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Hochschild Cohomology of a Sullivan Algebra

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Abstract. Let $A = (\wedge V, d)$ be a minimal Sullivan algebra where V is finite dimensional. We show that the Hochschild cohomology $HH^*(A; A)$ can be computed in terms of derivations of A. This provides another method to compute the loop space homology of a simply connected space for which $\pi_*(X) \otimes \mathbb{Q}$ is finite dimensional.

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1. Introduction

Let $A = \oplus_{n\geq 0} A^n$ be a commutative graded algebra over a commutative ring $\Bbbk$, and M a $\mathbb{Z}$-graded A-module. The A-tensor algebra $T_A(M)$ is defined by $T_A(M) = \oplus_{k\geq 0} T_A^k(M)$, where

$$T_A^k(M) = M \otimes_A M \otimes \cdots \otimes_A M \quad (k \geq 1 \text{ factors}) \text{ and } T_A^0(M) = A.$$

The exterior algebra $\wedge_A M$ is the commutative graded algebra obtained as the quotient of $T_A(M)$ by the ideal generated by elements of the form $x \otimes y - (-1)^{|x||y|} y \otimes x$, where $x, y \in T_A(M)$. The exterior product induces a graded commutative algebra structure on $\wedge_A M$.

Let $Z = \oplus_i Z_i$ be a $\mathbb{Z}$-graded free $\Bbbk$-module. There is a canonical isomorphism of commutative graded algebras

$$\varphi : \wedge_A (A \otimes Z) \to A \otimes \wedge_{\Bbbk} Z.$$

We assume that (A, d) is a differential graded algebra with a differential $d : A^n \to A^{n+1}$, and $A \otimes Z$ is an (A, d)-differential graded module; then $(\wedge_A (A \otimes Z), d)$ and $(A \otimes \wedge Z, d)$ are endowed with canonical differential graded algebra structures and φ becomes a homomorphism of differential graded algebras.

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A derivation θ of degree k is a linear mapping $A^n \to A^{n-k}$, such that $\theta(ab) = \theta(a)b + (-1)^{k|a|}a\theta(b)$. Let $\mathrm{Der}_k\, A$ denote the vector space of all derivations of degree k and $\mathrm{Der}\, A = \oplus_k \mathrm{Der}_k\, A$. With the commutator bracket, $\mathrm{Der}\, A$ becomes a graded Lie algebra. Using the grading convention $A^n = A_{-n}$, we may regard a derivation of degree k as increasing the lower degree by k. There is a differential $\delta : \mathrm{Der}_k\, A \to \mathrm{Der}_{k-1}\, A$ defined by $\delta\theta = [d, \theta]$.

Moreover, $\mathrm{Der}\, A$ is a differential graded A-module with the action $(a\theta)(x) = a\theta(x)$. With the grading convention $A_{-n} = A^n$, if $\theta, \theta' \in \mathrm{Der}_k\, A$ and $a \in A^i$, then $a\theta \in \mathrm{Der}_{k-i}\, A$ and

$$[a\theta, b\theta'] = (-1)^{|b||\theta|}ab[\theta, \theta'] + a\theta(b)\theta' + (-1)^{|a\theta||b\theta'|}b\theta'(a)\theta. \tag{1}$$

If $A = \wedge V$ and V is finite dimensional, then $\mathrm{Der}\, A \cong A \otimes V^{\#}$, where $V^{\#}$ is the graded dual of V (Lemma 5). With the above grading convention, $V^{\#} = \oplus_{i\geq 1}(V^{\#})_i$ is positively graded.

On $L = s^{-1}\,\mathrm{Der}\, A$, we define a bracket of degree 1 by

$$\{\alpha, \beta\} = s^{-1}[s\alpha, s\beta] \tag{2}$$

and a differential $\delta'(\alpha) = -\{d', \alpha\}$, where $d' = s^{-1}d \in L_{-2}$. We extend the bracket to $\wedge_A L = A \oplus L \oplus \wedge_A^2 L \oplus \dots$ by $\{a, b\} = 0$ for $a, b \in \wedge_A^0 L = A$, and $\{\alpha, a\} = -(-1)^{|\alpha|}(s\alpha)(a)$, $\alpha \in L$. It is then defined inductively on $\wedge_A^{\geq 2} L$ by forcing the Leibniz rule

$$\begin{aligned}\{\alpha, \beta\wedge\gamma\} &= \{\alpha, \beta\}\wedge\gamma + (-1)^{(|\alpha|+1)|\beta|}\beta\wedge\{\alpha, \gamma\}\\ &= \{\alpha, \beta\}\wedge\gamma + (-1)^{(|\beta||\gamma|}\{\alpha, \gamma\}\wedge\beta.\end{aligned} \tag{3}$$

Hence for α_i, β_i in L,

$$\begin{aligned}\{\ &\alpha_1\wedge\cdots\wedge\alpha_m, \beta_1\wedge\cdots\wedge\beta_n\}\\ &= \sum_{i,j}(-1)^{\epsilon_{ij}}\alpha_1\wedge\cdots\hat{\alpha_i}\wedge\alpha_m\wedge\{\alpha_i, \beta_j\}\wedge\cdots\wedge\hat{\beta_j}\wedge\cdots\beta_n,\end{aligned} \tag{4}$$

where ˆ means omitted and $\epsilon_{ij} = \sum_{k>i}|\alpha_k||\alpha_i| + \sum_{k<j}|\beta_k||\beta_j|$. The above bracket (called Nijenhuis–Schouten bracket) turns $\wedge_A L$ into a Gerstenhaber algebra. See [15, Sect. 2] for instance.

The differential δ' extends into an algebra differential d_0 on $\wedge_A L$ in the same way, that is, $d_0\alpha = -\{d', \alpha\}$, for $\alpha \in \wedge_A L$. It comes from the Leibniz rule (3) that d_0 is a derivation. Moreover, the Jacobi identity ensures that d_0 is compatible with the bracket. Hence, $(\wedge_A L, d_0)$ becomes a differential graded Gerstenhaber algebra [11, Lemma 5].

From now on, we assume that $\Bbbk$ is a field of characteristic 0. Let $A = (\wedge V, d)$ be a Sullivan algebra where V is finite dimensional and $Z = s^{-1}V^{\#}$. The isomorphism $s^{-1}\,\mathrm{Der}\wedge V \cong A \otimes Z$ transfers a bracket of degree 1 on $A \otimes Z$. Moreover if $s^{-1}\theta, s^{-1}\theta' \in Z$ and $a, b \in A$, one uses Eq. (1) to obtain

$$\begin{aligned}\{\ &a\otimes s^{-1}\theta, b\otimes s^{-1}\theta'\}\\ &= (-1)^{|a|+|b|}\left(a\theta(b)\otimes s^{-1}\theta' + (-1)^{|a\theta||b\theta'|}b\theta'(a)\otimes s^{-1}\theta\right).\end{aligned} \tag{5}$$

The bracket is then extended to $A \otimes \wedge Z$ by the Leibniz rule. In the same way, the differential is extended to $A \otimes \wedge Z$ by $D\alpha = -\{d', \alpha\}$ where $d' = s^{-1}d$.

The main result states:

Theorem 1. *Let $A = (\wedge V, d)$ be a Sullivan algebra over a field $\Bbbk$ of characteristic 0, where $V = \oplus_{i\geq 2} V^i$ is finite dimensional and L the desuspension of* Der $\wedge V$ *with the desuspended differential and $Z = s^{-1}V^{\#}$. Then, $\varphi : (\wedge_A L, d_0) \to (A\otimes\wedge Z, D)$ extends to an isomorphism of differential graded Gerstenhaber algebras.*

Let $\bar{A}$ be the kernel of the augmentation $\epsilon : A \to \Bbbk$. We denote by $C^*(A; A) = \mathrm{Hom}(T(s\bar{A}), A)$ (resp. $HH^*(A; A)$) the Hochschild complex (resp. cohomology) of the cochain algebra A with coefficients in A [12]. We recall the following result.

Theorem 2 ([11]). *If $A = (\wedge V, d)$ is a Sullivan algebra, then there is a mapping $\phi : (\wedge_A L, d_0) \to C^*(A; A)$ which induces an isomorphism of graded Gerstenhaber algebras in homology.*

Note that if A is not a Sullivan algebra, then ϕ does not necessarily induce a bijective map in homology [1, Theorem 6.2].

By combining Theorems 1 and 2, we get an easy method to compute the Gerstenhaber bracket on $HH^*(A; A)$, when $A = \wedge V$ is a Sullivan algebra for which V is finite dimensional.

2. Resolutions to Compute $HH^*(A; A)$

The Hochschild cohomology is usually computed using a semifree resolution [5]. Let (A, d) be an augmented differential graded algebra, not necessarily commutative. The bar construction $\mathbb{B}(A; A; A)$ is defined as follows (see for instance [7,10]).

$$\mathbb{B}_k(A; A; A) = A \otimes T^k(s\bar{A}) \otimes A.$$

An element $a[a_1|a_2|\cdots a_k]b \in A\otimes T^k(s\bar{A})\otimes A$ is of degree $|a|+|b|+\sum_{i=1}^k |sa_i|$. The differential $d = d_0 + d_1$ is defined as follows.

$$d_0 : \mathbb{B}_k(A; A; A) \to \mathbb{B}_k(A; A; A), \quad d_1 : \mathbb{B}_k(A; A; A) \to \mathbb{B}_{k-1}(A; A; A),$$

$$\begin{aligned} d_0(a[a_1|a_2|\cdots a_k]b) &= (da)[a_1|a_2|\cdots a_k]b - \sum_{i=1}^k (-1)^{\epsilon(i)} a[a_1|\cdots|da_i|\cdots|a_k]b \\ &\quad + (-1)^{\epsilon(k+1)} a[a_1|a_2|\cdots a_k](db), \end{aligned}$$

$$\begin{aligned} &d_1(a[a_1|a_2|\cdots a_k]b)\\ &\quad = (aa_1)[a_2|\cdots a_k]b - \sum_{i=2}^{k}(-1)^{\epsilon(i)}a[a_1|\cdots|a_{i-1}a_i|\cdots|a_k]b\\ &\qquad - (-1)^{\epsilon(k)}a[a_1|a_2|\cdots a_{k-1}](a_kb), \end{aligned}$$

where $\epsilon(i) = |a| + \sum_{j=1}^{i-1}|sa_j|$. There is a quasi-isomorphism $\mathbb{B}(A;A;A) \to (A,d)$ which provides a semifree resolution of A as an $A \otimes A^{op}$-module [5, Lemma 4.3]. Therefore the Hochschild cochain complex is given by

$$(C^*(A;A),D) = \mathrm{Hom}_{A\otimes A^{op}}(\mathbb{B}(A;A;A),A) \cong (\mathrm{Hom}(T(s\bar{A}),A),D_0+D_1),$$

where the differential is expressed as follows [9]:

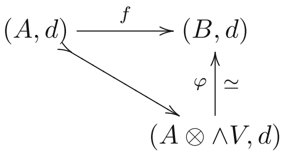
$$\begin{aligned} (D_0f)([a_1|a_2|\ldots|a_k]) = d\left(f([a_1|a_2|\ldots|a_k])\right)&\\ +\sum_{i=1}^{k}(-1)^{\bar{\epsilon}(i)}f([a_1|\ldots|da_i|\ldots|a_k])& \end{aligned}$$

and

$$\begin{aligned} (D_1f)([a_1|a_2|\ldots|a_k]) = &-(-1)^{|sa_1||f|}a_1f([a_2|\ldots|a_k])\\ &+(-1)^{\bar{\epsilon}(k)}f([a_1|\ldots|a_{k-1}])a_k\\ &+\sum_{i=2}^{k}(-1)^{\bar{\epsilon}(i)}f([a_1|\ldots|a_{i-1}a_i|\ldots|a_k]), \end{aligned}$$

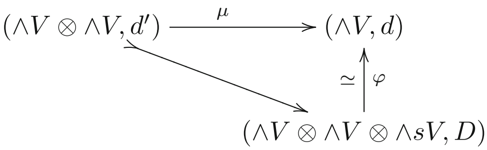
where $\bar{\epsilon}(i) = |f| + |sa_1| + \cdots + |sa_{i-1}|$.

We now define another resolution for a Sullivan algebra $(\wedge V,d)$. Let $f : (A,d) \to (B,d)$ be a map between commutative differential graded algebras. There exists a relative Sullivan algebra $(A \otimes \wedge V,d)$ and a quasi-isomorphism φ such that the following diagram commutes [13, 16].



(6)

Given a Sullivan algebra $(\wedge V,d)$, the multiplication $\mu : (\wedge V \otimes \wedge V,d') \to (\wedge V,d)$ is a morphism of differential graded algebras, where $d' = d\otimes 1+1\otimes d$. There exists a commutative differential graded algebra $(\wedge V \otimes \wedge V \otimes \wedge sV,D)$, where $sV^n = V^{n+1}$ such that the following diagram commutes [6].



(7)

Moreover, the differential on $\wedge V \otimes \wedge V \otimes \wedge sV$ is defined by

$$D(v\otimes 1\otimes 1) = dv\otimes 1\otimes 1, \quad D(1\otimes v\otimes 1) = 1\otimes dv\otimes 1,$$

and $D(1 \otimes 1 \otimes sv)$ is defined by induction on the degree of v by the formula

$$D(1 \otimes 1 \otimes sv) = v \otimes 1 \otimes 1 - 1 \otimes v \otimes 1 + \sum_{i=1}^{\infty} \frac{(sD)^i}{i!}(v \otimes 1 \otimes 1).$$

Here, s is the derivation of degree -1 on $\wedge V \otimes \wedge V \otimes \wedge sV$ defined as

$$s(v \otimes 1 \otimes 1) = s(1 \otimes v \otimes 1) = 1 \otimes 1 \otimes sv, \quad s(1 \otimes 1 \otimes sv) = 0.$$

Let

$$\alpha = \sum_{i=1}^{\infty} \frac{(sD)^i}{i!}(v \otimes 1 \otimes 1) \in \wedge^{\geq 1}(V \oplus V) \otimes sV.$$

The condition $D^2 = 0$ yields

$$(dv \otimes 1 - 1 \otimes dv) \otimes 1 = D\alpha.$$

Proposition 3. *The quasi-isomorphism*

$$\varphi : (\wedge V \otimes \wedge V \otimes \wedge sV, D) \to (\wedge V, d)$$

is a semifree resolution of $(\wedge V, d)$ *as a* $\wedge V \otimes \wedge V$*-differential module.*

Proof. From the commutativity of the above diagram, one deduces that the quasi-isomorphism

$$\varphi : (\wedge V \otimes \wedge V \otimes \wedge sV, D) \to (\wedge V, d)$$

is a morphism of $A \otimes A$-modules. Let $V = V_1 \oplus V_2 \oplus \cdots$ be a decomposition of V such that $dV_i \subset \wedge(V_1 \oplus \cdots \oplus V_{i-1})$. The filtration of $\wedge V \otimes \wedge V \otimes \wedge sV$ by submodules $P(n) = \wedge V \otimes \wedge V \otimes \wedge s(V_1 \oplus \cdots \oplus V_n)$ shows that the above quasi-isomorphism $(\wedge V \otimes \wedge V \otimes \wedge sV, D) \to (\wedge V, d)$ is a semifree resolution. □

Therefore, the Hochschild cohomology $HH^*(\wedge V; \wedge V)$ is given by

$$\begin{aligned} HH^*(\wedge V; \wedge V) &\cong \mathrm{Ext}_{\wedge V \otimes \wedge V}(\wedge V, \wedge V) \\ &\cong H^*(\mathrm{Hom}_{\wedge V \otimes \wedge V}(\wedge V \otimes \wedge V \otimes \wedge sV, \wedge V), \tilde{D}). \end{aligned}$$

Proposition 4. *There is a quasi-isomorphism*

$$(\wedge V \otimes \wedge V \otimes \wedge sV, D) \to \mathbb{B}(\wedge V; \wedge V; \wedge V)$$

such that the following diagram commutes.

$$\begin{array}{ccc} (\wedge V \otimes \wedge V \otimes \wedge sV, D) & \xrightarrow{\quad \jmath \quad} & \mathbb{B}(\wedge V; \wedge V; \wedge V) \,. \\ & {\scriptstyle \varphi}\searrow^{\simeq} \quad {}^{\simeq}\swarrow{\scriptstyle \psi} & \\ & (\wedge V, d) & \end{array}$$

Proof. Recall that the map $\psi : \mathbb{B}(\wedge V; \wedge V; \wedge V) \to (\wedge V, d)$ satisfies $\psi([\,]) = 1$ and $\psi([a_1|\cdots|a_k]) = 0$, for $k > 0$ [6, Lemma 4.3]. Moreover, $\varphi(sV) = 0$ [4].

As $(\wedge V \otimes \wedge V \otimes \wedge sV, D)$ and $\mathbb{B}(\wedge V; \wedge V; \wedge V)$ are semifree resolutions of $\wedge V$ as $\wedge V \otimes \wedge V$-modules, there is a quasi-isomorphism

$$\jmath : (\wedge V \otimes \wedge V \otimes \wedge sV, D) \to \mathbb{B}(\wedge V; \wedge V; \wedge V).$$

However, we give an explicit formula for $\jmath$.

Let $v \in V_1$, then $dv = 0$. Define $\jmath(sv) = 1 \otimes [v] \otimes 1$. As $d([v]) = v[\,] - [\,]v$,

$$\jmath(D(sv)) = \jmath((v \otimes 1 - 1 \otimes v) \otimes 1) = v[\,] - [\,]v = D\jmath(sv).$$

Let $V_{\leq i} = V_1 \oplus V_2 \oplus \cdots \oplus V_i$. Now assume that $\jmath$ has been defined on $sV_{\leq i}$, such that $\jmath$ commutes with differentials. We need to define $\jmath$ on sV_{i+1}. Let $v \in V_{i+1}$. Recall that

$$D(sv) = v \otimes 1 \otimes 1 - 1 \otimes v \otimes 1 + \alpha, \quad \text{where} \quad \alpha = \sum_{k=1}^{\infty} \frac{(sD)^k(v \otimes 1 \otimes 1)}{k!}.$$

As $\alpha \in \wedge V \otimes \wedge V \otimes s(V_{\leq i})$, by hypothesis, $d(\jmath(Dsv)) = \jmath(D^2(sv)) = 0$. But $\mathbb{B}_{\geq 1}(\wedge V; \wedge V; \wedge V)$ is acyclic; hence, $\jmath(Dsv)$ is a boundary. Moreover,

$$d([v]) = v[\ ] - [\ ]v - [dv]$$

and

$$\jmath(Dsv) = v[\ ] - [\ ]v + \jmath(\alpha) = d([v]) + [dv] + \jmath(\alpha).$$

Therefore, $\jmath(Dsv) = d([v] + \beta)$, where $\beta \in \wedge(V_{\leq i}) \otimes T^{\geq 2} s \wedge^+ (V_{\leq i}) \otimes \wedge(V_{\leq i})$. Define $\jmath(sv) = [v] + \beta$, then $d\jmath(sv) = \jmath(Dsv)$ as required.

We extend $\jmath$ to $\wedge^{\geq 2}(sV)$ by

$$\jmath(sv_1 \wedge \ldots \wedge sv_n) = \frac{1}{n!} \sum_{\sigma \in S_n} \epsilon(\sigma)[\jmath(v_{\sigma(1)})| \ldots |\jmath(v_{\sigma(n)})],$$

where $v_i \in V$. □

3. Gerstenhaber Structure on the Hochschild Cohomology

Recall that the tensor algebra TV is endowed with a coalgebra structure when the reduced diagonal is defined

$$\bar{\Delta}(v_1 \otimes \cdots \otimes v_n) = \sum_{i=1}^{n-1} (v_1 \otimes \cdots \otimes v_i) \otimes (v_{i+1} \otimes \cdots \otimes v_n).$$

The multiplicative structure on

$$HH^*(A; A) = H^*(\mathrm{Hom}_{A \otimes A}(B(A; A; A), A)) = H_*(\mathrm{Hom}(T(\bar{A}), A))$$

derives from the above defined comultiplication on $T(\bar{A})$. Moreover, $\wedge V$ is endowed with a cocommutative coalgebra structure defined by

$$\bar{\Delta}(x_1 \wedge \cdots \wedge x_n) = \sum_{i=1}^{n-1} \sum_{\sigma} \epsilon(\sigma)(x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(i)}) \otimes (x_{\sigma(i+1)} \wedge \cdots \wedge x_{\sigma(n)}),$$

where σ is an $(i, n-i)$ shuffle and $\epsilon(\sigma)$ its Koszul sign. The restriction of $\jmath$ to $\wedge sV \to T(\bar{A})$ is a morphism of coalgebras. Therefore,

$$\mathrm{Hom}(\jmath) : (\mathrm{Hom}(T(\bar{A}), A), D) \to (\mathrm{Hom}_{\wedge V}(\wedge V \otimes \wedge sV, \wedge V), D)$$

is a map of differential graded algebras. As $\mathrm{Hom}(\jmath)$ is a quasi-isomorphism, we deduce that $H_*(\mathrm{Hom}(\jmath))$ is an isomorphism of algebras. Moreover, there is a mapping $(\wedge_A L, d) \to (C^*(A; A), D_0 + D_1)$ which induces a morphism of Gerstenhaber algebras in homology [11].

Lemma 5. *Assume that V is finite dimensional and let (v_i), $i \in \{1, \ldots, n\}$, be a homogeneous linear basis of V. For $i \in \{1, \ldots, n\}$, let θ_i be the derivation of $\wedge V$ uniquely determined by*

$$\theta_i(v_j) = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases}$$

The graded $\wedge V$-module $\mathrm{Der} \wedge V$ *is freely generated by the derivations θ_i, $(i = 1, \ldots, n)$.*

Proof. Let us denote by $V^{\#}$ the graded dual of V. The restriction of each θ_i to V is an element of $V^{\#}$ of upper degree $-|v_i|$. Thus, we have an isomorphism of graded $\wedge V$-modules

$$\mathrm{Der} \wedge V \cong \mathrm{Hom}(V, \wedge V) \cong (\wedge V) \otimes V^{\#}.$$

□

The derivation θ_i referred to in the proof of the above lemma will be denoted by $(v_i, 1)$.

Proof of Theorem 1. Let $A = \wedge V$ where V is finite dimensional. The isomorphism $\mathrm{Der}\, A \cong A \otimes V^{\#}$ extends to an isomorphism of graded algebras

$$\varphi : \wedge_A s^{-1}(\mathrm{Der}\, A) \cong \wedge_A s^{-1}(A \otimes V^{\#}) \cong A \otimes \wedge(s^{-1}V^{\#}).$$

It is explicitly defined by

$$\varphi(s^{-1}(a_1\theta_1) \ldots s^{-1}(a_n\theta_n)) = (-1)^{|a_1|+\cdots+|a_n|}(-1)^{\epsilon} a_1 \ldots a_n s^{-1}\theta_1 \ldots s^{-1}\theta_n,$$

where $(-1)^{\epsilon}$ satisfies

$$a_1 s^{-1}\theta_1 \ldots a_n s^{-1}\theta_n = (-1)^{\epsilon} a_1 \ldots a_n s^{-1}\theta_1 \ldots s^{-1}\theta_n.$$

The next two lemmas will complete the proof.

Lemma 6. *The map φ is a morphism of Gerstenhaber algebras.*

Proof. Clearly, φ commutes with brackets. Denote the wedge product $\alpha \wedge \beta$ by $\alpha\beta$.

$$\begin{aligned} \varphi(\{\alpha, \beta\gamma\}) &= \varphi(\{\alpha, \beta\}\gamma) + (-1)^{(|\alpha|+1)|\beta|}\varphi(\beta\{\alpha, \gamma\}) \\ &= \varphi(\{\alpha, \beta\})\varphi(\gamma) + (-1)^{(|\alpha|+1)|\beta|}\varphi(\beta)\varphi(\{\alpha, \gamma\}) \\ &= \{\varphi(\alpha), \varphi(\beta)\}\varphi(\gamma) + (-1)^{(|\alpha|+1)|\beta|}\varphi(\beta)\{\varphi(\alpha), \varphi(\gamma)\} \\ &= \{\varphi(\alpha), \varphi(\beta)\varphi(\gamma)\} \\ &= \{\varphi(\alpha), \varphi(\beta\gamma)\}. \end{aligned}$$

By an induction argument, one deduces that φ is a morphism of Gerstenhaber algebras. □

Lemma 7. *The map $\varphi : (\wedge_A s^{-1}(\mathrm{Der} \wedge V), d_0) \to (\wedge V \otimes \wedge(s^{-1}V^{\#}), D)$ commutes with differentials.*

Proof. The differential D on $\wedge V \otimes \wedge(s^{-1}V^{\#})$ is defined by $D\alpha = -\{\varphi(d'), \alpha\}$, where $d' = s^{-1}d$. As φ is compatible with brackets, we deduce that

$$\varphi(d_0\alpha) = -\varphi(\{d', \alpha\}) = -\{\varphi(d'), \varphi(\alpha)\} = D(\varphi(\alpha)).$$

Hence, φ commutes with differentials. □

Remark 8. The Gerstenhaber structure on $\mathrm{Hom}_{\wedge V}(\wedge V \otimes \wedge sV, \wedge V)$ is defined through the isomorphism

$$\mathrm{Hom}_{\wedge V}(\wedge V \otimes \wedge sV, \wedge V) \cong \wedge V \otimes \wedge(s^{-1}V^{\#}).$$

Moreover, the $\wedge V$-module $(\wedge V \otimes \wedge(s^{-1}V^{\#}), D)$ is the "dual" of the Sullivan model of LX described by Sullivan and Vigué-Poirrier [17]. However, the former carries the Gerstenhaber structure of the free loop space homology.

4. Computation of the Free Loop Space Homology

We apply the above result in the computation of the free loop space homology. Let X be a closed oriented manifold of dimension m and $LX = \mathrm{map}(S^1, X)$ the space of free loops on X. The loop homology of X is the homology of LX with a shift of degrees, $\mathbb{H}_*(LX) = H_{*+m}(LX)$ and an associative and graded commutative product

$$\mu : \mathbb{H}_p(LX) \otimes \mathbb{H}_q(LX) \to \mathbb{H}_{p+q}(LX)$$

called loop product [2]. When coefficients are taken in a field, there is an isomorphism of graded vector spaces [14]

$$HH_*(C^*X; C^*X) \cong H^*(LX),$$

which dualizes in

$$HH^*(C^*X; C_*X) \cong H_*(LX).$$

If $\mathbb{k}$ is of characteristic 0 and X is simply connected, there is an isomorphism of Gerstenhaber algebras [8–10]

$$\Phi : \mathbb{H}_*(LX) \to HH^*(C^*X; C^*X).$$

Moreover, if $A = (\wedge V, d)$ is the minimal Sullivan model of X, then one has an isomorphism of Gerstenhaber algebras [7, Proposition 3.3]

$$HH^*(A; A) \cong HH^*(C^*X; C^*X).$$

We assume that $\pi_*(X) \otimes \mathbb{Q}$ is finite dimensional; hence, the minimal Sullivan model of X is of the from $(\wedge V, d)$, where V is finite dimensional. There are isomorphisms of Gerstenhaber algebras

$$\mathbb{H}_*(LX, \mathbb{Q}) \xrightarrow{\cong} HH^*(A; A) \xleftarrow{\cong} H_*(\wedge_A L, d_0) \xrightarrow{\cong} H_*(\wedge V \otimes \wedge Z, d),$$

where $L = s^{-1}(\mathrm{Der} \wedge V)$ and $Z = s^{-1}V^{\#}$. In this section, we describe a spectral sequence of $\wedge V \otimes \wedge Z$ that simplifies the computation of $\mathbb{H}_*(LX, \mathbb{Q})$ in some cases.

Proposition 9. *If $\pi_*(X) \otimes \mathbb{Q}$ is finite dimensional and $(\wedge V, d)$ is the minimal Sullivan model of X, then $\mathcal{L} = \mathrm{Der}(\wedge V, d)$ is semifree over $(\wedge V, d)$.*

Proof. The minimal Sullivan model is of the form $(\wedge(V_1 \oplus \cdots \oplus V_n), d)$ such that $dV_1 = 0$ and $dV_i \subset \wedge(V_1 \oplus \cdots \oplus V_{i-1})$ for $1 \leq i \leq n$.

Define a filtration on $\mathcal{L}$ as follows.

$$F_p\mathcal{L} = \{\theta \in \mathcal{L} : \theta(V_1 \oplus \cdots \oplus V_{n-p}) = 0\}.$$

We get a filtration

$$\{0\} = F_0 \subset F_1\mathcal{L} \subset \cdots \subset F_{n-1}\mathcal{L} \subset F_n\mathcal{L} = \mathcal{L}.$$

For instance, if $V_n = <v_{n,1}, \ldots, v_{n,k}>$, then $F_1\mathcal{L} = \wedge V \otimes Z^1$, where Z^1 is spanned by $\{\theta_{1,1}, \ldots, \theta_{1,k}\}$ and $\theta_{1,i} = (v_{n,i}, 1)$. If $V_{n-1} = <v_{n-1,1}, \ldots, v_{n-1,l}>$, then $F_2\mathcal{L}/F_1\mathcal{L} = \wedge V \otimes Z^2$ is spanned by $\{\theta_{2,1}, \ldots, \theta_{2,l}\}$, such that $\theta_{2,j} = (v_{n-1,j}, 1)$. Moreover, $\delta Z^2 \subset (\wedge V) \otimes Z^1 = F_1\mathcal{L}$. In general, $F_k\mathcal{L}/F_{k-1}\mathcal{L} = \wedge V \otimes Z^k$, where Z^k is spanned by derivations $\{\theta_{k,1}, \ldots,\}$ with $\theta_{k,i} = (v_{n-k+1,i}, 1)$ and $\delta Z^k \subset (\wedge V) \otimes (Z^1 \oplus \cdots \oplus Z^{k-1})$. This defines a semifree filtration of $\mathcal{L}$; hence, $(\mathcal{L}, \delta)$ is a semifree differential module over $(\wedge V, d)$. □

It comes from the definition that $[F_p\mathcal{L}, F_q\mathcal{L}] \subset F_r\mathcal{L}$, where $r = \max\{p, q\}$. Hence $[F_p\mathcal{L}, F_q\mathcal{L}] \subset F_{p+q}\mathcal{L}$. The filtration induces a spectral sequence of differential Lie algebras such that $E^0_{m,*} = F_m\mathcal{L}/F_{m-1}\mathcal{L} \cong A \otimes Z^{m,*}$ and $d_0 = d_A \otimes 1$. Hence, $E^1_{m,*} \cong H(A) \otimes Z^m$. The E^1-term, together with differentials, yields

$$\begin{array}{ccccc}
E^1_{n,*} & \xrightarrow{d_1} & E^1_{n-1,*} \cdots & \xrightarrow{d_1} & E^1_{1,*} \\
\| & & \| & & \| \\
H(A) \otimes Z^n_* & \xrightarrow{d_1} & H(A) \otimes Z^{n-1}_* \cdots & \xrightarrow{d_1} & H(A) \otimes Z^1_*.
\end{array}$$

In particular, if $(\wedge V, d) = (\wedge(V_1 \oplus V_2), d)$ with $dV_1 = 0$ and $dV_2 \subset \wedge V_1$, then the above spectral sequence collapses at the E^2-level.

Example 10. Consider the Sullivan algebra $(\wedge(x, y), d)$ with $|x| = 2$, $|y| = 5$ and $dy = x^3$. Here, $H = (\wedge x)/(x^3)$ and Z^1 (resp. Z^2) is spanned by $z_1 = (y, 1)$ (resp. $z_2 = (x, 1)$). Hence $E^1 = H \otimes Z$. Moreover, $d_1 z_1 = 0$, $d_1 z_2 = 3x^2 z_1$ and $d_1(xz_2) = 0$. Therefore, the E^2-term is spanned by $\{z_1, xz_1, xz_2, x^2 z_2\}$ as a vector space. We note that xz_2 and $x^2 z_2$ are of respective degrees 0 and -2.

We can now define a spectral sequence that is useful to compute the loop space homology for certain spaces. Let X be a simply connected compact oriented m-manifold of which $\pi_*(X) \otimes \mathbb{Q}$ is finite dimensional and $A = (\wedge(V_1 \oplus \cdots \oplus V_n), d)$ its minimal Sullivan model, where $dV_i \subset \wedge(V_i \oplus \cdots \oplus V_{i-1})$. Let $Z = s^{-1}V^{\#}$ and $Z^k = s^{-1}V^{\#}_{n-k+1}$. We define a filtration on $A \otimes \wedge Z$ by $F_p = A \otimes \wedge(Z^1 \oplus \cdots \oplus Z^p)$. It verifies

$$A = F_0 \subset F_1 \subset \cdots \subset F_n = A \otimes \wedge Z.$$

As $F_p F_q \subset F_r$, where $r = \max\{p, q\}$, $F_p F_q \subset F_{p+q}$. Moreover, $\{F_p, F_q\} \subset F_s$, where $s = \max\{p, q\}$, $\{F_p, F_q\} \subset F_{p+q}$. This filtration yields a spectral sequence of Gerstenhaber algebras for which $E^1 = H^*(A) \otimes \wedge Z$ and which converges to $H_*(A \otimes \wedge Z, d) \cong \mathbb{H}_*(LX, \mathbb{Q})$. Using this technique, we can recover the loop space homology of complex projective spaces and perform computations for other homogeneous spaces.

Example 11 [3,7,9]. Consider $X = \mathbb{C}P(n)$ of which the minimal Sullivan model is $(\wedge(x, y), d)$, $dx = 0$, $dy = x^{n+1}$. Therefore,

$$\mathbb{H}_*(\mathbb{C}P(n), \mathbb{Q}) \cong H_*(\wedge x/(x^{n+1}) \otimes \wedge(z_1, z_{2n}), d),\ dz_{2n} = 0,\ dz_1 = (n+1)x^n z_{2n}.$$

Here, z_1 and z_{2n} are of respective degrees 1and $2n$. Homology classes are

$$\{x^j z_{2n}^k,\ x^i z_1,\ x^i z_1 z_{2n}^k, \quad k \geq 0, \quad 0 \leq j \leq n-1, \quad 1 \leq i \leq n\}.$$

Brackets can be computed from the Lie algebra structure of derivations on $(\wedge(x, y), d)$. For instance, $\{x^i z_{2n}, x^j z_{2n}\} = 0$, $\{x^i z_1, x^j z_{2n}\} = jx^{i+j-1} z_{2n}$, $\{xz_1 x^i z_{2n}^k, xz_1 x^j z_{2n}^l\} = (i-j)xz_1 x^{i+j} z_{2n}^{k+l}$. In particular, $\{xz_1, x^j z_{2n}\} = jx^j z_{2n}$; hence, $\mathrm{ad}^k(xz_1) \neq 0$, for $k \geq 1$.

Example 12. We consider the minimal Sullivan model of $X = Sp(5)/SU(5)$ which is given by $A = (\wedge(x_6, x_{10}, y_{11}, y_{15}, y_{19}, d)$ with $dx_i = 0$, $dy_{11} = x_6^2$, $dy_{15} = x_6 x_{10}$, $dy_{19} = x_{10}^2$. The rational cohomology is given by classes of $\{1, x_6, x_{10}, x_6 y_{15} - x_{10} y_{11}, x_{10} y_{15} - x_6 y_{19}, x_6(x_{10} y_{15} - x_6 y_{19})\}$. The loop space homology is computed from the complex

$$\begin{aligned} &(A \otimes \wedge(z_{10}, z_{14}, z_{18}, w_5, w_9), d), \quad dz_i = 0, \\ &dw_5 = 2x_6 z_{10} + x_{10} z_{14}, \\ &dw_9 = x_6 z_{14} + 2x_{10} z_{18}. \end{aligned}$$

It contains $H^*(X) \otimes \wedge(z_{10}, z_{14}, z_{18})/I$, where I is the ideal generated by $\{dw_5, dw_9\}$, but also $x_6 w_i$ and $x_{10} w_i$. Nonzero brackets include

$$\begin{aligned} &\{x_6 w_5, x_6 z_i^k\} = x_6 z_i^k, \quad \{x_6 w_9, x_{10} z_i^k\} = x_6 z_i^k, \\ &\{x_{10} w_5, x_6 z_i^k\} = x_{10} z_i^k, \quad \{z_{10}, (x_6 y_{15} - x_{10} y_{11}) z_i^k\} = -x_{10} z_i^k, \\ &\{z_{14}, (x_6 y_{15} - x_{10} y_{11}) z_i^k\} = x_6 z_i^k, \quad \{z_{18}, (x_{10} y_{15} - x_6 y_{19}) z_i^k\} = -x_6 z_i^k. \end{aligned}$$

Hence, for $\alpha = x_6 w_5$, $\mathrm{ad}^k \alpha \neq 0$, $k \geq 1$. It is the same for $\beta = x_{10} w_9$.

We have the more general result.

Theorem 13. *Let X be a homogeneous space of which the minimal Sullivan model is given by $(A, d) = (\wedge(x_1, \ldots, x_n, y_1, \ldots, y_m), d)$, where $dx_i = 0$ and $dy_i \in \wedge(x_1, \ldots, x_n)$. Then the graded Lie algebra $s\mathbb{H}_*(LX, \mathbb{Q})$ is not nilpotent.*

Proof. We consider the complex $(A \otimes \wedge(z_1, \ldots, z_m, w_1, \ldots, w_n), d)$ where $z_j = s^{-1}(y_j, 1)$, $w_i = s^{-1}(x_i, 1)$, $dz_j = 0$ and $dw_i = \sum_j \frac{\partial f_j}{\partial x_i} z_j$. We need to find coefficients $q_i \in \mathbb{Q}$ such that $\alpha = \sum_i q_i x_i w_i$ is a d_1-cocycle.

$$\begin{aligned} d_1(\sum_i q_i x_i w_i) &= \sum_i \sum_j q_i x_i \frac{\partial f_j}{\partial x_i} z_j \\ &= \sum_j (\sum_i q_i x_i \frac{\partial f_j}{\partial x_i}) z_j. \end{aligned}$$

In particular, $d_1 \alpha = 0$ if $\sum_i q_i x_i \frac{\partial f_j}{\partial x_i} = c_j f_j$, for $j = 1, 2, \ldots, m$. It is the case if one takes $q_i = |x_i|$ as $\sum_i |x_i| x_i \frac{\partial f_j}{\partial x_i} = c_j f_j$, where c_j is the degree of the homogeneous polynomial f_j. This is the Euler Theorem for homogeneous functions in the graded case.

If we denote by Z^1 and Z^2, the respective spans of $\{z_j\}$ and $\{w_i\}$, and $H = H^*(X, \mathbb{Q})$, then $d_1 Z^1 = 0$ and $d_1 Z^2 \subset H \otimes Z^1$. As $\alpha \in H \otimes Z^2$ and $d_1(H \otimes \wedge^2 Z^2) \subset H \otimes \wedge^+ Z^1 \otimes Z^2$, then α cannot be a d_1-coboundary. Moreover, $\{\alpha, x_i z_i\} = |x_i| x_i z_i$; hence, $s\mathbb{H}_*(LX, \mathbb{Q})$ is not nilpotent. □

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