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Einstein-like Pseudo-Riemannian Homogeneous Manifolds of Dimension Four

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Abstract. A full classification of invariant Einstein-like metrics on four-dimensional pseudo-Riemannian homogeneous spaces with non-trivial isotropy is given. Specially, proper examples of Codazzi manifolds which are not conformally flat have been presented.

Mathematics Subject Classification. 53C50, 53C30.

Keywords. Non-trivial isotropy, invariant metrics, Einstein-like manifold, homogeneous space, Ricci tensor.

1. Introduction

A pseudo-Riemannian manifold (M, g) is called *homogeneous*, if $I(M)$, the group of isometries of M, acts transitively on M. Equivalently, for any given points $p, q \in M$, an isometry ϕ of M exists such that $\phi(p) = q$ [15].

Homogeneous manifolds of dimension three have been completely studied through Riemannian and Lorentzian manifolds. Until recent years, most of the works in dimension four were focused on the Riemannian spaces (see for example [2]), while little was studied about different signatures [8,9]. A classification of four-dimensional pseudo-Riemannian homogeneous manifolds with non-trivial isotropy has been given in [14] to present the solutions of the Einstein–Maxwell equation.

Einstein metrics and spaces with constant scalar curvature are important families of pseudo-Riemannian metrics with wide applications in geometry and applied physics. It is well known that every Einstein metric is necessarily Ricci-parallel (i.e., $(\nabla_X Ric) = 0$, for all vector fields X tangent to M). Other properties, based on the Ricci tensor, were introduced by Gray [13]:

(M, g) admits a *cyclic-parallel Ricci tensor*, or belongs to class $\mathscr{A}$, if

$$(\nabla_X Ric)(Y, Z) + (\nabla_Y Ric)(Z, X) + (\nabla_Z Ric)(X, Y) = 0, \qquad (1.1)$$

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for all vector fields X, Y and Z tangent to M. The Equation (1.1) is equivalent to requiring that Ric is a *Killing tensor*, that is,

$$(\nabla_X Ric)(X,X) = 0, \quad \forall X \in \mathfrak{X}(M). \tag{1.2}$$

The Eq. (1.2) is also known as the first odd Ledger condition. It is a necessary condition for a (pseudo-)Riemannian manifold to be a D'Atri space. Hence, to classify homogeneous pseudo-Riemannian manifolds of a given dimension satisfying (1.2) is the first step to understand D'Atri spaces of that dimension [1].

(M,g) admits a *Codazzi Ricci tensor*, or belongs to class $\mathscr{B}$, if

$$(\nabla_X Ric)(Y,Z) = (\nabla_Y Ric)(X,Z), \tag{1.3}$$

for all X, Y, $Z \in \mathfrak{X}(M)$. These two conditions which are known as *Einstein-like property*, are in fact generalization of Einstein and Ricci-parallel metrics.

Several authors investigated Einstein-like metrics through different kinds of homogeneous spaces, in both Riemannian and pseudo-Rimannian signatures. In the pseudo-Riemannian case, most of investigations focused on the three-dimensional case and few works also have been done in dimension four [7]. In [3], the complete classification of Einstein-like metrics on three-dimensional homogeneous Lorentzian manifolds was obtained. The same author classified three-dimensional curvature homogeneous Lorentzian manifolds equipped with either Einstein-like or conformally flat metric [4]. Einstein-like metrics have been studied in the class of three-dimensional Lorentzian manifolds admitting a parallel null vector field [11]. In order to study Einstein-like homogeneous four-dimensional spaces, it is necessary to consider this matter through four-dimensional homogeneous spaces with non-trivial isotropy and this is the subject of the present work.

This paper is organized in the following way: In Sect. 2 we report some basic facts on the pseudo-Riemannian homogeneous spaces. Pseudo-Riemannian four-dimensional homogeneous spaces with non-trivial isotropy, equipped with a Ricci-parallel invariant metric, will be presented in Sect. 3. Section 4 is devoted to study non-Ricci-parallel homogeneous pseudo-Riemnnian four-dimensional manifolds which are cyclic parallel. Finally, the non-Ricci-parallel invariant metrics g which are Codazzi, also strict examples of class $\mathscr{B}$ will be presented in the last section. The Obtained results are summarized in the following diagram (Fig. 1):

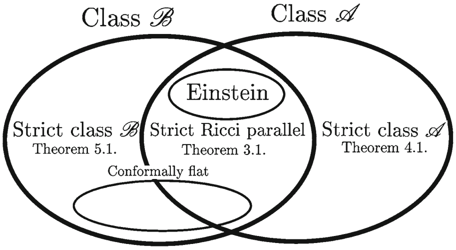

Figure 1. Set specification of the results

2. Preliminaries

Working on a homogeneous space permits us to see the whole manifold, as a single point! In fact, geometrical properties of the manifold can be studied by expressing the (pseudo-)Riemannian manifold as a coset space $M = G/H$ with an invariant metric. At this point of view, the geometrical objects on M will be expressed in Lie terms, in the sense that we apply the Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ and the factor space $\mathfrak{m}$, to study different objects on $M = G/H$. Many authors worked on the different subjects of homogeneous spaces. As a sample, a three-dimensional complete, connected and simply connected homogeneous Riemannian manifold is either symmetric or it is a Lie group equipped with a left-invariant Riemannian metric [16]. Similar results were proved for the Lorentzian signature in [5,6]. In dimension four, pseudo-Riemannian homogeneous manifolds were studied in [8,12], where it is proved that a complete, connected and simply connected homogeneous manifold with non-degenerate Ricci operator is isometric to a Lie group equipped with a left-invariant metric of the appropriate signature. Ricci-parallel examples of the pseudo-Riemannian homogeneous manifolds were also classified in the same articles.

Let $M = G/H$ be a four dimensional homogeneous space with non-trivial isotropy. The condition $\mathfrak{h} \neq 0$ permits us to suppose that M corresponds to one of the examples listed in [14], where the complete classification of pseudo-Riemannian homogeneous four-manifolds is presented and used for classifying four-dimensional Einstein–Maxwell homogeneous spaces.

In order to classify four-dimensional Einstein-like homogeneous pseudo-Riemannian manifolds, the first step is to consider case by case the classification of homogeneous pseudo-Riemannian manifolds given in [14].

Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{h}$ be the isotropy subalgebra. Let $\mathfrak{m} = \mathfrak{g}/\mathfrak{h}$ be the factor space which is the subspace of $\mathfrak{g}$ complementary to $\mathfrak{h}$. Now, the isotropy representation is defined uniquely by the pair $(\mathfrak{g}, \mathfrak{h})$ as following:

$$\phi : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{m}), \quad \phi(x)(y) = [x, y]_{\mathfrak{m}}, \quad \text{for all} \quad x \in \mathfrak{g}, \quad y \in \mathfrak{m}.$$

Let g be a matrix with respect to a basis $\{h_1, \ldots, h_r, u_1, \ldots, u_n\}$ of $\mathfrak{g}$, where $\{h_j\}$ and $\{u_i\}$ are bases of $\mathfrak{h}$ and $\mathfrak{m}$ for $1 \leq j \leq r = \dim H$ and $1 \leq i \leq n = \dim M$, respectively. We can present a bilinear form on $\mathfrak{m}$ by the matrix g. Using the isotropy representation a bilinear form is invariant if and only if ${}^t\phi(x) \circ g + g \circ \phi(x) = 0$, for all $x \in \mathfrak{h}$. From [14], every invariant pseudo-Riemannian metric $\bar{g}$ on the homogeneous space $M = G/H$ corresponds uniquely with a nondegenerate invariant symmetric bilinear form g on $\mathfrak{m}$. For the invariant bilinear form g, the corresponding Levi–Civita connection Λ is determined by the identity

$$\Lambda(x)(y_{\mathfrak{m}}) = \frac{1}{2}[x, y]_{\mathfrak{m}} + \nu(x, y), \quad \text{for all} \quad x, y \in \mathfrak{g}, \tag{2.1}$$

where $\nu : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{m}$ is the $\mathfrak{h}$-invariant symmetric mapping uniquely determined by

$$2g(\nu(x, y), z_{\mathfrak{m}}) = g(x_{\mathfrak{m}}, [z, y]_{\mathfrak{m}}) + g(y_{\mathfrak{m}}, [z, x]_{\mathfrak{m}}), \quad \text{for all } x, y, z \in \mathfrak{g}.$$

The curvature tensor is then determined by

$$\begin{aligned} R:\mathfrak{m}\times\mathfrak{m} &\to \mathfrak{gl}(\mathfrak{m}) \\ (x,y) &\to [\Lambda(x),\Lambda(y)]-\Lambda([x,y]). \end{aligned} \tag{2.2}$$

Finally, the Ricci tensor Ric of g will be deduced by contraction on the first and third indices of the curvature tensor.

We then have all the needed information to check whether Eqs. (1.1) and (1.3) hold, that is, if $M = G/H$ is Einstein-like. We have applied the above argument to all the spaces included in Komrakov's classification [14] of four-dimensional homogeneous pseudo-Riemannian manifolds with nontrivial isotropy and checked the possible forms for the covariant derivative of Ricci tensor to hold on (1.1) and (1.3). The results we obtained are resumed in the following sections.

3. Four-Dimensional Homogeneous Ricci-Parallel Examples

In this section we study the Ricci-parallel examples of pseudo-Riemannian homogeneous spaces with non-trivial isotropy. It is obvious that such examples belong to $\mathscr{A}$ and $\mathscr{B}$.

A pseudo-Riemannian manifold (M, g) is called *Einstein* if

$$Ric = \eta g, \tag{3.1}$$

for a real constant η. It is well known that every Einstein pseudo-Riemannian manifold is Ricci-parallel. The Einstein examples of homogeneous pseudo-Riemannian four-spaces with non-trivial isotropy are completely determined in [14]. In the following theorem we explicitly determine the *strictly Ricci-parallel* (that is, non-Einstein) examples of homogeneous pseudo-Riemannian four-spaces with non-trivial isotropy.

Theorem 3.1. *Let $(G/H, g)$ be arbitrary pseudo-Riemannian four-dimensional homogeneous spaces with non-trivial isotropy, equipped with an invariant metric g. Then $(G/H, g)$ is strictly Ricci-parallel, if and only if it belongs to one of the cases of the Table 1.*

In Table 1, $\underline{N}$ stands for "neutral metric", $\underline{L}$ for "Lorentzian metric", $\{\theta^1,\dots,\theta^4\}$ is the dual basis of $\{u_1,\dots,u_4\}$ and $\underline{\checkmark}$ means that all of the invariant metrics are strictly Ricci-parallel.

Proof. We apply the notation of [14] to identify four dimensional homogeneous spaces with non-trivial isotropy. A homogeneous space of type $n.m^k{:}q$ is the one corresponding to the q-th pair $(\mathfrak{g},\mathfrak{h})$ of type $n.m^k$, where $n = \dim(\mathfrak{h})$ $(= 1,\dots,6)$, m is the number of the complex subalgebra $\mathfrak{h}^{\mathbb{C}}$ of $\mathfrak{so}(4,\mathbb{C})$ and k is the number of the real form of $\mathfrak{h}^{\mathbb{C}}$ [10]. We considered all the spaces included in Komrakov's classification of $M = G/H$, four-dimensional homogeneous pseudo-Riemannian with nontrivial isotropy which appeared in Theorem 2 of [14], and checked the possibility of being Ricci-parallel. We summarize below the full details of the case $1.1^1 : 1$ in Komrakov's list. We studied the other cases in the same way. For this homogeneous pseudo-Riemannian

Table 1. Strictly Ricci-parallel examples of homogeneous spaces G/H with non-trivial isotropy

Case	Invariant metric	Signature	Non-Einstein
$1.1^1:2$	$2a\theta^1\theta^3+b\theta^2\theta^2+2c\theta^2\theta^4+d\theta^4\theta^4, a(c^2-bd)\neq 0$	N/L	$p=0$
$1.1^1:3$	$2a\theta^1\theta^3+b\theta^2\theta^2+2c\theta^2\theta^4+d\theta^4\theta^4, a(c^2-bd)\neq 0$	N/L	$b=a$
$1.1^1:4$	$2a\theta^1\theta^3+b\theta^2\theta^2+2c\theta^2\theta^4+d\theta^4\theta^4, a(c^2-bd)\neq 0$	N/L	$b=0$
$1.1^1:5$	$2a\theta^1\theta^3+b\theta^2\theta^2+2c\theta^2\theta^4+d\theta^4\theta^4, a(c^2-bd)\neq 0$	N/L	$ab\neq bd-c^2$
$1.1^1:6$	$2a\theta^1\theta^3+b\theta^2\theta^2+2c\theta^2\theta^4+d\theta^4\theta^4, a(c^2-bd)\neq 0$	N/L	$b\neq 0$
$1.1^1:7$	$2a\theta^1\theta^3+b\theta^2\theta^2+2c\theta^2\theta^4+d\theta^4\theta^4, a(c^2-bd)\neq 0$	N/L	✓
$1.1^2:2$	$a\theta^1\theta^1+b\theta^2\theta^2+2c\theta^2\theta^4+a\theta^3\theta^3+d\theta^4\theta^4, a(c^2-bd)\neq 0$	N/L	$p=0$
$1.1^2:3$	$a\theta^1\theta^1+b\theta^2\theta^2+2c\theta^2\theta^4+a\theta^3\theta^3+d\theta^4\theta^4, a(c^2-bd)\neq 0$	N/L	$b=a$
$1.1^2:4$	$a\theta^1\theta^1+b\theta^2\theta^2+2c\theta^2\theta^4+a\theta^3\theta^3+d\theta^4\theta^4, a(c^2-bd)\neq 0$	N/L	$b=-a$
$1.1^2:5$	$a\theta^1\theta^1+b\theta^2\theta^2+2c\theta^2\theta^4+a\theta^3\theta^3+d\theta^4\theta^4, a(c^2-bd)\neq 0$	N/L	$b=0$
$1.1^2:6$	$a\theta^1\theta^1+b\theta^2\theta^2+2c\theta^2\theta^4+a\theta^3\theta^3+d\theta^4\theta^4, a(c^2-bd)\neq 0$	N/L	$ab\neq c^2-bd$
$1.1^2:7$	$a\theta^1\theta^1+b\theta^2\theta^2+2c\theta^2\theta^4+a\theta^3\theta^3+d\theta^4\theta^4, a(c^2-bd)\neq 0$	N/L	$ab\neq bd-c^2$
$1.1^2:8$	$a\theta^1\theta^1+b\theta^2\theta^2+2c\theta^2\theta^4+a\theta^3\theta^3+d\theta^4\theta^4, a(c^2-bd)\neq 0$	N/L	$b\neq 0$
$1.1^2:9-10$	$a\theta^1\theta^1+b\theta^2\theta^2+2c\theta^2\theta^4+a\theta^3\theta^3+d\theta^4\theta^4, a(c^2-bd)\neq 0$	N/L	✓
$1.3^1:2$	$-2a\theta^1\theta^4+2a\theta^2\theta^3+b\theta^3\theta^3+2c\theta^3\theta^4+d\theta^4\theta^4, a\neq 0$	N	$\lambda=0$
$1.3^1:3,6,8,$ 10,11,20	$-2a\theta^1\theta^4+2a\theta^2\theta^3+b\theta^3\theta^3+2c\theta^3\theta^4+d\theta^4\theta^4, \quad a\neq 0$	N	✓
$1.3^1:5$	$-2a\theta^1\theta^4+2a\theta^2\theta^3+b\theta^3\theta^3+2c\theta^3\theta^4+d\theta^4\theta^4, a\neq 0$	N	$l=\mu=0$
$1.3^1:9$	$-2a\theta^1\theta^4+2a\theta^2\theta^3+b\theta^3\theta^3+2c\theta^3\theta^4+d\theta^4\theta^4, a\neq 0$	N	$l\neq 1$
$1.3^1:24,25$	$-2a\theta^1\theta^4+2a\theta^2\theta^3+b\theta^3\theta^3+2c\theta^3\theta^4+d\theta^4\theta^4, a\neq 0$	N	$l=0$
$1.4^1:1,5$	$-2a\theta^1\theta^3+a\theta^2\theta^2+b\theta^3\theta^3+2c\theta^3\theta^4+d\theta^4\theta^4, ad\neq 0$	N/L	$b=0$
$1.4^1:13$	$-2a\theta^1\theta^3+a\theta^2\theta^2+b\theta^3\theta^3+2c\theta^3\theta^4+d\theta^4\theta^4, ad\neq 0$	N/L	$a(l+1)\neq -\frac{d}{2}$
$1.4^1:14$	$-2a\theta^1\theta^3+a\theta^2\theta^2+b\theta^3\theta^3+2c\theta^3\theta^4+d\theta^4\theta^4, ad\neq 0$	N/L	$l\neq -1$
$1.4^1:15,18$	$-2a\theta^1\theta^3+a\theta^2\theta^2+b\theta^3\theta^3+2c\theta^3\theta^4+d\theta^4\theta^4, ad\neq 0$	N/L	$d\neq -2a$
$1.4^1:16,19$	$-2a\theta^1\theta^3+a\theta^2\theta^2+b\theta^3\theta^3+2c\theta^3\theta^4+d\theta^4\theta^4, ad\neq 0$	N/L	$d\neq 2a$

Table 1. continued

Case	Invariant metric	Signature	Non-Einstein
$1.4^1 : 17, 20-22$, 24-25	$-2a\theta^1\theta^3 + a\theta^2\theta^2 + b\theta^3\theta^3 + 2c\theta^3\theta^4 + d\theta^4\theta^4, ad \neq 0$	N/L	✓
$2.1^1 : 1$	$2a\theta^1\theta^3 + 2b\theta^2\theta^4, ab \neq 0$	N	$b \neq a$
$2.1^1 : 2$	$2a\theta^1\theta^3 + 2b\theta^2\theta^4, ab \neq 0$	N	✓
$2.1^2 : 1$	$2a\theta^1\theta^3 + b\theta^2\theta^2 + b\theta^4\theta^4, ab \neq 0$	L	$b \neq -a$
$2.1^2 : 2$	$2a\theta^1\theta^3 + b\theta^2\theta^2 + b\theta^4\theta^4, ab \neq 0$	L	$b \neq a$
$2.1^2 : 3-5$	$2a\theta^1\theta^3 + b\theta^2\theta^2 + b\theta^4\theta^4, ab \neq 0$	L	✓
$2.1^3 : 1, 3$	$a\theta^1\theta^1 + a\theta^3\theta^3 + b\theta^2\theta^2 + b\theta^4\theta^4, ab \neq 0$	N	$b \neq a$
$2.1^3 : 2$	$a\theta^1\theta^1 + a\theta^3\theta^3 + b\theta^2\theta^2 + b\theta^4\theta^4, ab \neq 0$	N	$b \neq -a$
$2.1^3 : 4, 5$	$a\theta^1\theta^1 + a\theta^3\theta^3 + b\theta^2\theta^2 + b\theta^4\theta^4, ab \neq 0$	N	✓
$2.1^4 : 1$	$2a\theta^1\theta^3 + 2b\theta^1\theta^4 + 2b\theta^2\theta^3 - 2a\theta^2\theta^4, a^2 + b^2 \neq 0$	N	$b \neq 0$
$2.2^1 : 2$	$2a\theta^1\theta^3 + 2a\theta^2\theta^4 + b\theta^2\theta^2, a \neq 0$	N	$p = 0$
$2.2^1 : 4$	$2a\theta^1\theta^3 + 2a\theta^2\theta^4 + b\theta^2\theta^2, a \neq 0$	N	✓
$2.2^2 : 1, 2$	$2a\theta^1\theta^4 - 2a\theta^2\theta^3 + b\theta^3\theta^3 + b\theta^4\theta^4, a \neq 0$	N	✓
$2.4^1 : 1$	$2a\theta^1\theta^3 - a\theta^2\theta^2 + b\theta^4\theta^4, ab \neq 0$	L	✓
$2.5^1 : 5, 6$	$2a\theta^1\theta^3 + 2a\theta^2\theta^4 + b\theta^3\theta^3, a \neq 0$	N	$\alpha \neq \frac{1}{4}$
$2.5^1 : 7-10$	$2a\theta^1\theta^3 + 2a\theta^2\theta^4 + b\theta^3\theta^3, a \neq 0$	N	✓
$2.5^2 : 3$	$2a\theta^1\theta^3 + a\theta^2\theta^2 + b\theta^3\theta^3 + a\theta^4\theta^4, a \neq 0$	L	$\beta \neq \frac{1}{4}$
$2.5^2 : 4-5$	$2a\theta^1\theta^3 + a\theta^2\theta^2 + b\theta^3\theta^3 + a\theta^4\theta^4, a \neq 0$	L	✓
$3.3^1 : 2-4$	$2a\theta^1\theta^3 + 2a\theta^2\theta^4 + b\theta^3\theta^3, a \neq 0$	N	✓
$3.3^2 : 2, 3$	$2a\theta^1\theta^3 + a\theta^2\theta^2 + b\theta^3\theta^3 + a\theta^4\theta^4, a \neq 0$	L	✓
$3.5^1 : 2, 3$	$4a\theta^1\theta^3 + a\theta^2\theta^2 + b\theta^4\theta^4, ab \neq 0$	N/L	✓
$3.5^2 : 3, 4$	$a\theta^1\theta^1 + a\theta^2\theta^2 + a\theta^3\theta^3 + b\theta^4\theta^4, ab \neq 0$	L	✓
$6.1^3 : 1$	$a\theta^1\theta^1 + a\theta^2\theta^2 + a\theta^3\theta^3 - a\theta^4\theta^4, a \neq 0$	L	✓

four-manifold $M = G/H$, there exists a basis $\{h_1, u_1, \dots u_4\}$ of $\mathfrak{g}$, where the non-zero brackets are

$$[h_1, u_1] = u_1, \quad [h_1, u_3] = -u_3, \quad [u_1, u_3] = [u_2, u_4] = u_2, \quad [u_3, u_4] = u_3,$$

and the isotropy is $\mathfrak{h} = \mathrm{span}\{h_1\}$ [14]. Then, by taking $\mathfrak{m} = \mathrm{span}\{u_1, \dots u_4\}$ we have the following isotropy representation for h_1 and invariant metrics with respect to $\{u_i\}$:

$$H_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad g = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & b & 0 & c \\ a & 0 & 0 & 0 \\ 0 & c & 0 & d \end{pmatrix}, \tag{3.2}$$

for some real constants a, b, c, d. The metric g in this case is non-degenerate if and only if $a^2(c^2 - bd) \neq 0$. It is obvious that if $bd > c^2$ the manifold is Lorentzian, otherwise if $bd < c^2$ then g is of signature $(2, 2)$. Setting $\Lambda_i = \Lambda(u_i)$, the invariant Levi–Civita connection is completely determined using Eq. (2.1):

$$\Lambda_1 = \begin{pmatrix} 0 & -\frac{b}{2a} & 0 & \frac{a-c}{2a} \\ 0 & 0 & \frac{bd+ac-c^2}{2(bd-c^2)} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{ab}{2(bd-c^2)} & 0 \end{pmatrix}, \quad \Lambda_2 = \begin{pmatrix} -\frac{b}{2a} & 0 & 0 & 0 \\ 0 & \frac{bc}{bd-c^2} & 0 & \frac{bd}{bd-c^2} \\ 0 & 0 & \frac{b}{2a} & 0 \\ 0 & \frac{-b^2}{bd-c^2} & 0 & \frac{bc}{bd-c^2} \end{pmatrix},$$
$$\Lambda_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{-bd+ac+c^2}{2(bd-c^2)} & 0 & 0 & 0 \\ 0 & \frac{b}{2a} & 0 & \frac{a+c}{2a} \\ -\frac{ab}{2(bd-c^2)} & 0 & 0 & 0 \end{pmatrix}, \quad \Lambda_4 = \begin{pmatrix} \frac{a-c}{2a} & 0 & 0 & 0 \\ 0 & \frac{c^2}{bd-c^2} & 0 & \frac{dc}{bd-c^2} \\ 0 & 0 & -\frac{a-c}{2a} & 0 \\ 0 & -\frac{bc}{bd-c^2} & 0 & -\frac{c^2}{bd-c^2} \end{pmatrix}. \tag{3.3}$$

Next, we use (2.2) to calculate $R_{ij} := R(u_i, u_j)$. The curvature tensor of $(G/H, g)$ is completely determined by

$$R_{12} = \begin{pmatrix} 0 & -\frac{b^2(2a^2+bd-c^2)}{4a^2(bd-c^2)} & 0 & -\frac{b((bd-c^2)(a+c)+2a^2c)}{4a^2(bd-c^2)} \\ 0 & 0 & \frac{(bd-ac-c^2+2a^2)b}{4a(bd-c^2)} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{b^2}{4(bd-c^2)} & 0 \end{pmatrix},$$

$$R_{13} = \begin{pmatrix} \frac{b(3bd-3c^2-a^2)}{4a(bd-c^2)} & 0 & 0 & 0 \\ 0 & -\frac{cb}{2(bd-c^2)} & 0 & -\frac{db}{2(bd-c^2)} \\ 0 & 0 & \frac{b(-3bd+3c^2+a^2)}{4a(bd-c^2)} & 0 \\ 0 & \frac{b^2}{2(bd-c^2)} & 0 & \frac{cb}{2(bd-c^2)} \end{pmatrix},$$

$$R_{14} = \begin{pmatrix} 0 & -\frac{b((c^2-bd)(a-c)+2a^2c)}{4a^2(bd-c^2)} & 0 & -\frac{c^2(a^2-c^2)+bd(a^2+c^2)}{4a^2(bd-c^2)} \\ 0 & 0 & \frac{-bad+cbd+a^2c-c^3}{4a(bd-c^2)} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{b(a+c)}{4(bd-c^2)} & 0 \end{pmatrix},$$

$$R_{23}=\begin{pmatrix} 0 & 0 & 0 & 0 \\ -\frac{(bd+ac+2a^2-c^2)b}{4a(bd-c^2)} & 0 & 0 & 0 \\ 0 & \frac{b^2(2a^2+bd-c^2)}{4a^2(bd-c^2)} & 0 & \frac{b((c^2-bd)(a-c)+2a^2c)}{4a^2(bd-c^2)} \\ \frac{b^2}{4(bd-c^2)} & 0 & 0 & 0 \end{pmatrix},$$

$$R_{24}=\begin{pmatrix} \frac{b}{2a} & 0 & 0 & 0 \\ 0 & -\frac{cb}{bd-c^2} & 0 & -\frac{bd}{bd-c^2} \\ 0 & 0 & -\frac{b}{2a} & 0 \\ 0 & \frac{b^2}{bd-c^2} & 0 & \frac{bc}{bd-c^2} \end{pmatrix},$$

$$R_{34}=\begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{bd(a+c)+a^2c-c^3}{4a(bd-c^2)} & 0 & 0 & 0 \\ 0 & -\frac{b((bd-c^2)(a+c)+2a^2c)}{4a^2(bd-c^2)} & 0 & -\frac{c^2(a^2-c^2)+bd(a^2+c^2)}{4a^2(bd-c^2)} \\ \frac{(a-c)b}{4(bd-c^2)} & 0 & 0 & 0 \end{pmatrix}.$$

Consequently, the Ricci tensor Ric is given by

$$Ric=\begin{pmatrix} 0 & 0 & \frac{b}{2a}+\frac{ab}{c^2-bd} & 0 \\ 0 & \frac{1}{2}b^2\left(-\frac{1}{a^2}+\frac{4}{c^2-bd}\right) & 0 & -\frac{bc}{2a^2}+\frac{2bc}{c^2-bd} \\ \frac{b}{2a}+\frac{ab}{c^2-bd} & 0 & 0 & 0 \\ 0 & -\frac{bc}{2a^2}+\frac{2bc}{c^2-bd} & 0 & -\frac{3}{2}-\frac{c^2}{2a^2}+\frac{2c^2}{c^2-bd} \end{pmatrix}. \tag{3.4}$$

Moreover, using (3.3) and (3.4), we have the following nonzero component for the covariant derivatives of the Ricci tensor

$$\begin{aligned} \Lambda_1 Ric_{23} &= \frac{b^2(a^2-c^2+bd)}{2a^2(bd-c^2)}, & \Lambda_1 Ric_{34} &= \frac{b(a-2c)(a^2-c^2+bd)}{4a^2(c^2-bd)}, \\ \Lambda_2 Ric_{24} &= \frac{b^2(a^2-c^2+bd)}{2a^2(bd-c^2)}, & \Lambda_2 Ric_{44} &= \frac{bc(a^2-c^2+bd)}{a^2(bd-c^2)}, \\ \Lambda_3 Ric_{12} &= \frac{b^2(a^2-c^2+bd)}{2a^2(c^2-bd)}, & \Lambda_3 Ric_{14} &= \frac{b(a+2c)(a^2-c^2+bd)}{4a^2(c^2-bd)}, \\ \Lambda_4 Ric_{24} &= \frac{bc(a^2-c^2+bd)}{2a^2(bd-c^2)}, & \Lambda_4 Ric_{44} &= \frac{c^2(a^2-c^2+bd)}{a^2(bd-c^2)}, \end{aligned} \tag{3.5}$$

where by $\Lambda_i Ric_{jk}$ we mean $(\Lambda(u_i)Ric)(u_j,u_k)$. According to (3.5) it is easy to check that g is Ricci-parallel if and only if

$$bd=c^2-a^2, \tag{3.6}$$

but this is the necessary and sufficient condition to be Einstein, according to the Eq. (3.1). Thus, this case is not contained in the Table 1. □

One can easily find in the Table 1 some examples which are not always strict Ricci-parallel. For example, case $1.1^1.2$ with $p\neq 0,\frac{1}{2}$ is not even Ricci-parallel; same is the case $1.1^1.3$ when $b\neq a$.

4. Examples of Class $\mathscr{A}$

In the previous section we studied the Ricci-parallel examples of pseudo-Riemannian homogeneous 4-spaces with non-trivial isotropy. It is obvious

Table 2. Strict examples of class $\mathscr{A}$

Case	Class $\mathscr{A}$
$1.1^1 : 3$	$b \neq a$
$1.1^1 : 4$	$b \neq 0$
$1.1^2 : 3$	$b \neq a$
$1.1^2 : 4$	$b \neq -a$
$1.1^2 : 5$	$b \neq 0$
$1.4^1 : 1$	$d = -4a,\ b \neq 0$
$1.4^1 : 2$	$p = 0,\ b \neq 0$
$1.4^1 : 5$	$b \neq 0$
$2.5^1 : 1$	$b \neq 0$
$2.5^2 : 1$	$b \neq 0$

that such examples belong to both classes of Einstein-like spaces. Strict examples of class $\mathscr{A}$, i. e. non-Ricci-parallel spaces of this kind, are determined in the following theorem:

Theorem 4.1. *Let $(G/H, g)$ be an arbitrary non-Ricci-parallel pseudo-Riemannian four-dimensional homogeneous spaces with non-trivial isotropy, equipped with an invariant metric g. Then $(G/H, g)$ is in class $\mathscr{A}$ if it belongs to one of the cases of the following Table 2.*

Proof. Like for Theorem 3.1 we bring the details of the cases $1.1^1 : 1$ and $1.1^1 : 3$ and the other cases were treated by similar arguments. For the case $1.1^1 : 1$, Eq. (3.5) enables us to consider Eq. (1.1). Straightforward calculations yield that a homogeneous space of type $1.1^1 : 1$ belongs to class $\mathscr{A}$ if and only if $bd = c^2 - a^2$ which is equivalent to be Ricci-parallel. This shows that we do not have any strict cyclic parallel space of this type.

Now, let $(G/H, g)$ be a homogeneous space of type $1.1^1 : 3$. With respect to a basis $\{h_1, u_1, \ldots, u_4\}$ for $\mathfrak{g}$, the non-zero brackets are

$$[h_1, u_1] = u_1, \quad [h_1, u_3] = -u_3, \quad [u_1, u_3] = h_1 + u_2,$$

where $\mathfrak{h} = \mathrm{span}\{h_1\}$ [14]. The isotropy representation and invariant metric g with respect to the basis $\{u_1, \ldots, u_4\}$ are the same as Eq. (3.2). The Levi–Civita connection will be determined by applying the Eq. (2.1) and we get

$$\Lambda_1 = \begin{pmatrix} 0 & -\frac{b}{2a} & 0 & -\frac{c}{2a} \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \Lambda_2 = \begin{pmatrix} -\frac{b}{2a} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{b}{2a} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\Lambda_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{b}{2a} & 0 & \frac{c}{2a} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \Lambda_4 = \begin{pmatrix} -\frac{c}{2a} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{c}{2a} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We then specify the curvature tensor from the Eq. (2.2) and the non-zero components are

$$R_{12} = \begin{pmatrix} 0 & -\frac{b^2}{4a^2} & 0 & -\frac{bc}{4a^2} \\ 0 & 0 & \frac{b}{4a} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad R_{13} = \begin{pmatrix} \frac{3b-4a}{4a} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-3b+4a}{4a} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$R_{14} = \begin{pmatrix} 0 & -\frac{bc}{4a^2} & 0 & -\frac{c^2}{4a^2} \\ 0 & 0 & \frac{c}{4a} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad R_{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\frac{b}{4a} & 0 & 0 & 0 \\ 0 & \frac{b^2}{4a^2} & 0 & \frac{bc}{4a^2} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$R_{34} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{c}{4a} & 0 & 0 & 0 \\ 0 & -\frac{bc}{4a^2} & 0 & -\frac{c^2}{4a^2} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The Ricci tensor will be deduced by applying Equation (**??**) and have

$$Ric = \begin{pmatrix} 0 & 0 & \frac{b-2a}{2a} & 0 \\ 0 & -\frac{b^2}{2a^2} & 0 & -\frac{bc}{2a^2} \\ \frac{b-2a}{2a} & 0 & 0 & 0 \\ 0 & -\frac{bc}{2a^2} & 0 & -\frac{c^2}{2a^2} \end{pmatrix}.$$

To check when ϱ is Killing, we first determine the covariant derivative of the Ricci tensor.

$$\Lambda_1\varrho = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{b(b-a)}{2a^2} & 0 \\ 0 & \frac{b(b-a)}{2a^2} & 0 & \frac{c(b-a)}{2a^2} \\ 0 & 0 & \frac{c(b-a)}{2a^2} & 0 \end{pmatrix}, \qquad \Lambda_3\varrho = \begin{pmatrix} 0 & \frac{b(a-b)}{2a^2} & 0 & \frac{c(a-b)}{2a^2} \\ \frac{b(a-b)}{2a^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{c(a-b)}{2a^2} & 0 & 0 & 0 \end{pmatrix},$$

$$\Lambda_4\varrho = \Lambda_2\varrho = 0.$$

It is clear that the Killing equation always holds and so any homogeneous space of type $1.1^1 : 3$ belongs to class $\mathscr{A}$. The strict examples of this type were determined by excluding the Ricci-parallel examples of the Table 1. □

5. Examples of Class $\mathscr{B}$

Finally, we will consider 4-dimensional homogeneous Einstein-like spaces of class $\mathscr{B}$ which are not Ricci-parallel. All of this kind of spaces is determined in the following theorem.

Theorem 5.1. *Let $(G/H, g)$ be an arbitrary non-Ricci-parallel pseudo-Riemannian four-dimensional homogeneous space with non-trivial isotropy, equipped with an invariant metric g. Then $(G/H, g)$ is in class $\mathscr{B}$ if it belongs to one of the cases of the following Table 3:*

Table 3. Strict examples of class $\mathscr{B}$

Case	Class $\mathscr{B}$
$1.1^1 : 1$	$b = 0, a \neq \pm c$
$1.1^1 : 2$	$b = 0, p \neq 0, \frac{1}{2}$
$1.1^2 : 1$	$b = 0$
$1.1^2 : 2$	$b = 0, p \neq 0, 1$
$1.3^1 : 2$	$l \neq 0$
$1.3^1 : 4$	✓
$1.3^1 : 5$	$l \neq 0$ or $l = 0, \mu \neq 0, 2$
$1.3^1 : 7$	✓
$1.3^1 : 12$	$\mu \neq l \pm 1$
$1.3^1 : 13$	$l \neq -\frac{1}{2}, \frac{3}{2}$
$1.3^1 : 14$	$l \neq 0, 1$
$1.3^1 : 15, 16, 19, 22, 26 - 29$	✓
$1.3^1 : 21, 24, 25$	$l \neq 0, 2$
$1.3^1 : 30$	$l \neq 1$ or $l = 1, \mu \neq \pm 1$
$1.4^1 : 2$	$p = 3, b \neq 0$
$1.4^1 : 9$	$d \neq -2a(h^2 + h + r)$
$1.4^1 : 10$	$r \neq -h - h^2$
$1.4^1 : 11$	$d \neq -2ar$
$1.4^1 : 12$	$r \neq 0$
$2.2^1 : 2$	$p \neq 0, \pm 2$
$2.2^1 : 3$	✓
$2.5^1 : 3 - 4$	$2h - h^2 + 4g \neq 0$
$2.5^2 : 2$	$r^2 + p \neq 0$
$3.3^1 : 1$	$p \neq 0$
$3.3^2 : 1$	$p \neq 0$

Proof. The proof is based on case by case study of Komrakow's list. We bring the details of the case $1.1^1 : 1$ and just apply the similar arguments for the other cases. According to the Eqs. (3.5), (1.3) satisfies if and only if either $b = 0$ or $d = \frac{c^2 - a^2}{b}$. The second solution yields that the Ricci tensor is parallel and we must exclude the Ricci-parallel solutions from the first solution. Clearly $c \neq \pm a$ in this case according to the Eq. (3.6). □

Two pseudo-Riemannian manifolds (M, g) and $(M, \tilde{g})$ are called *conformally equivalent*, if $g = e^{2f}\tilde{g}$, for $f \in C^\infty(M)$. By this definition, (M, g) is called *conformally flat* if that is conformally equivalent to a flat manifold $(M, \tilde{g})$. In dimension $n \geq 4$, conformal flatness translates into the following system of algebraic equations:

$$\begin{aligned} W_{ijkh} = R_{ijkh} &- \frac{1}{2}(g_{ik}\varrho_{jh} + g_{jh}\varrho_{ik} - g_{ih}\varrho_{jk} - g_{jk}\varrho_{ih}) \\ &+ \frac{r}{6}(g_{ik}g_{jh} - g_{ih}g_{jk}) = 0, \\ &\text{for all indices } i, j, k, h = 1, \dots, 4, \end{aligned} \tag{5.1}$$

where W denotes the *Weyl tensor* and r is the scalar curvature. To belong to class $\mathscr{B}$ is a necessary condition for being conformally flat. A complete classification of four-dimensional conformally flat homogeneous pseudo-Riemannian manifolds was obtained in [10]. As a conclusion of the Theorem 5.1, we have the following corollary:

Corollary 5.2. *Let $(G/H, g)$ be a pseudo-Riemannian four-dimensional homogeneous space of the Table 3. Then $(G/H, g)$ properly belongs to strict class $\mathscr{B}$ (i.e., it is not conformally flat), if it is one of the cases of the following Table 4:*

Proof. We consider case by case the strict examples of class $\mathscr{B}$, which are presented in the Table 3. For the case $1.1^1 : 1$, the non-zero components of the Weyl tensor are

$$W_{1223} = -\frac{b^2(a^2-2bd+2c^2)}{12a(bd-c^2)}, \quad W_{1234} = \frac{b(a^2c-2bcd+2c^3-3abd+3ac^2)}{12a(bd-c^2)},$$
$$W_{1313} = \frac{b(a^2-2bd+2c^2)}{6(bd-c^2)}, \quad W_{1324} = -\frac{b}{2},$$
$$W_{1423} = -\frac{b(a^2c-2bcd+2c^3+3abd-3ac^2)}{12a(bd-c^2)}, \quad W_{1434} = \frac{bd(a^2-2bd+2c^2)}{12a(bd-c^2)},$$
$$W_{2424} = -\frac{b(a^2-2bd+2c^2)}{6a^2}.$$

Thus, the Weyl tensor vanishes identically if and only if $b = 0$ and so the strict examples of class $\mathscr{B}$, belonging to the case $1.1^1 : 1$, are conformally flat and so not contained in the Table 4. The other cases were checked by similar arguments. □

Table 4. Proper examples of strict class $\mathscr{B}$

Case	Proper class $\mathscr{B}$
$1.3^1 : 2$	$d \neq 0$
$1.3^1 : 4$	$d \neq 0$
$1.3^1 : 5$	$b\mu(\mu-1) \neq 2cl(\mu-1) - d(l^2+\mu)$ and $(2c+dl)^2 + \mu^2 \neq 0$
$1.3^1 : 7$	$d \neq b\lambda - 2c$
$1.3^1 : 12$	$b(l+\mu-1)(\mu-\frac{1}{2}) \neq 0$
$1.3^1 : 15$	$b \neq -d$
$1.3^1 : 16$	$b \neq d$
$1.3^1 : 19$	$b \neq 0$
$1.3^1 : 21$	$b(l-\frac{1}{2}) \neq 0$
$1.3^1 : 24$	$(b-2d(l^2-l))(l-\frac{2}{3}) \neq 0$
$1.3^1 : 25$	$(b+2d(l^2-l))(l-\frac{2}{3}) \neq 0$
$1.3^1 : 28$	$b \neq 2d$
$1.3^1 : 29$	$b \neq -2d$
$1.3^1 : 30$	$b+d-ld-\mu b \neq 2c$
$1.4^1 : 9$	$d^2 + (p^2+p-r)^2 \neq 0$ and $(p+\frac{1}{2})^2 + (4ar+a+4d)^2 \neq 0$
$1.4^1 : 10$	$r \neq h + h^2$
$2.5^2 : 2$	$s \neq 0$

Note that, as also shown by the above Corollary 5.2, differently from the Riemannian case, a (locally) homogeneous conformally flat pseudo-Riemannian manifold need not be (locally) symmetric (see also [10]).

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