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On a Result of Turpin

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Abstract. We consider several formulations of the equicontinuity of the sequence of power maps $(x \longmapsto x^n)_n$ in the non-commutative case. We give some analog of a result of Turpin for a locally convex algebra not necessarily commutative. The link with the operation of entire functions is also examined.

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1. Preliminaries and Introduction

A locally convex algebra (*l.c.a.* for short) is a Hausdorff locally convex space which is an algebra over a field $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$) with separately continuous multiplication. If the multiplication is continuous in both variables, it is said to be with jointly continuous multiplication. Let (A, τ) be a locally convex algebra, the topology of which is given by a family of $(p_i)_{i\in I}$ of seminorms. It is said to be multiplicatively m-convex (m-convex for short) if

$$p_i\,(xy) \leq p_i\,(x)\,p_i\,(y)\,, \quad \text{for all } x, y \in A,\ i \in I.$$

A B_0-algebra is an *l.c.a.* whose underlying locally convex space is a completely metrisable space. An entire function $f(z) = \sum_{n=0}^{+\infty} a_n z^n$, $a_n \in \mathbb{K}$, operates in an *l.c.a.* (A, τ) if, for every x in A, $f(x) = \sum_{n=0}^{+\infty} a_n x^n$, converges in (A, τ). A unital topological algebra is said to be Q-algebra if and only if the set of all invertible elements of A is open. Let E be a locally convex space. The space E is said to be Mackey-complete (M-complete for short) if and only if every bounded and closed disk B is completant, i.e., the space $(E_B, \|.\|_B)$ is Banach, where $E_B = \bigcup_{\lambda>0} \lambda B$ is the span of B and $\|.\|_B$ is the gauge of B. For a detailed account of basis properties of general locally m-convex algebras and B_0-algebras, we refer the reader to [7,11].

In [9], Turpin showed that a commutative *l.c.a.* (A, τ), in which the sequence $(x \longmapsto x^n)_n$ of power maps is equicontinuous at zero is necessarily m-convex. In the non-commutative case, Żelazko gives in [13] an example

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of a complete non-m-convex locally convex algebra, in which the sequence $(x \longmapsto x^n)_n$ of power maps is equicontinuous at zero. Analysis of the situation in the non-commutative case leads us to consider the following properties in a locally convex algebra (A, τ):

($\mathcal{P}_1$) $\forall V \in V(0)$, $\exists U \in V(0) : U^{(n)} \subset V$, for every $n \in \mathbb{N}^*$, where $U^{(n)} = \{x_1 \cdots x_n : \ x_1, \ldots, x_n \in U\}$, and $V(0)$ be a fundamental system of neighborhood of zero.

($\mathcal{P}_2$) $\forall V \in V(0)$, $\exists U \in V(0)$: $\frac{1}{n!} \sum x_{j_1} \cdots x_{j_n} \in V$, for every $x_1, \ldots, x_n \in U$, $n \in \mathbb{N}^*$, and where the sum is taken over all permutations $(j_1, \ldots, j_n)$ of the sequence $(1, \ldots, n)$.

($\mathcal{P}_3$) The topology τ can be given by a family of seminorms $(p_i)_{i \in I}$, such that, for every $i \in I$, there exists $j \in I$, such that:

$$p_i(x^n) \leq (p_j(x))^n, \quad \text{for all } x \in A, \ n \in \mathbb{N}^*.$$

($\mathcal{P}_4$) The sequence $(x \longmapsto x^n)_n$ of power maps is equicontinuous at zero.

($\mathcal{P}_5$) Entire functions operate in (A, τ).

In [4], using a Baire argument and the Mazur Orlicz formula, the first author obtained that, $(\mathcal{P}_5) \Longrightarrow (\mathcal{P}_4)$, in any unital Baire locally convex algebra with continuous multiplication. Whence, the m-convexity in the commutative case [6], and hence the result of Mitiagin, Rolewics and Żelazko for commutative B_0-algebra [8]. He also showed in [5] that $(\mathcal{P}_4) \Longrightarrow (\mathcal{P}_5)$, in any unital and M-complete $l.c.a.$ In particular, entire functions operate in any unital and M-complete $l.m.c.a.$ In this note, we show that, in a locally convex algebra, the properties $(\mathcal{P}_2)$, $(\mathcal{P}_3)$ and $(\mathcal{P}_4)$ are all equivalent (Proposition 2.1). As a consequence, the properties $(\mathcal{P}_3)$, $(\mathcal{P}_4)$, and $(\mathcal{P}_5)$ are equivalent in any unital (commutative or not) B_0-algebra (Corollary 2.3). In the case, where the topology is given by a family of seminorms $(p_i)_{i \in I}$, such that $p_i(x^2) \leq (p_i(x))^2$, for every $x \in A$, $i \in I$, the inverse map $x \longmapsto x^{-1}$ is continuous. If moreover, (A, τ) is a Q-algebra, then the properties $(\mathcal{P}_4)$ and $(\mathcal{P}_5)$ are equivalent (Proposition 2.5). In the general case of a topological algebra, we show that the property $(\mathcal{P}_1)$ is equivalent to the fact that the algebra is locally idempotent (namely, it has a base of idempotent neighborhoods of zero [10]). Whence, the property $(\mathcal{P}_1)$ is equivalent to the m-convexity in the locally convex case.

2. Properties $(\mathcal{P}_i)$, $i = 1, \ldots, 5$, in Locally Convex Algebras

In the commutative case, properties $(\mathcal{P}_1)$, $(\mathcal{P}_2)$, and $(\mathcal{P}_4)$ are equivalent. In fact, the implication $(\mathcal{P}_1) \Longrightarrow (\mathcal{P}_2)$ is due to the fact that:

$$x_1 \cdots x_n = \frac{1}{n!} \sum x_{j_1} \cdots x_{j_n}, \quad \text{for every } (x_1, \ldots, x_n) \in A^n, \ n \in \mathbb{N}^*.$$

To see that $(\mathcal{P}_2) \Longrightarrow (\mathcal{P}_4)$, just take $x_1 = x_2 = \cdots = x_n = x$ in $(\mathcal{P}_2)$. The implication $(\mathcal{P}_4) \Longrightarrow (\mathcal{P}_1)$ is due to Turpin [9]. In the non-commutative case, the example of a non-m-convex non-commutative B_0-algebra, on which all entire functions operate, constructed in [12], by Żelazko, verifies $(\mathcal{P}_4)$ but not $(\mathcal{P}_1)$. In the general case, $(\mathcal{P}_2)$, $(\mathcal{P}_3)$, and $(\mathcal{P}_4)$ are equivalent as the following result shows.

Proposition 2.1. *Let (A,τ) be a locally convex algebra. Then, the following assertions are equivalent.*

1. *(A,τ) satisfies $(\mathcal{P}_2)$.*
2. *(A,τ) satisfies $(\mathcal{P}_3)$.*
3. *(A,τ) satisfies $(\mathcal{P}_4)$.*

Proof. **(1)** $\Longrightarrow$ **(3)** It is obvious.

(3) $\Longrightarrow$ **(1)** Let V be an absolutely convex and closed neighborhood of zero. Since (A,τ) verifies $(\mathcal{P}_4)$, there is an absolutely convex neighborhood U of zero, such that

$$x^n \in V, \quad x \in U, \quad n \in \mathbb{N}^*.$$

By Mazur–Orlics theorem (see [3]), we have:

$$\frac{1}{n!}\sum x_{j_1}\cdots x_{j_n} = \frac{1}{n!}\sum_{\varepsilon_1,\ldots,\varepsilon_n=0}^{1}(-1)^{n-(\varepsilon_1+\cdots+\varepsilon_n)}\left(x_0+\varepsilon_1x_1+\cdots+\varepsilon_nx_n\right)^n,$$

where $x_1,\ldots,x_n \in A$, $n \in \mathbb{N}^*$, the sum is taken over all permutations $(j_1,\ldots,j_n)$ of the sequence $(1,\ldots,n)$, and x_0 is an arbitrary point of A. Let us take $x_0 = 0$. Since U is convex, we have $\varepsilon_1x_1+\cdots+\varepsilon_nx_n \in nU$, for every $x_1,\ldots,x_n \in U$, and thus $(\varepsilon_1x_1+\cdots+\varepsilon_nx_n)^n \in n^nV$. Whence

$$\frac{1}{n!}\sum_{\varepsilon_1,\ldots,\varepsilon_n=0}^{1}(-1)^{n-(\varepsilon_1+\cdots+\varepsilon_n)}\left(\varepsilon_1x_1+\cdots+\varepsilon_nx_n\right)^n \in \frac{2^nn^n}{n!}V.$$

But, there exists $C>0$, such that $\frac{(2n)^n}{n!} \leq C^n$, for every $n \in \mathbb{N}^*$. Then

$$\frac{1}{n!}\sum x_{j_1}\cdots x_{j_n} \in V, \quad \text{for every } x_1,\ldots,x_n \in \frac{1}{C}U.$$

Thus (A,τ) verifies $(\mathcal{P}_2)$.

(3) $\Longrightarrow$ **(2)** Let V be an absolutely convex and closed neighborhood of zero. Since (A,τ) verifies $(\mathcal{P}_4)$, there is an absolutely convex and closed neighborhood U of zero, such that

$$x^n \in V, \quad x \in U, \quad n \in \mathbb{N}^*.$$

Let p be the gauge associated to V and q that associated to U. We have

$$p(x^n) = \inf\{\lambda>0 : x^n \in \lambda V\} \quad \text{and} \quad q(x) = \inf\{\lambda>0 : x \in \lambda U\}.$$

Let $\alpha>0$ be such that $x \in \alpha U$. Then $x^n \in \alpha^nV$. So, $\alpha^n \in \{\lambda>0 : x^n \in \lambda V\}$. Whence

$$p(x^n) \leq (q(x))^n.$$

Thus, considering the family of gauges associated to all absolutely convex and closed neighborhood of zero, we obtain the result.

(2) $\Longrightarrow$ (3) We know that the family $\left(K_i(0,\frac{1}{k})\right)_{k\in\mathbb{N}^*,\, i\in I}$ constitutes a fundamental system of neighborhood of zero, where

$$K_i\left(0,\frac{1}{k}\right)=\left\{x\in A: p_i(x)<\frac{1}{k}\right\},\quad k=1,2,\ldots$$

It follows that $p_j(x)<\dfrac{1}{k}$, for every $x\in K_j(0,\frac{1}{k})$. Whence

$$p_i(x^n)\le (p_j(x))^n<\frac{1}{k^n}\le\frac{1}{k},\quad \text{for every } n\in\mathbb{N}^*.$$

Consequently, $x^n\in K_i(0,\frac{1}{k})$, for every $n\in\mathbb{N}^*$. Hence (A,τ) verifies $(\mathcal{P}_4)$. $\square$

Remark 2.2. The algebra L^ω of Arens [2] is a commutative non-m-convex B_0-algebra on which only polynomial functions operate. Consequently, this algebra does not satisfy the properties $(\mathcal{P}_2)$, $(\mathcal{P}_3)$, and $(\mathcal{P}_4)$.

Using [4,5], we obtain the following result:

Corollary 2.3. *Let (A,τ) be a unital (commutative or not) B_0-algebra. Then the following assertions are equivalent.*

1. *(A,τ) satisfies $(\mathcal{P}_3)$.*
2. *(A,τ) satisfies $(\mathcal{P}_4)$.*
3. *(A,τ) satisfies $(\mathcal{P}_5)$.*

Remark 2.4.

1. Completeness is not superfluous in the previous result. Indeed, take the algebra $A=(C([0,1]),(\|.\|_p)_{p\in\mathbb{N}^*})$, where

$$\|f\|_p=\left(\int_0^1|f(t)|^p\,\mathrm{d}t\right)^{\frac{1}{p}},\quad \text{for every } f\in A.$$

It is a non-complete unital and metrizable locally convex algebra. Since

$$\|f\|_p=\left(\int_0^1|f(t)|^p\,\mathrm{d}t\right)^{\frac{1}{p}}\le\|f\|_\infty,\quad \text{for every } p\in\mathbb{N}^* \text{ and } f\in A.$$

This algebra satisfies $(\mathcal{P}_5)$. On the other hand, by the fact that $\overline{A}=L^\omega$ and by a result of Turpin [9], the algebra A is not m-convex. Then, it does not satisfy $(\mathcal{P}_4)$ and thus neither $(\mathcal{P}_3)$ by Proposition 2.1.

2. In the unital M-complete locally m-convex case, the properties of Corollary 2.3. are satisfied. Pseudo-completeness (i.e.; if every bounded and closed idempotent disk is Banach) with the m-convexity is not sufficient in the previous corollary. Indeed, let A be the algebra of all complex polynomials P endowed with the topology of uniform on the compact subsets of the positive real line $\mathbb{R}_+$. It is an *l.m.c.a.* It is pseudo-complete, non-M-complete, and the exponential function does not operate in it [1]. Whence, A verifies $(\mathcal{P}_3)$ and $(\mathcal{P}_4)$ but not $(\mathcal{P}_5)$.

In the particular case when $i=j$ and $n=2$ in $(\mathcal{P}_3)$ i.e.:

$(\mathcal{P}_6)$ $p_i(x^2)\le(p_i(x))^2$, for every $i\in I$ and $x\in A$,

we have the following result.

Proposition 2.5. *Let (A, τ) be a unital locally convex algebra satisfying $(\mathcal{P}_6)$. Then*

1. *The inverse map $x \longmapsto x^{-1}$ is continuous.*
2. *If moreover (A, τ) is a Q-algebra, then properties $(\mathcal{P}_4)$ and $(\mathcal{P}_5)$ are satisfied.*

Proof. 1. Let x_1, y_1 be elements in A and $i \in I$, such that

$$x_1 y_1 = y_1 x_1, \quad p_i(x_1) \leq 1 \quad \text{and} \quad p_i(y_1) \leq 1.$$

One has

$$2p_i(x_1 y_1) \leq [p_i(x_1) + p_i(y_1)]^2 + [p_i(x_1)]^2 + [p_i(y_1)]^2 \leq 6.$$

Whence, $p_i(x_1 y_1) \leq 3$. Applying the previous inequality for $x_1 = \frac{x}{p_i(x)+\varepsilon}$ and $y_1 = \frac{y}{p_i(y)+\varepsilon}$, where $x, y \in A$, such that $xy = yx$ and $\varepsilon > 0$, one has

$$p_i(xy) \leq 3\left(p_i(x) + \varepsilon\right)\left(p_i(y) + \varepsilon\right).$$

This implies that

$$p_i(xy) \leq 3p_i(x)p_i(y).$$

Now, let $(x_\alpha)_\alpha$ be a net of invertible elements, such that $\lim\limits_{\alpha} x_\alpha = e$ (the unit of A). Since $x_\alpha^{-1}(x_\alpha - e) = (x_\alpha - e)\, x_\alpha^{-1}$, we have

$$\left|p_i(x_\alpha^{-1}) - p_i(e)\right| \leq p_i(x_\alpha^{-1} - e) \leq p_i\left[x_\alpha^{-1}(x_\alpha - e)\right] \leq 3p_i(x_\alpha^{-1})p_i(x_\alpha - e).$$

Take $\varepsilon > 0$. Since $\lim_\alpha x_\alpha = e$, there exists α_0, such that

$$p_i(x_\alpha - e) < \varepsilon', \quad \text{for every } \alpha \geq \alpha_0,$$

where $\varepsilon' < \frac{\varepsilon}{3(1+\varepsilon)}$. Whence

$$p_i(x_\alpha^{-1}) \leq \frac{1}{1 - 3\varepsilon'} \quad \text{for every } \alpha \geq \alpha_0.$$

It follows that

$$p_i(x_\alpha^{-1} - e) \leq \frac{3\varepsilon'}{1 - 3\varepsilon'} \leq \varepsilon, \quad \text{for every } \alpha \geq \alpha_0.$$

This implies that $\lim_\alpha x_\alpha^{-1} = e$. Finally, for $x \in G(A)$ and $(x_\alpha)_\alpha \in G(A)$, such that $\lim_\alpha x_\alpha = x$. We have $x_\alpha^{-1} = (x^{-1}x_\alpha)^{-1}x^{-1}$ and $x^{-1}x_\alpha \longrightarrow e$. Whence, $x_\alpha^{-1} \longrightarrow x^{-1}$, from which the claim **(1)** follows.

2. It follows from Proposition 2 in [9].

Proposition 2.6. *Let (A, τ) be a topological algebra. Then, the following assertions are equivalent.*

1. *(A, τ) satisfies $(\mathcal{P}_1)$.*
2. *(A, τ) is locally idempotent.*

Proof. **(1)** $\Longrightarrow$ **(2)** Let V a neighborhood of zero. Since (A, τ) satisfies $(\mathcal{P}_1)$, there exists a neighborhood U of zero, such that

$$U^{(n)} \subset V, \quad \text{for every } n \in \mathbb{N}^*.$$

Put $\Omega = \bigcup_{n=1}^{\infty} U^{(n)}$. Then, Ω is an idempotent neighborhood of zero and $\Omega \subset V$. Whence, (A, τ) is locally idempotent.

(2) $\Longrightarrow$ **(1)** Let V be a neighborhood of zero. Since (A, τ) is locally idempotent, there exists an idempotent neighborhood U of zero, such that $U \subset V$. Whence, $U^{(n)} \subset V$, for every $n \in \mathbb{N}^*$.

Corollary 2.7. *Let (A, τ) be a locally convex algebra. Then, the following assertions are equivalent.*

1. *(A, τ) satisfies $(\mathcal{P}_1)$.*
2. *(A, τ) is m-convex.*

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