delta\varphi(r,x,t).$$

For every $(r,x,t) \in \Omega$, we have

$$\begin{aligned}&\Delta_{\frac{n-1}{2}}\varphi(r,x,t)\\ &\quad= \left(16r^2 + 4|x-x_0|^2 + 4(t-t_0)^2 + 2n + 2\right) e^{2r^2-|x-x_0|^2-(t-t_0)^2} > 0\end{aligned}$$

Since $\forall (r,x,t) \in \Omega$, $\Delta_{\frac{n-1}{2}} u(r,x,t) \geqslant 0$, then for every $(r,x,t) \in \Omega$

$$\Delta_{\frac{n-1}{2}}\phi(r,x,t) > 0. \tag{4.3}$$

Let $(r,x,t) \in \partial B_{(0,x_0,t_0),\varepsilon}$,
– If $(r,x,t) \notin H$ then $\varphi(r,x,t) < 0$ and, therefore,

$$\phi(r,x,t) = u(r,x,t) + \delta\varphi(r,x,t) < u(r,x,t) < M.$$

– If $(r,x,t) \in H$ then $(r,x,t) \in K$, hence

$$\phi(r,x,t) = u(r,x,t) + \delta\varphi(r,x,t) \leqslant M' + \delta M'' < M.$$

Hence,

$$\forall (r,x,t) \in \partial B_{(0,x_0,t_0),\varepsilon},\ \phi(r,x,t) < M. \tag{4.4}$$

Let $(r_3, x_3, t_3) \in \overline{B_{(0,x_0,t_0),\varepsilon}}$ such that $\sup_{(r,x,t)\in\overline{B_{(0,x_0,t_0),\varepsilon}}} \phi(r,x,t) = \phi(r_3, x_3, t_3)$, then

$$\phi(r_3, x_3, t_3) \geqslant \phi(0, x_0, t_0) = u(0, x_0, t_0) = M,$$

and by relation (4.4) we deduce that the function ϕ attains its maximum in $(r_3, x_3, t_3) \in B_{(0,x_0,t_0),\varepsilon}$.
• If $r_3 \neq 0$, then

$$\Delta_{\frac{n-1}{2}}\phi(r_3, x_3, t_3) = \frac{\partial^2\phi}{\partial r^2}(r_3, x_3, t_3) + \Delta\phi(r_3, x_3, t_3) + \frac{\partial^2\phi}{\partial t^2}(r_3, x_3, t_3) \leqslant 0. \tag{4.5}$$

• If $r_3 = 0$, since u is even with respect to the first variable, then for every $(x,t) \in \mathbb{R}^n \times \mathbb{R}$, $\frac{\partial u}{\partial r}(0,x,t) = 0$, and therefore $\frac{\partial^2 u}{\partial r^2}(0,x,t) = \lim_{r\to 0} \frac{1}{r}\frac{\partial u}{\partial r}(r,x,t)$, in particular $\ell_{\frac{n-1}{2}} u(0,x,t) = (n+1)\frac{\partial^2 u}{\partial r^2}(0,x,t)$, hence

$$\Delta_{\frac{n-1}{2}} \phi(0,x_3,t_3) = (n+1)\frac{\partial^2 \phi}{\partial r^2}(0,x_3,t_3) + \Delta\phi(0,x_3,t_3) + \frac{\partial^2 \phi}{\partial t^2}(0,x_3,t_3) \leqslant 0. \tag{4.6}$$

Relations (4.5) and (4.6) show that $\Delta_{\frac{n-1}{2}} \phi(r_3,x_3,t_3) \leqslant 0$ which contradicts relation (4.3) and prove that assertion (4.2) is not true. □

Theorem 4.4. *Let $a_0, a_1, \ldots, a_n$ and T be positive real numbers and let*

$$\Omega =]-a_0, a_0[\times]-a_i, a_i[\times]0, T[.$$

Let $u \in \mathscr{C}^2(\Omega) \cap \mathscr{C}(\overline{\Omega})$, be a function even with respect to the first variable satisfying

$$\forall (r,x,t) \in \Omega, \quad \Delta_{\frac{n-1}{2}} u(r,x,t) \geqslant 0.$$

If u attains its maximum in Ω, then u is constant.

Proof. The result follows immediately from Proposition 4.2 and Proposition 4.3. □

Theorem 4.5. *Let $h \in \mathscr{C}^2(\mathbb{R}\times\mathbb{R}^n\times]0,+\infty[) \cap \mathscr{C}(\mathbb{R}\times\mathbb{R}^n\times[0,+\infty[)$, be a function even with respect to the first variable. If*
(i) $\forall (r,x) \in \mathbb{R}\times\mathbb{R}^n$; $\quad h(r,x,0) \geqslant 0$.
(ii) $\lim_{r^2+|x|^2+t^2 \to +\infty} h(r,x,t) = 0$.
(iii) $\forall (r,x,t) \in \mathbb{R}\times\mathbb{R}^n\times[0,+\infty[$, $\quad \Delta_{\frac{n-1}{2}} h(r,x,t) \leqslant 0$. *Then, h is nonnegative.*

Proof. Suppose that there is $(r_1,x_1,t_1) \in \mathbb{R}\times\mathbb{R}^n\times[0,+\infty[$ such that

$$h(r_1,x_1,t_1) < 0. \tag{4.7}$$

Since h is continuous on $\mathbb{R}\times\mathbb{R}^n\times[0,+\infty[$, and according to ii), we deduce that h is bounded on $\mathbb{R}\times\mathbb{R}^n\times[0,+\infty[$ and attains its minimum in $(r_0,x_0,t_0) \in \mathbb{R}\times\mathbb{R}^n\times[0,+\infty[$; furthermore, we have $h(r_0,x_0,t_0) \leqslant h(r_1,x_1,t_1) < 0$, hence according to i) we have $t_0 > 0$. Now, let $b_0, b_1, \ldots, b_n, \varepsilon$ be positive real numbers such that $\varepsilon > \sup(t_0,t_1)$ and such that the set $\Omega_1 = \prod_{j=0}^n]-b_j, b_j[\times]0, 2\varepsilon[$ contains (r_0,x_0,t_0). Let $g = -h$, then g satisfies the hypothesis of Theorem 4.4 on Ω_1, and attains its maximum in $(r_0,x_0,t_0) \in \Omega_1$. This implies that

$$\forall (r,x,t) \in \Omega_1,\ h(r,x,t) = h(r_0,x_0,t_0) < 0.$$

In particular

$$h(r_0,x_0,\varepsilon) = h(r_0,x_0,t_0) < 0. \tag{4.8}$$

Relation (4.8) holds for every $\varepsilon > \sup(t_0,t_1)$ and consequently

$$\lim_{\varepsilon\to+\infty} h(r_0,x_0,\varepsilon) = h(r_0,x_0,t_0) < 0,$$

which contradicts the hypothesis ii). □

Theorem 4.6. *Let $f \in \mathscr{D}_e(\mathbb{R}\times\mathbb{R}^n)$, then for every $(r,x,t) \in \mathbb{R}\times\mathbb{R}^n\times]0,+\infty[$, we have*

$$W(f)(r,x,2t) \leqslant \mathscr{U}\left(W(f)(.,.,t)\right)(r,x,t). \tag{4.9}$$

Proof. Let $f \in \mathscr{D}_e(\mathbb{R}\times\mathbb{R}^n)$, s be a positive real number, and $\Theta_s(f)$ be the function defined on $\mathbb{R}\times\mathbb{R}^n\times[0,+\infty[$, by $\Theta_s(f)(r,x,t) = \mathscr{U}(W(f)(.,.,s))(r,x,t) - W(f)(r,x,s+t)$. Our goal is to show that the function $\Theta_s(f)$ satisfies the assumptions of Theorem 4.5. According to Lemma 3.3, it is clear that for every $s>0$, the function $(r,x,t) \longmapsto W(f)(r,x,s+t) \in \mathscr{C}_e^{\infty}(\mathbb{R}\times\mathbb{R}^n\times]0,+\infty[) \cap \mathscr{C}_e(\mathbb{R}\times\mathbb{R}^n\times[0,+\infty[)$. On the other hand by Lemma 3.4, $\mathscr{U}(W(f)(.,.,s)) \in \mathscr{C}_e^{\infty}(\mathbb{R}\times\mathbb{R}^n\times]0,+\infty[)$; furthermore by Lemma 3.3, we deduce that for every $s>0$, $W(f)(.,.,s) \in L^1(d\nu_{n+1}) \cap \mathscr{C}_{0,e}(\mathbb{R}\times\mathbb{R}^n)$ which implies by Lemma 3.1 that $\mathscr{U}(W(f)(.,.,s)) \in \mathscr{C}_e(\mathbb{R}\times\mathbb{R}^n\times[0,+\infty[)$; this means that $\Theta_s(f) \in \mathscr{C}_e^2(\mathbb{R}\times\mathbb{R}^n\times]0,+\infty[) \cap \mathscr{C}_e(\mathbb{R}\times\mathbb{R}^n\times[0,+\infty[)$ and by relation (3.3) we have $\Theta_s(f)(r,x,0)=0$. From Lemmas 3.3 3.4, we have

$$\lim_{r^2+|x|^2+t^2\to+\infty} \Theta_s(f)(r,x,t) = 0$$

Now, according to relations (3.3) and (3.2), we have

$$\Delta_{\frac{n-1}{2}}(\mathscr{U}(W(f)(.,.,s))(r,x,t) = 0. \tag{4.10}$$

We have, for every $g,h \in \mathscr{C}_e^2(\mathbb{R}\times\mathbb{R}^n\times]0,+\infty[)$,

$$\Delta_{\frac{n-1}{2}}(fg) = g\Delta_{\frac{n-1}{2}}(f) + f\Delta_{\frac{n-1}{2}}(g) + 2\left(\frac{\partial f}{\partial r}\frac{\partial g}{\partial r} + \sum_{j=1}^{n}\frac{\partial f}{\partial x_j}\frac{\partial g}{\partial x_j} + \frac{\partial f}{\partial t}\frac{\partial g}{\partial t}\right). \tag{4.11}$$

We know that $\Delta_{\frac{n-1}{2}}(\mathscr{U}(f)) = 0$, and by the same way for every $1 \leqslant j \leqslant n$,

$$\Delta_{\frac{n-1}{2}}\left(\frac{\partial \mathscr{U}(f)}{\partial x_j}\right) = \frac{\partial}{\partial x_j}\left(\Delta_{\frac{n-1}{2}}\mathscr{U}(f)\right) = 0. \tag{4.12}$$

and also

$$\Delta_{\frac{n-1}{2}}\left(\frac{\partial \mathscr{U}(f)}{\partial t}\right) = \frac{\partial}{\partial t}\left(\Delta_{\frac{n-1}{2}}\mathscr{U}(f)\right) = 0. \tag{4.13}$$

Then, by a standard calculus, we get

$$\Delta_{\frac{n-1}{2}}\left(\frac{\partial \mathscr{U}(f)}{\partial r}\right) = \frac{\partial}{\partial r}\left(\Delta_{\frac{n-1}{2}}\mathscr{U}(f)\right) + \frac{n}{r^2}\frac{\partial \mathscr{U}(f)}{\partial r} = \frac{n}{r^2}\frac{\partial \mathscr{U}(f)}{\partial r}. \tag{4.14}$$

Combining relations (3.8),(4.11),(4.12), (4.13) and (4.14), we deduce that

$$\begin{aligned}
&\Delta_{\frac{n-1}{2}} W(f)\\
&= \Delta_{\frac{n-1}{2}} \Big(\frac{\partial(\mathscr{U}(f))}{\partial r}\Big)^2 + \sum_{j=1}^{n} \Delta_{\frac{n-1}{2}} \Big(\frac{\partial(\mathscr{U}(f))}{\partial x_j}\Big)^2 + \Delta_{\frac{n-1}{2}} \Big(\frac{\partial(\mathscr{U}(f))}{\partial t}\Big)^2\\
&= \frac{2n}{r^2}\left(\frac{\partial \mathscr{U}(f)}{\partial r}\right)^2 + 2\left|\nabla\left(\frac{\partial(\mathscr{U}(f))}{\partial r}\right)\right|^2 + 4\Big(\sum_{j=1}^{n}\left(\frac{\partial}{\partial x_j}\left(\frac{\partial \mathscr{U}(f)}{\partial r}\right)\right)^2\\
&\quad + \left(\frac{\partial}{\partial t}\left(\frac{\partial \mathscr{U}(f)}{\partial x_j}\right)\right)^2 + \left(\frac{\partial}{\partial t}\left(\frac{\partial \mathscr{U}(f)}{\partial r}\right)\right)^2\Big) \geqslant 0. \qquad (4.15)
\end{aligned}$$

Relations (4.10) and (4.15) imply that

$$\forall (r,x,t) \in \mathbb{R} \times \mathbb{R}^n \times [0,+\infty[,\ \Delta_{\frac{n-1}{2}} \Theta_s(f)(r,x,t) \leqslant 0,$$

and Corollary 4.5 achieves then the proof. □

According to [3], the Littlewood–Paley g-function associated with the spherical mean operator is defined for $f \in \mathscr{D}_e(\mathbb{R}\times\mathbb{R}^n)$ by

$$\forall (r,x) \in [0,+\infty[\times\mathbb{R}^n,\ g(f)(r,x) = \left(\int_0^{+\infty} |\nabla(\mathscr{U}(f))(r,x,t)|^2\, tdt\right)^{1/2}.$$

Lemma 4.7. *For every nonnegative functions $f, h \in \mathscr{D}_e(\mathbb{R}\times\mathbb{R}^n)$, we have*

$$\begin{aligned}
&\int_0^{+\infty}\int_{\mathbb{R}^n} \big(g(f)(r,x)\big)^2 h(r,x)d\nu_{n+1}(r,x)\\
&\quad \leqslant 4\int_0^{+\infty}\int_0^{+\infty}\int_{\mathbb{R}^n} |\nabla \mathscr{U}(f)(r,x,t)|^2\, \mathscr{U}(h)(r,x,t)d\nu_{n+1}(r,x)tdt.
\end{aligned}$$

Proof. By relations (3.8) and (4.9) and using Fubini's Theorem we get

$$\begin{aligned}
&\int_0^{+\infty}\int_{\mathbb{R}^n} \big(g(f)(r,x)\big)^2 h(r,x)d\nu_{n+1}(r,x)\\
&\leqslant \int_0^{+\infty} t\left(\int_0^{+\infty}\int_{\mathbb{R}^n} h(r,x)\mathscr{P}^{\frac{t}{2}}\Big(\left|\nabla \mathscr{P}^{\frac{t}{2}}(f)\right|^2\Big)(r,x)d\nu_{n+1}(r,x)\right)dt.
\end{aligned}$$

Since $\mathscr{P}^{\frac{t}{2}}$ is a self-adjoint operator in $L^2(d\nu_{n+1})$, then we deduce that

$$\begin{aligned}
&\int_0^{+\infty}\int_{\mathbb{R}^n} \big(g(f)(r,x)\big)^2 h(r,x)d\nu_{n+1}(r,x)\\
&\leqslant \int_0^{+\infty} t\left(\int_0^{+\infty}\int_{\mathbb{R}^n} \mathscr{P}^{\frac{t}{2}}(h)(r,x)\left|\nabla \mathscr{P}^{\frac{t}{2}}(f)(r,x)\right|^2 d\nu_{n+1}(r,x)\right)dt.
\end{aligned}$$

The result follows by the change of variables $s = \frac{t}{2}$. □

Lemma 4.8. *For every positive functions $f, g \in \mathscr{D}_e(\mathbb{R}\times\mathbb{R}^n)$ such that $Supp(f)\cup Supp(g) \subset B_\eta$, we have*

$$\int_0^{+\infty}\int_0^{+\infty}\int_{\mathbb{R}^n} \Delta_{\frac{n-1}{2}}\left(\mathscr{U}(f)^2\mathscr{U}(g)\right)(r,x,t)tdtd\nu_{n+1}(r,x) = \int_0^{+\infty}\int_{\mathbb{R}^n} f^2(r,x)g(r,x)d\nu_{n+1}(r,x). \tag{4.16}$$

Proof. By the relation (4.11), we get

$$\Delta_{\frac{n-1}{2}}\left(\mathscr{U}(f)^2\mathscr{U}(g)\right) = \mathscr{U}(g)\Delta_{\frac{n-1}{2}}(\mathscr{U}(f)^2) + \mathscr{U}(f)^2\Delta_{\frac{n-1}{2}}(\mathscr{U}(g)) + 2\frac{\partial(\mathscr{U}(f)^2)}{\partial r}\frac{\partial(\mathscr{U}(g))}{\partial r} + 2\sum_{j=1}^{n}\frac{\partial(\mathscr{U}(f)^2)}{\partial x_j}\frac{\partial(\mathscr{U}(g))}{\partial x_j} + 2\frac{\partial(\mathscr{U}(f)^2)}{\partial t}\frac{\partial(\mathscr{U}(g))}{\partial t}. \tag{4.17}$$

Using the fact that $\Delta_{\frac{n-1}{2}}\big(\mathscr{U}(f)\big) = \Delta_{\frac{n-1}{2}}\big(\mathscr{U}(g)\big) = 0$, we obtain

$$\Delta_{\frac{n-1}{2}}\Big(\mathscr{U}(f)^2\Big) = 2\left|\nabla\big(\mathscr{U}(f)\big)\right|^2. \tag{4.18}$$

On the other hand,

$$\left|2\frac{\partial\big(\mathscr{U}(f)^2\big)}{\partial r}\frac{\partial(\mathscr{U}(g))}{\partial r} + 2\sum_{j=1}^{n}\frac{\partial\big(\mathscr{U}(f)^2\big)}{\partial x_j}\frac{\partial(\mathscr{U}(g))}{\partial x_j} + 2\frac{\partial\big(\mathscr{U}(f)^2\big)}{\partial t}\frac{\partial(\mathscr{U}(g))}{\partial t}\right| \leqslant 2\mathscr{U}(f)\left|\nabla\mathscr{U}(f)\right|^2 + 2\mathscr{U}(f)\left|\nabla\mathscr{U}(g)\right|^2. \tag{4.19}$$

Since $\mathscr{U}(f)$ and $\mathscr{U}(g)$ are bounded, then using relations (4.11), (4.18) and (4.19) we deduce that there is a positive constant C such that

$$\begin{aligned}\int_0^{+\infty}\int_0^{+\infty}\int_{\mathbb{R}^n} &\left|\Delta_{\frac{n-1}{2}}\left(\mathscr{U}(f)^2\mathscr{U}(g)\right)(r,x,t)\right| tdtd\nu_{n+1}(r,x)\\ &\leqslant C\int_0^{+\infty}\int_0^{+\infty}\int_{\mathbb{R}^n}\left|\nabla\mathscr{U}(f)(r,x,t)\right|^2 tdtd\nu_{n+1}(r,x)\\ &\quad + C\int_0^{+\infty}\int_0^{+\infty}\int_{\mathbb{R}^n}\left|\nabla\mathscr{U}(g)(r,x,t)\right|^2 tdtd\nu_{n+1}(r,x).\end{aligned}$$

Using now the relation (4.18) and [11, proposition 4.3], we get

$$\begin{aligned}\int_0^{+\infty}\int_0^{+\infty}\int_{\mathbb{R}^n} &\left|\Delta_{\frac{n-1}{2}}\left(\mathscr{U}(f)^2\mathscr{U}(g)\right)(r,x,t)\right| tdtd\nu_{n+1}(r,x)\\ &\leqslant \frac{C}{2}\left(\|f\|^2_{2,\nu_{n+1}} + \|g\|^2_{2,\nu_{n+1}}\right) < +\infty.\end{aligned}$$

Then, we have

$$\int_0^{+\infty}\int_0^{+\infty}\int_{\mathbb{R}^n} \Delta_{\frac{n-1}{2}}\left(\mathscr{U}(f)^2\mathscr{U}(g)\right)(r,x,t)tdtd\nu_{n+1}(r,x) = \lim_{\xi\to+\infty}\int_0^{\xi}\int_0^{\xi}\int_{[-\xi,\xi]^n} \Delta_{\frac{n-1}{2}}\left(\mathscr{U}(f)^2\mathscr{U}(g)\right)(r,x,t)tdtd\nu_{n+1}(r,x). \tag{4.20}$$

By Fubini's theorem, we obtain

$$\int_0^\xi\int_0^\xi\int_{[-\xi,\xi]^n}\Delta_{\frac{n-1}{2}}\left(\mathscr{U}(f)^2\mathscr{U}(g)\right)(r,x,t)tdtd\nu_{n+1}(r,x)$$
$$=I(\xi)+\sum_{j=1}^n I_j(\xi)+K(\xi). \tag{4.21}$$

where

- $I(\xi)=\int_0^\xi\int_0^\xi\int_{[-\xi,\xi]^n}\ell_{\frac{n-1}{2}}\Big(\mathscr{U}(f)^2\mathscr{U}(g)\Big)(r,x,t)d\nu_{n+1}(r,x)tdt$
  $=\frac{\xi^n}{(\pi)^{\frac{n}{2}}2^{n-\frac{1}{2}}\Gamma(\frac{n+1}{2})}\int_0^\xi\int_{[-\xi,\xi]^n}\frac{\partial\Big(\mathscr{U}(f)^2\mathscr{U}(g)\Big)}{\partial r}(\xi,x,t)tdtdx.$

For $\xi\geqslant 2\eta$, and from [11, Lemma 4.2], we deduce that

$$|\frac{\partial\Big(\mathscr{U}(f)^2\mathscr{U}(g)\Big)}{\partial r}(\xi,x,t)|\leqslant\frac{C}{\xi^{6n+4}}.$$

Hence,

$$|I(\xi)|\leqslant\frac{C_1}{\xi^{4n+2}}\longrightarrow_{\xi\to+\infty}0. \tag{4.22}$$

- $I_j(\xi)=\int_0^\xi\int_0^\xi\int_{[-\xi,\xi]^n}\frac{\partial^2}{\partial x_j^2}\left(\mathscr{U}(f)^2\mathscr{U}(g)\right)(r,x,t)tdtd\nu_{n+1}(r,x).$

As the same way and using again [11, Lemma 4.2], we show that

$$|I_j(\xi)|\leqslant\frac{C_2}{\xi^{4n+2}}\longrightarrow_{\xi\to+\infty}0. \tag{4.23}$$

And $K(\xi)=\displaystyle\int_0^\xi\int_{[-\xi,\xi]^n}\left(\int_0^\xi\frac{\partial^2\Big(\mathscr{U}(f)^2\mathscr{U}(g)\Big)}{\partial t^2}(r,x,t)tdt\right)d\nu_{n+1}(r,x)$. Integrating by parts and using relation (3.3), we get

$$\int_0^\xi\frac{\partial^2\Big(\mathscr{U}(f)^2\mathscr{U}(g)\Big)}{\partial t^2}(r,x,t)tdt$$
$$=\xi\frac{\partial\Big(\mathscr{U}(f)^2\mathscr{U}(g)\Big)}{\partial t}(r,x,\xi)-\Big(\mathscr{U}(f)^2\mathscr{U}(g)\Big)(r,x,\xi)+\Big(f(r,x)\Big)^2g(r,x),$$

by [11, lemma 4.1], for $\xi>0$, and $(r,x)\in[0,+\infty[\times\mathbb{R}^n$

$$\Big|\xi\frac{\partial\Big(\mathscr{U}(f)^2\mathscr{U}(g)\Big)}{\partial t}(r,x,\xi)-\Big(\mathscr{U}(f)^2\mathscr{U}(g)\Big)(r,x,\xi)\Big|\leqslant\frac{C}{\xi^{6n+3}}.$$

Consequently,

$$\lim_{\xi\to+\infty}K(\xi)=\int_0^{+\infty}\int_{\mathbb{R}^n}\Big(f(r,x)\Big)^2g(r,x)d\nu_{n+1}(r,x). \tag{4.24}$$

Then, the result follows from relations (4.20), (4.21), (4.22), (4.23) and (4.24).

□

In [3], Hleili and Omri defined the maximal function f^* associated with the spherical mean operator by

$$f^*(r,x) = \sup_{t>0} \left|\mathscr{P}^t(f)(r,x)\right|, \tag{4.25}$$

and they showed that for every $f \in \mathscr{D}_e(\mathbb{R}\times\mathbb{R}^n)$, the maximal function f^* belongs to the space $L^p(d\nu_{n+1})$ and satisfies

$$\|f^*\|_{p,\nu_{n+1}} \leqslant 2\left(\frac{p}{p-1}\right)^{\frac{1}{p}} \|f\|_{p,\nu_{n+1}}. \tag{4.26}$$

Lemma 4.9. *For every nonnegative functions $f, h \in \mathscr{D}_e(\mathbb{R}\times\mathbb{R}^n)$, we have*

$$\begin{aligned}&\int_0^{+\infty}\int_{\mathbb{R}^n} \big(g(f)(r,x)\big)^2 h(r,x)d\nu_{n+1}(r,x)\\ &\leqslant 2\int_0^{+\infty}\int_{\mathbb{R}^n} \big|f(r,x)\big|^2 h(r,x)d\nu_{n+1}(r,x) + 8\\ &\int_0^{+\infty}\int_{\mathbb{R}^n} f^*(r,x)g(f)(r,x)g(h)(r,x)d\nu_{n+1}(r,x).\end{aligned}$$

Proof. Using relation (4.18) and Lemma 4.7, we get

$$\begin{aligned}&\int_0^{+\infty}\int_{\mathbb{R}^n} \big(g(f)(r,x)\big)^2 h(r,x)d\nu_{n+1}(r,x)\\ &\leqslant 2\int_0^{+\infty}\int_0^{+\infty}\int_{\mathbb{R}^n} \Delta_{\frac{n-1}{2}}\left((\mathscr{P}^t(f))^2\right)(r,x)\mathscr{P}^t(h)(r,x)d\nu_{n+1}(r,x)tdt.\end{aligned} \tag{4.27}$$

Now, by a standard calculus, we have

$$\begin{aligned}&\Delta_{\frac{n-1}{2}}\left((\mathscr{P}^t(f))^2\right)\mathscr{P}^t(h)\\ &\quad= \Delta_{\frac{n-1}{2}}\left((\mathscr{P}^t(f))^2\mathscr{P}^t(h)\right) - 4\mathscr{P}^t(f)\langle\nabla(\mathscr{P}^t(f)) \mid \nabla(\mathscr{P}^t(h))\rangle.\end{aligned} \tag{4.28}$$

Combining relations (4.16), (4.25), (4.27) and (4.28), and using again Cauchy-Schwarz inequality, we get the desired result. □

Proposition 4.10. *For every $p \in [4,+\infty[$ and for every nonnegative function $f \in \mathscr{D}_e(\mathbb{R}\times\mathbb{R}^n)$, the function $g(f)$ belongs to the space $L^p(d\nu_{n+1})$ and we have*

$$\|g(f)\|_{p,\nu_{n+1}} \leqslant A_p\|f\|_{p,\nu_{n+1}},$$

where $A_p = \sqrt{2}\left(8\frac{p^{\frac{2-p}{2p}}(p-2)^{\frac{3p-4}{2p}}}{(p-1)^{\frac{1}{p}}} + \sqrt{1+64\frac{p^{\frac{2-p}{p}}(p-2)^{\frac{3p-4}{p}}}{(p-1)^{\frac{2}{p}}}}\right)$.

Proof. Let q be the conjugate exponent of $\frac{p}{2}$, that is $\frac{2}{p}+\frac{1}{q}=1$, and let $f,h \in \mathscr{D}_e(\mathbb{R}\times\mathbb{R}^n)$ be nonnegative functions, such that $\|h\|_{q,\nu_{n+1}} \leqslant 1$.

From Lemma 4.9 and the generalized Hölder's inequality, we get

$$\int_0^{+\infty}\int_{\mathbb{R}^n}\big(g(f)(r,x)\big)^2 h(r,x)d\nu_{n+1}(r,x)$$
$$\leqslant 2\|f\|^2_{p,\nu_{n+1}} + 8\|f^*\|_{p,\nu_{n+1}}\|g(f)\|_{p,\nu_{n+1}}\|g(h)\|_{q,\nu_{n+1}}.$$

Therefore, using relation (4.26) and [3, Proposition4.6], we deduce that

$$\int_0^{+\infty}\int_{\mathbb{R}^n}\big(g(f)(r,x)\big)^2 h(r,x)d\nu_{n+1}(r,x)$$
$$\leqslant 2\|f\|^2_{p,\nu_{n+1}} + 8\frac{2^{\frac{3}{2}}p^{\frac{2-p}{2p}}(p-2)^{\frac{3p-4}{2p}}}{(p-1)^{\frac{1}{p}}}\|f\|_{p,\nu_{n+1}}\ \|g(f)\|_{p,\nu_{n+1}}.$$

Hence,

$$\|g(f)\|^2_{p,\nu_{n+1}} = \sup_{\|h\|_{q,\nu_{n+1}}\leqslant 1}\Big(\int_0^{+\infty}\int_{\mathbb{R}^n} g^2(f)(r,x)h(r,x)d\nu_{n+1}(r,x)\Big)$$
$$\leqslant \Big(\sqrt{2}\|f\|_{p,\nu_{n+1}} + \frac{8p^{\frac{2-p}{2p}}(p-2)^{\frac{3p-4}{2p}}}{(p-1)^{\frac{1}{p}}}\|g(f)\|_{p,\nu_{n+1}}\Big)^2$$
$$-\frac{64p^{\frac{2-p}{p}}(p-2)^{\frac{3p-4}{p}}}{(p-1)^{\frac{2}{p}}}\|g(f)\|^2_{p,\nu_{n+1}}.$$

□

Proposition 4.11. *For every $p\in[4,+\infty[$ and for every $f\in\mathscr{D}_e(\mathbb{R}\times\mathbb{R}^n)$, the function $g(f)$ belongs to the space $L^p(d\nu_{n+1})$ and we have*

$$\|g(f)\|_{p,\nu_{n+1}} \leqslant B_p\|f\|_{p,\nu_{n+1}}.$$

Proof. Let $p\in[4,+\infty[$ and $f\in\mathscr{D}_e(\mathbb{R}\times\mathbb{R}^n)$ such that $Supp(f)\subset B_a$, without loss of generality, we can assume that f is real valued and we consider $f^+ = \dfrac{f+|f|}{2}$, $f^- = \dfrac{-f+|f|}{2}$. Then, f^+ is nonnegative, belongs to $\mathscr{C}_{e,c}(\mathbb{R}\times\mathbb{R}^n)$ and satisfies $Supp(f^+)\subset B_a$. From [11, Lemma 4.5], we know that for every real number $0<\varepsilon<1$, there is a nonnegative function $h_1\in\mathscr{D}_e(\mathbb{R}\times\mathbb{R}^n)$ such that $Supp(h_1)\subset B_{a+2}$ and

$$\forall (r,x)\in[0,+\infty[\times\mathbb{R}^n,\quad 0\leqslant h_1(r,x)-f^+(r,x)\leqslant\varepsilon. \tag{4.29}$$

On the other hand, the function $h_2 = h_1 - f = h_1 - f^+ + f^-$ is nonnegative, belongs to the space $\mathscr{D}_e(\mathbb{R}\times\mathbb{R}^n)$, and satisfies $Supp(h_2)\subset B_{a+2}$. Moreover,

$$\forall (r,x)\in[0,+\infty[\times\mathbb{R}^n,\quad 0\leqslant h_2(r,x)-f^-(r,x) = h_1(r,x)-f^+(r,x)\leqslant\varepsilon,$$

and $f = h_1 - h_2$. Since, the mapping $f\mapsto g(f)$ is sub-linear in the sense $g(f_1+f_2)\leqslant g(f_1)+g(f_2)$, we deduce that $g(f)\leqslant g(h_1)+g(h_2)$. Using Proposition 4.10, it follows that for every $f\in\mathscr{D}_e(\mathbb{R}\times\mathbb{R}^n)$

$$\|g(f)\|_{p,\nu_{n+1}} \leqslant \|g(h_1)\|_{p,\nu_{n+1}} + \|g(h_2)\|_{p,\nu_{n+1}}$$
$$\leqslant A_p\left(\|h_1\|_{p,\nu_{n+1}} + \|h_2\|_{p,\nu_{n+1}}\right).$$

Now, from relation (4.29) and Minkowski's inequality, we get

$$\|h_1\|_{p,\nu_{n+1}} = \left(\int\int_{B^+_{a+2}} (h_1(r,x))^p d\nu_{n+1}(r,x)\right)^{\frac{1}{p}}$$
$$\leqslant \|f\|_{p,\nu_{n+1}} + \varepsilon\left(\nu_{n+1}(B^+_{a+2})\right)^{\frac{1}{p}},$$

and by the same way $\|h_2\|_{p,\nu_{n+1}} \leqslant \|f\|_{p,\nu_{n+1}} + \varepsilon\left(\nu_{n+1}(B^+_{a+2})\right)^{\frac{1}{p}}$. This means that for every $\varepsilon \in \mathbb{R}$, $0 < \varepsilon < 1$,

$$\|g(f)\|_{p,\nu_{n+1}} \leqslant 2A_p\left(\|f\|_{p,\nu_{n+1}} + \varepsilon\left(\nu_{n+1}(B^+_{a+2})\right)^{\frac{1}{p}}\right),$$

and, consequently, $\|g(f)\|_{p,\nu_{n+1}} \leqslant 2A_p\|f\|_{p,\nu_{n+1}} = B_p\|f\|_{p,\nu_{n+1}}$. □

Theorem 4.12. *For every $p \in [4,+\infty[$ and for every $f \in L^p(d\nu_{n+1})$, the function $g(f)$ belongs to the space $L^p(d\nu_{n+1})$ and we have*

$$\|g(f)\|_{p,\nu_{n+1}} \leqslant B_p\|f\|_{p,\nu_{n+1}}.$$

Proof. Let $p \in [4,+\infty[$ and let $f \in L^p(d\nu_{n+1})$, then there is a sequence $(f_k)_{k\in\mathbb{N}} \subset \mathscr{D}_e(\mathbb{R}\times\mathbb{R}^n)$ such that $\lim_{k\to+\infty}\|f - f_k\|_{p,\nu_{n+1}} = 0$. Since the mapping $g \longmapsto g(f)$ is sub-linear, then for every $k, k' \in \mathbb{N}$, we have

$$\left|g(f_k) - g(f_{k'})\right| \leqslant g(f_k - f_{k'}),$$

hence, by Proposition 4.11, we get

$$\|g(f_k) - g(f_{k'})\|_{p,\nu_{n+1}} \leqslant \|g(f_k - f_{k'})\|_{p,\nu_{n+1}} \leqslant B_p\|f_k - f_{k'}\|_{p,\nu_{n+1}} \underset{k,k'\to+\infty}{\longrightarrow} 0.$$

This means that $((g(f_k))_{k\in\mathbb{N}}$ is a Cauchy sequence which converges in L^p $(d\nu_{n+1})$. We put $g(f) = \lim_{k\to+\infty} g(f_k)$.

It is clear that $g(f)$ is independent of the choice of $(f_k)_{k\in\mathbb{N}}$, and we have $\|g(f)\|_{p,\nu_{n+1}} \leqslant \lim_{k\to+\infty} B_p\|f_k\|_{p,\nu_{n+1}} = B_p\|f\|_{p,\nu_{n+1}}$. □

Theorem 4.13. *For every $p \in]1,+\infty[$ and for every function $f \in L^p(d\nu_{n+1})$, the function $g(f)$ belongs to the space $L^p(d\nu_{n+1})$, and we have*

$$\|g(f)\|_{p,\nu_{n+1}} \leqslant D_p\|f\|_{p,\nu_{n+1}}, \tag{4.30}$$

where

$$D_p = \begin{cases} C_p, & \text{if } p \in]1,2] \\ B_p, & \text{if } p \in [4,+\infty[\\ 2^{\frac{4-p}{2p}} B_4^{\frac{2(p-2)}{p}}, & \text{if } p \in [2,4] \end{cases}$$

Proof. The result is a consequence of Theorem [3, Theorem4.7], Theorem 4.12 and Marcinkiewicz interpolation theorem. □

Theorem 4.14. *For every $p \in]1,+\infty[$ and for every $f \in L^p(d\nu_{n+1})$, the function $g(f)$ belongs to the space $L^p(d\nu_{n+1})$ and we have*

$$\|f\|_{p,\nu_{n+1}} \leqslant 4D_q\|g(f)\|_{p,\nu_{n+1}},$$

where D_q is the constant given by relation (4.30) and $q = \dfrac{p}{p-1}$.

Proof. Let $f \in \mathscr{D}_e(\mathbb{R} \times \mathbb{R}^n)$ and $g_1(f)$ be the function defined on $[0,+\infty[\times\mathbb{R}^n$ by

$$g_1(f)(r,x) = \left(\int_0^{+\infty} \left|\frac{\partial \mathscr{U}(f)}{\partial t}(r,x,t)\right|^2 tdt\right)^{\frac{1}{2}}.$$

From relation (3.2), we know that

$$\mathscr{U}(f)(r,x,t) = \int_0^{+\infty}\int_{\mathbb{R}^n} e^{-t\sqrt{s^2+|y|^2}}\widetilde{\mathscr{F}}(f)(s,y)j_{\frac{n-1}{2}}(rs)e^{i\langle y|x\rangle}d\nu_{n+1}(s,y).$$

Hence, for every $(r,x,t) \in \mathbb{R}\times\mathbb{R}^n\times]0,+\infty[$, we have

$$\frac{\partial \mathscr{U}(f)}{\partial t}(r,x,t) = \widetilde{\mathscr{F}}^{-1}\left(-\sqrt{s^2+|y|^2}e^{-t\sqrt{s^2+|y|^2}}\widetilde{\mathscr{F}}(f)\right)(r,x),$$

and, therefore, using Plancherel's theorem, we get

$$\begin{aligned}&\|g_1(f)\|^2_{2,\nu_{n+1}}\\ &= \int_0^{+\infty}\left(\int_0^{+\infty}\int_{\mathbb{R}^n}(r^2+|x|^2)e^{-2t\sqrt{r^2+|x|^2}}\left|\widetilde{\mathscr{F}}(f)(r,x)\right|^2 d\nu_{n+1}(r,x)\right)tdt.\end{aligned}$$

$$\|g_1(f)\|^2_{2,\nu_{n+1}} = \frac{1}{4}\|\widetilde{\mathscr{F}}(f)\|^2_{2,\nu_{n+1}} = \frac{1}{4}\|f\|^2_{2,\nu_{n+1}}.$$

Let $h \in \mathscr{D}_e(\mathbb{R}\times\mathbb{R}^n)$, then by Schwarz's inequality, we have

$$\begin{aligned}\int_0^{+\infty}&\int_{\mathbb{R}^n} f(r,x)h(r,x)d\nu_{n+1}(r,x)\\ &= \|g_1(f+h)\|^2_{2,\nu_{n+1}} - \|g_1(f-h)\|^2_{2,\nu_{n+1}}\\ &\leqslant 4\int_0^{+\infty}\int_{\mathbb{R}^n} g_1(f)(r,x)g_1(h)(r,x)d\nu_{n+1}(r,x),\end{aligned}$$

and, therefore, for every $p,q \in]1,+\infty[$ such that $\frac{1}{p}+\frac{1}{q}=1$, we get

$$\int_0^{+\infty}\int_{\mathbb{R}^n} f(r,x)h(r,x)d\nu_{n+1}(r,x) \leqslant 4\|g_1(f)\|_{p,\nu_{n+1}}\|g_1(h)\|_{q,\nu_{n+1}}.$$

In particular if $\|h\|_{q,\nu_{n+1}} \leqslant 1$ then by relation (4.30), we deduce that

$$\int_0^{+\infty}\int_{\mathbb{R}^n} f(r,x)h(r,x)d\nu_{n+1}(r,x) \leqslant 4D_q\|g(f)\|_{p,\nu_{n+1}}.$$

Since,

$$\|f\|_{p,\nu_{n+1}} = \sup_{\|h\|_{q,\nu_{n+1}}\leqslant 1}\left\{\int_0^{+\infty}\int_{\mathbb{R}^n} f(r,x)h(r,x)d\nu_{n+1}(r,x)\right\},$$

then, $\|f\|_{p,\nu_{n+1}} \leqslant 4D_q\|g(f)\|_{p,\nu_{n+1}}$. □

References

[1] Achour, A., Trimèche, K.: La g-fonction de Littlewood–Paley associée à un opérateur différentiel singulier sur $]0, +\infty[$. Ann. Inst. Fourier, Grenoble, **33**, 203–226 (1983)

[2] Annabi, H., Fitouhi, A.: La g-fonction de, Littlewood–Paley associée à une classe d'opérateurs différentiels sur $]0, +\infty[$ contenant l'opérateur de Bessel. C. R. Acad. Sci. Paris **303**, 411–413 (1986)

[3] Hleili, K., Omri, S.: The Littlwood–Paley g-function associated with the spherical mean operator. Mediterr. J. Math. **10**(2), 887–907 (2013)

[4] Hopf, E.: A remark on linear elliptic differential equations of the second order. Proc. Am. Math. Soc. **3**, 791–793 (1952)

[5] Lamouchi, H., Majjaouli, B., Omri, S.: Localization of orthonormal sequences in the spherical mean setting, Mediterr. J. Math. (2015). doi:10.1007/s00009-015-0572-9

[6] Lebedev, N.N.: Special Functions and their Applications. Dover publications, New York (1972)

[7] Nessibi, M.M., Rachdi, L.T., Trimèche, K.: Ranges and inversion formulas for spherical mean operator and its dual. J. Math. Anal. Appl. **196**(3), 861–884 (1995)

[8] Omri, S.: Uncertainty principle in terms of entropy for the spherical mean operator. J. Math. Inequal **5**(4), 473–490 (2011)

[9] Rachdi, L.T., Trimèche, K.: Weyl transforms associated with the spherical mean operator. Anal. Appl. **1**(N2), 141–164 (2003)

[10] Stein, E.M.: Topics in Harmonic Analysis Related to the Littlewood–Paley Theory, Annals of Mathematical Studies. Vol. 63, Princeton Univ. Press, Princeton (1970)

[11] Stempak, K.: La thorie de Littlewood–Paley pour la transformation de Fourier-Bessel. C.R.A.S. Paris, Srie I, Math. **303**(3), 15–18 (1986)

A. Hammami
Departement de mathmatiques
Facult des sciences de Tunis
Campus universitaire El Manar
2092, Tunis, Tunisia
e-mail: hammami2016@gmail.com

S. Omri
Departement de mathmatiques
Institut prparatoire aux études d'ingnieurs El Manar Campus
universitaire El Manar
2092, Tunis, Tunisia
e-mail: slim.omri@ipein.rnu.tn

Received: February 2, 2016.

Revised: May 18, 2016.

Accepted: May 23, 2016.