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Focal Copulas: A Common Framework for Various Classes of Semilinear Copulas

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Abstract. A new method to construct semi-copulas is introduced. These semi-copulas are called *focal semi-copulas* and their construction is based on linear interpolation on segments connecting the diagonal of the unit square with two focal points. Several classes of semilinear semi-copulas, such as lower semilinear semi-copulas, upper semilinear semi-copulas, ortholinear semi-copulas and biconic semi-copulas with a given diagonal section, turn out to be special cases of focal semi-copulas. Subclasses of focal semi-copulas, such as focal (quasi-)copulas are characterized as well.

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1. Introduction

Aggregation functions are used in several fields of applied sciences as tools to convert finitely many input values into a representative output value. These values are usually assumed to belong to the unit interval $[0, 1]$. Two properties are fundamental for any aggregation function. They coincide in the point $(0, \ldots, 0)$ as well as in the point $(1, \ldots, 1)$, and they are increasing. Subclasses of aggregation functions are obtained by adding more properties. For instance, semi-copulas are binary aggregation functions that have 1 as neutral element. Semi-copulas turn out to be appropriate tools for capturing the relation between multivariate aging and dependence [2,9,14]. Some other properties such as 1-Lipschitz continuity and 2-increasingness have respectively led to the notions of quasi-copulas and copulas [24]. Quasi-copulas appear in fuzzy set theoretical approaches to preference modeling and similarity measurement [5–7]. Copulas turn out to be appropriate tools for linking a joint distribution function with its margins [27]. For this reason, they have become very popular in statistics and probability theory [18].

Recall that a semi-copula [12,13] is a function $S: [0, 1]^2 \to [0, 1]$ satisfying the following conditions:

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(i) for any $x \in [0,1]$, it holds that

$$S(x,0) = S(0,x) = 0, \quad S(x,1) = S(1,x) = x\,;$$

(ii) for any $x, x', y, y' \in [0,1]$ such that $x \leq x'$ and $y \leq y'$, it holds that $S(x,y) \leq S(x',y')$.

The semi-copulas $T_{\mathbf{M}}$ and $T_{\mathbf{D}}$ given by $T_{\mathbf{M}}(x,y) = \min(x,y)$ and $T_{\mathbf{D}}(x,y) = \min(x,y)$ whenever $\max(x,y) = 1$, and $T_{\mathbf{D}}(x,y) = 0$ elsewhere, are respectively the greatest and the smallest semi-copula, i.e. for any semi-copula S, it holds that $T_{\mathbf{D}} \leq S \leq T_{\mathbf{M}}$.

A semi-copula S is semilinear [19] if for any $\mathbf{x} \in [0,1]^2$, there exists $\mathbf{y} \in [0,1]^2$, $\mathbf{y} \neq \mathbf{x}$, such that S is linear on the segment connecting the points $\mathbf{x}$ and $\mathbf{y}$. The semi-copulas $T_{\mathbf{M}}$ and $T_{\mathbf{D}}$ are semilinear.

A semi-copula Q is a quasi-copula [16,17,23] if it is 1-Lipschitz continuous, i.e. for any $x, x', y, y' \in [0,1]$, it holds that

$$|Q(x',y') - Q(x,y)| \leq |x' - x| + |y' - y|.$$

A semi-copula C is a copula [1,22,24] if it is 2-increasing, i.e. for any $x, x', y, y' \in [0,1]$ such that $x \leq x'$ and $y \leq y'$, it holds that

$$V_C([x,x'] \times [y,y']) := C(x',y') + C(x,y) - C(x',y) - C(x,y') \geq 0.$$

$V_C([x,x'] \times [y,y'])$ is called the C-volume of the rectangle $[x,x'] \times [y,y']$. The copulas $T_{\mathbf{M}}$ and $T_{\mathbf{L}}$ with $T_{\mathbf{L}}(x,y) = \max(x+y-1,0)$ are respectively the greatest and the smallest copula, i.e. for any copula C, it holds that $T_{\mathbf{L}} \leq C \leq T_{\mathbf{M}}$. Note that for any copula C it holds that

$$\frac{\partial^2 C(x,y)}{\partial x \partial y} \geq 0, \tag{1}$$

for any $(x,y) \in [0,1]^2$ where the mixed partial derivative exists. If the mixed partial derivative of a semi-copula exists everywhere, then (1) guarantees that it is a copula [3].

The diagonal section of a $[0,1]^2 \to [0,1]$ function F is the function $\delta_F \colon [0,1] \to [0,1]$ defined by $\delta_F(x) = F(x,x)$. A diagonal function [10] is a function $\delta \colon [0,1] \to [0,1]$ satisfying the following conditions:

(D1) $\delta(0) = 0$, $\delta(1) = 1$;
(D2) δ is increasing;
(D3) for any $x \in [0,1]$, it holds that $\delta(x) \leq x$;
(D4) δ is 2-Lipschitz continuous, i.e. for any $x, x' \in [0,1]$, it holds that

$$|\delta(x') - \delta(x)| \leq 2|x' - x|.$$

The functions $\delta_{\mathbf{M}}(x) = x$ and $\delta_{\mathbf{W}}(x) = \max(2x-1,0)$ are examples of diagonal functions. Moreover, for any diagonal function δ, it holds that

$$\delta_{\mathbf{W}} \leq \delta \leq \delta_{\mathbf{M}}.$$

The set of all diagonal functions will be denoted by $\mathcal{D}$; the subset of *twice differentiable* functions in $\mathcal{D}$ will be denoted by $\mathcal{D}^{\mathrm{d}}$. The set of all $[0,1] \to [0,1]$ functions that satisfy conditions (D1)–(D3) will be denoted by $\mathcal{D}_{\mathrm{S}}$; the subset of *absolutely continuous* functions in $\mathcal{D}_{\mathrm{S}}$ will be denoted by $\mathcal{D}_{\mathrm{S}}^{\mathrm{ac}}$.

The diagonal section of a copula C is a diagonal function. Conversely, for any diagonal function δ, there exists at least one copula C with diagonal section $\delta_C = \delta$. For instance, the copula C_δ defined by

$$C_\delta(x, y) = \min\left(x, y, \frac{\delta(x) + \delta(y)}{2}\right)$$

is the greatest symmetric copula with diagonal section δ [8,11,25].

Interpolation is a basic method for constructing semi-copulas, quasi-copulas and copulas. Linear interpolation has led to several classes of semi-linear semi-copulas with a given diagonal section. For instance, biconic semi-copulas [20] are obtained by linear interpolation on segments that connect the diagonal of the unit square to the vertices $(0, 1)$ and $(1, 0)$, while ortholinear semi-copulas [21] are obtained by linear interpolation on segments that are perpendicular to the diagonal of the unit square. Some other classes of semilinear semi-copulas, such as lower and upper semilinear semi-copulas [10] and horizontal and vertical semilinear semi-copulas [4], are obtained by linear interpolation on horizontal and/or vertical segments that connect the diagonal of the unit square to the boundaries of the unit square.

Characteristic for biconic semi-copulas with a given diagonal section is that always one of the vertices $(0, 1)$ and $(1, 0)$ is involved in the interpolation procedure. In other words, the segments on which the linear interpolation is performed always contain one of these vertices. Moreover, such segments cover the whole unit square. Inspired by the above observation, we introduce *focal semi-copulas with a given diagonal section* by considering linear interpolation on segments that connect the diagonal of the unit square with either one of two specific points located outside the unit square. These two points are called *focal points*. The segments on which the linear interpolation is performed cover the unit square for an appropriate choice of these focal points. Choosing the focal points to be symmetric leads to symmetric focal semi-copulas with a given diagonal section. For instance, choosing the symmetric focal points $(0, 1)$ and $(1, 0)$ [resp. $(-\infty, \infty)$ and $(\infty, -\infty)$] leads to the class of biconic (resp. ortholinear) semi-copulas with a given diagonal section, while choosing the symmetric focal points $(-\infty, 1)$ and $(1, -\infty)$ [resp. $(0, \infty)$ and $(\infty, 0)$] leads to the class of lower (resp. upper) semilinear semi-copulas.

This paper is organized as follows. In the following three sections, we introduce three different classes of focal functions with a given diagonal section. For each class, we characterize the corresponding subclasses of focal semi-copulas, focal quasi-copulas and focal copulas. Finally, we provide some conclusions.

2. Class 1: Focal Points $(-a, 1)$ and $(1, -a)$

In this section, we introduce functions that are constructed by linear interpolation on segments connecting points on the diagonal of the unit square to the focal points $(-a, 1)$ and $(1, -a)$, with $a \in [0, \infty]$. This interpolation scheme is depicted in Fig. 1 for some segments.

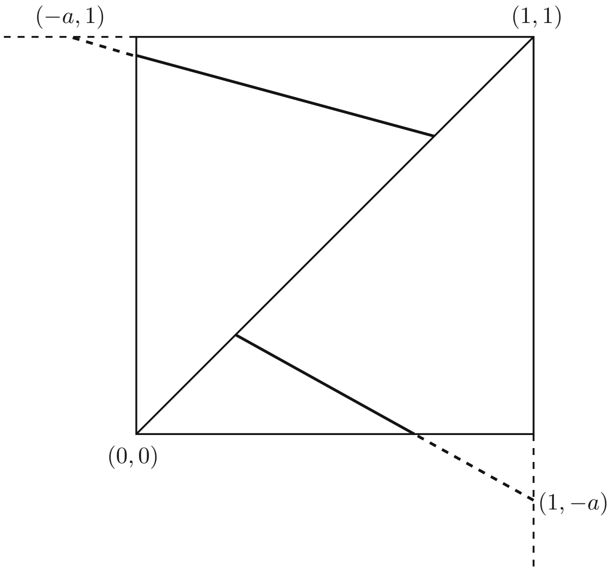

Figure 1. Some segments on which a focal function of class 1 is linear

Let us introduce the notations

$$s = \frac{x+ay}{1+a+x-y}, \tag{2}$$

$$t = \frac{y+ax}{1+a+x-y}, \tag{3}$$

with $a \in [0, \infty]$, and

$$I_1 := \{(x, y) \in [0, 1]^2 | y \leq x\},$$
$$I_2 := \{(x, y) \in [0, 1]^2 | x \leq y\},$$
$$D := I_1 \cap I_2.$$

Let $\delta \in \mathcal{D}_{\mathrm{S}}$ and $a \in [0, \infty]$. The function $A_{\delta,a} \colon [0, 1]^2 \to [0, 1]$ given by

$$A_{\delta,a}(x, y) = \begin{cases} y\dfrac{\delta(t)}{t}, & \text{if } (x, y) \in I_1, \\ x\dfrac{\delta(s)}{s}, & \text{if } (x, y) \in I_2, \end{cases} \tag{4}$$

is well defined. This function is called a *focal function* since it is linear on segments connecting the points (x, x) and $(-a, 1)$ as well as on segments connecting the points (x, x) and $(1, -a)$. Since the points $(-a, 1)$ and $(1, -a)$ are symmetric, any focal function $A_{\delta,a}$ is symmetric. For any focal function $A_{\delta,a}$, condition (i) of the definition of a semi-copula always holds. Note that a focal function $A_{\delta,a}$ is uniquely determined by its diagonal section. Clearly, a focal function $A_{\delta,a}$ is continuous if and only if δ is continuous.

Remark 1. (i) If $a = 0$, then the definition of a biconic function is retrieved [20].

(ii) If $a = \infty$, then the definition of a lower semilinear function is retrieved [10].

2.1. Focal Semi-Copulas of Class 1

In this subsection, we characterize the elements of $\mathcal{D}_{\mathrm{S}}^{\mathrm{ac}}$ for which the corresponding focal function $A_{\delta,a}$ is a semi-copula. For a focal function $A_{\delta,a}$, this characterization involves the use of the functions $\lambda_\delta, \mu_\delta \colon]0,1[\to \mathbb{R}$ defined by

$$\lambda_\delta(x) = \frac{\delta(x)}{x} \quad \text{and} \quad \mu_\delta(x) = \frac{x-\delta(x)}{1-x}. \tag{5}$$

Proposition 1. *Let $\delta \in \mathcal{D}_{\mathrm{S}}^{\mathrm{ac}}$ and $a \in [0,\infty]$. Then, the focal function $A_{\delta,a}$ defined in (4) is a semi-copula if and only if the function λ_δ is increasing.*

Proof. Suppose that λ_δ is increasing. To prove that $A_{\delta,a}$ is a semi-copula, it suffices to prove its increasingness in each variable. Since $A_{\delta,a}$ is symmetric, it suffices to prove its increasingness in each variable on I_2. We prove that $A_{\delta,a}$ is increasing in the second variable (the proof of the increasingness in the first variable is similar).

Let $(x,y),(x,y') \in I_2$ such that $y \le y'$. Let us introduce the notation $s_1 = \frac{x+ay'}{1+a+x-y'}$. The increasingness of $A_{\delta,a}$ is then equivalent to

$$x\frac{\delta(s_1)}{s_1} - x\frac{\delta(s)}{s} = x(\lambda_\delta(s_1) - \lambda_\delta(s)) \ge 0. \tag{6}$$

Since $s \le s_1$ and λ_δ is increasing, inequality (6) immediately follows.

Conversely, suppose that $A_{\delta,a}$ is a semi-copula. Let $y, y' \in]0,1[$ such that $y \le y'$ and $x \in [0,1]$ such that $x \le y$. Clearly, the points $(x, \frac{(x+a+1)y-x}{y+a})$ and $(x, \frac{(x+a+1)y'-x}{y'+a})$ are located in I_2. The increasingness of $A_{\delta,a}$ in the second variable implies

$$A_{\delta,a}\left(x, \frac{(x+a+1)y'-x}{y'+a}\right) - A_{\delta,a}\left(x, \frac{(x+a+1)y-x}{y+a}\right) \ge 0, \tag{7}$$

or, equivalently,

$$x(\lambda_\delta(y') - \lambda_\delta(y)) \ge 0.$$

Hence, the increasingness of λ_δ follows. □

Example 1. Consider the diagonal function $\delta_{\mathbf{M}}$. Clearly, $\delta_{\mathbf{M}} \in \mathcal{D}_{\mathrm{S}}^{\mathrm{ac}}$. One easily verifies that the function $\lambda_{\delta_{\mathbf{M}}}$ is increasing. The corresponding focal semi-copula is $T_{\mathbf{M}}$.

Example 2. Consider the diagonal function $\delta(x) = x^{1+\theta}$ with $\theta \in [0,1]$. Clearly, $\delta \in \mathcal{D}_{\mathrm{S}}^{\mathrm{ac}}$. One easily verifies that the function λ_δ is increasing. The corresponding family of focal semi-copulas is given by

$$A_{\delta,a}(x,y) = \begin{cases} yt^\theta, & \text{if } (x,y) \in I_1, \\ xs^\theta, & \text{if } (x,y) \in I_2, \end{cases}$$

where s and t are given in (2) and (3).

2.2. Focal Quasi-Copulas of Class 1

In this subsection, we characterize the elements of $\mathcal{D}$ for which the corresponding focal function $A_{\delta,a}$ is a quasi-copula.

Proposition 2. *Let $\delta \in \mathcal{D}$ and $a \in [0,\infty]$. Then, the focal function $A_{\delta,a}$ defined in (4) is a quasi-copula if and only if the following conditions are fulfilled:*

(i) *the function λ_δ defined in (5) is increasing;*
(ii) *the function $\psi_{\delta,a} \colon]0,1[\to \mathbb{R}$ defined by*

$$\psi_{\delta,a}(x) = \left(\frac{x}{1-x}\right)^{1+a} (1 - \lambda_\delta(x))$$

is increasing;
(iii) *the inequality*

$$1 - \frac{x(a+x)(1+c)}{1+c} \lambda'_\delta(x) \geq 0$$

holds for any $x \in]0,1[$ where the derivative exists.

Proof. Suppose that conditions (i)–(iii) are fulfilled. Due to Proposition 1, the function $A_{\delta,a}$ is increasing. Therefore, to prove that $A_{\delta,a}$ is a quasi-copula, we need to show that it is 1-Lipschitz continuous. Recall that the 1-Lipschitz continuity is equivalent to the 1-Lipschitz continuity in each variable. Since $A_{\delta,a}$ is symmetric, it is sufficient to show that $A_{\delta,a}$ is 1-Lipschitz continuous in each variable on I_2. We prove that $A_{\delta,a}$ is 1-Lipschitz continuous in the first variable on I_2 (the proof of the 1-Lipschitz continuity in the second variable is similar). Let $(x,y),(x',y) \in I_2$ such that $x \leq x'$. Let us introduce the notation $s_2 = \frac{x'+ay}{1+a+x'-y}$. The 1-Lipschitz continuity of $A_{\delta,a}$ is then equivalent to

$$x'\frac{\delta(s_2)}{s_2} - x\frac{\delta(s)}{s} \leq x' - x,$$

or, equivalently,

$$x'\left(\frac{1-s_2}{s_2}\right)^{1+a} \psi_{\delta,a}(s_2) - x\left(\frac{1-s}{s}\right)^{1+a} \psi_{\delta,a}(s) \geq 0.$$

Since $x \leq x'$, $s \leq s_2$ and $\psi_{\delta,a}$ is increasing, it holds that

$$\begin{aligned} x'\left(\frac{1-s_2}{s_2}\right)^{1+a} &\psi_{\delta,a}(s_2) - x\left(\frac{1-s}{s}\right)^{1+a} \psi_{\delta,a}(s) \\ &\geq x\left(\frac{1-s}{s}\right)^{1+a} (\psi_{\delta,a}(s_2) - \psi_{\delta,a}(s)) \geq 0. \end{aligned}$$

Conversely, suppose that $A_{\delta,a}$ is a quasi-copula. Proposition 1 implies the increasingness of λ_δ. Let $x, x' \in]0,1[$ such that $x \leq x'$ and $y \in [0,1]$ such that $x' \leq y$. The 1-Lipschitz continuity of $A_{\delta,a}$ in the first variable implies that

$$x'\frac{\delta(s_2)}{s_2} - x\frac{\delta(s)}{s} \leq x' - x. \tag{8}$$

Dividing by $x' - x$ and taking the limit $x' \to x$, inequality (8) becomes

$$\lambda_\delta(s) + x\frac{(1+a)(1-y)}{(1+a+x-y)^2}\lambda'_\delta(s) \leq 1,$$

where the derivative exists. Setting $x = y$, the last inequality reduces to

$$(1+a)(1-\lambda_\delta(x)) - x(1-x)\lambda'_\delta(x) \geq 0,$$

or, equivalently, $\psi'_{\delta,a}(x) \geq 0$, where the derivative exists. Since δ is absolutely continuous, it holds that $\psi_{\delta,a}$ is absolutely continuous. The fact that $\psi'_{\delta,a}(x) \geq 0$ on the interval $]0,1[$, where the derivative exists, then implies that $\psi_{\delta,a}$ is increasing. Similarly, the 1-Lipschitz continuity of $A_{\delta,a}$ in the second variable implies condition (iii). □

Example 3. Consider the diagonal function of Example 2. Clearly, conditions (i)–(iii) of Proposition 2 are fulfilled. The corresponding family of focal functions is a family of focal quasi-copulas.

2.3. Focal Copulas of Class 1

In this subsection, we characterize the elements of $\mathcal{D}^{\mathrm{d}}$ for which the corresponding focal function $A_{\delta,a}$ is a copula.

Theorem 1. *Let $\delta \in \mathcal{D}^{\mathrm{d}}$ and $a \in [0,\infty]$. Then, the focal function $A_{\delta,a}$ defined in (4) is a copula if and only if the following conditions are fulfilled:*

(i) *the function λ_δ defined in (5) is increasing;*

(ii) *the function $h_{\delta,a}\colon]0,1[\to \mathbb{R}$ defined by*

$$h_{\delta,a}(x) = x(a+x)\left(\frac{x}{1-x}\right)^a \lambda'_\delta(x)$$

is increasing;

(iii) *the inequality*

$$x(1-a-2x)\delta'(x) + 2(a+x)\delta(x) \geq 0 \tag{9}$$

holds for any $x \in]0,1]$.

Proof. Suppose that $A_{\delta,a}$ is a copula. Due to Proposition 1 condition (i) immediately follows. For any $(x,y) \in I_2 \backslash D$, the 2-increasingness of $A_{\delta,a}$ implies that

$$\frac{\partial^2 A_{\delta,a}(x,y)}{\partial x \partial y} \geq 0.$$

Substituting the expression of $A_{\delta,a}$, the latter inequality becomes

$$\left(\frac{x+a}{(1+a+x-y)^2} + \frac{x(1-a-x-y)}{(1+a+x-y)^3}\right)\lambda'_\delta(s)$$
$$+\frac{(1+a)x(a+x)(1-y)}{(1+a+x-y)^4}\lambda''_\delta(s) \geq 0, \tag{10}$$

where the derivatives exist. Setting $y = x$, the latter inequality reduces to

$$(a(1+a) + 2x(1-x))\lambda'_\delta(x) + x(1-x)(a+x)\lambda''_\delta(x) \geq 0, \tag{11}$$

where the derivatives exist. Some elementary manipulations yield $h'_{\delta,a}(x) \geq 0$ where the derivative exists. Hence, condition (ii) follows. Finally, the fact that $V_{A_{\delta,a}}(S) \geq 0$ for any square $S = [x,x'] \times [x,x']$ centered around the main diagonal yields

$$\begin{aligned} V_{A_{\delta,a}}([x,x'] \times [x,x']) &= \delta(x) + \delta(x') - A_{\delta,a}(x,x') - A_{\delta,a}(x',x) \\ &= \delta(x) + \delta(x') - 2x\frac{\delta(s)}{s} \geq 0. \end{aligned}$$

Dividing by $(x'-x)$ and taking the limit $x' \to x$, condition (iii) immediately follows.

Conversely, suppose that conditions (i)–(iii) are fulfilled. Due to Proposition 1, $A_{\delta,a}$ is a semi-copula. To prove that $A_{\delta,a}$ is a copula, we need to show its 2-increasingness. We distinguish the following three cases:

(a) If $(x,y) \in I_2 \backslash D$, then the 2-increasingness of $A_{\delta,a}$ is equivalent to inequality (10). Since $h_{\delta,a}$ is increasing, inequality (11) holds and the left-hand side of inequality (10) is greater than or equal to

$$\left(\frac{x+a}{(1+a+x-y)^2} + \frac{x(1-a-x-y)}{(1+a+x-y)^3}\right)\lambda'_\delta(s) \\ -\frac{(1+a)x(a+x)(1-y)}{(1+a+x-y)^4}\frac{a(1+a)+2s(1-s)}{s(1-s)(a+s)}\lambda'_\delta(s).$$

Some elementary manipulations show that the positivity of the latter expression is equivalent to

$$(1+a)(y-x)\lambda'_\delta(s) \geq 0.$$

Since λ_δ is increasing, the latter inequality clearly holds.

(b) If $(x,y) \in I_1 \backslash D$, then the proof of the 2-increasingness of $A_{\delta,a}$ is identical due to its symmetry.

(c) Finally, let $S = [x,x'] \times [x,x']$ be a square centered around the main diagonal. Its volume is given by

$$V_{A_{\delta,a}}(S) = \delta(x) + \delta(x') - 2A_{\delta,a}(x,x').$$

Due to (a), it holds that

$$\frac{\partial^2 A_{\delta,a}(u,v)}{\partial u \partial v} \geq 0,$$

for any $(u,v) \in I_2 \backslash D$, which implies that

$$V_1 = \int_x^{x'} \mathrm{d}u \int_u^{x'} \frac{\partial^2 A_{\delta,a}(u,v)}{\partial u \partial v}\,\mathrm{d}v \geq 0.$$

Some elementary manipulations yield

$$\begin{aligned} V_1 &= \int_x^{x'} \left(\frac{\partial A_{\delta,a}(u,x')}{\partial u} - \frac{\partial A_{\delta,a}(u,v)}{\partial u}|_{v=u} \right) \mathrm{d}u \\ &= \delta(x') - A_{\delta,a}(x,x') - \frac{1}{1+a} \int_x^{x'} \frac{u(1-u)\delta'(u) + (a+u)\delta(u)}{u} \mathrm{d}u \\ &= \delta(x') - A_{\delta,a}(x,x') - \frac{1}{1+a}(\delta(x') - \delta(x)) \\ &\quad + \frac{1}{1+a} \int_x^{x'} \frac{u^2\delta'(u) - (a+u)\delta(u)}{u} \mathrm{d}u. \end{aligned}$$

Note that inequality (9) is equivalent to

$$u(1-a-2u)\delta'(u) + 2(a+u)\delta(u) \geq 0,$$

for any $u \in]0,1]$. Dividing by $(1+a)u$, the latter inequality becomes

$$\frac{1}{1+a} \frac{u(1-a-2u)\delta'(u) + 2(a+u)\delta(u)}{u} \geq 0,$$

which implies that

$$V_2 = \frac{1}{1+a} \int_x^{x'} \frac{u(1-a-2u)\delta'(u) + 2(a+u)\delta(u)}{u} \, \mathrm{d}u \geq 0.$$

Some elementary manipulations yield

$$V_2 = \frac{1-a}{1+a} \left(\delta(x') - \delta(x)\right) - \frac{2}{1+a} \int_x^{x'} \frac{u^2\delta'(u) - (a+u)\delta(u)}{u} \, \mathrm{d}u.$$

It now follows that

$$2V_1 + V_2 = \delta(x) + \delta(x') - 2A_{\delta,a}(x,x') = V_{A_{\delta,a}}(S) \geq 0.$$

□

Example 4. Consider the diagonal function of Example 2. Clearly, the conditions of Theorem 1 are fulfilled. The corresponding family of focal functions is a family of focal copulas.

In the following proposition, we provide sufficient conditions for the conditions of Theorem 1.

Proposition 3. *Let $\delta \in \mathcal{D}^{\mathrm{d}}$ and $a \in [0,\infty]$. If the following conditions*

(i) *the function λ_δ defined in (5) is increasing;*
(ii) *the function λ_δ is convex;*
(iii) *the function $\rho_\delta \colon]0,1] \to \mathbb{R}$ defined by*

$$\rho_\delta(x) = \frac{\lambda_\delta(x)}{x} = \frac{\delta(x)}{x^2}$$

is decreasing,

are fulfilled, then the focal function $A_{\delta,a}$ is a copula.

Proof. Suppose that conditions (i)–(iii) are fulfilled. Condition (ii) clearly implies inequality (10) and hence condition (ii) of Theorem 1 follows. Inequality (9) can be written as

$$x(1-2x)\delta'(x) + 2x\delta(x) + a(2\delta(x) - x\delta'(x)) \geq 0. \tag{12}$$

Since ρ_δ is decreasing, it follows that

$$\rho_\delta'(x) = \frac{\delta'(x)x^2 - 2x\delta(x)}{x^4} \leq 0,$$

or, equivalently, $2\delta(x) \geq x\delta'(x)$. Denoting the left-hand side of inequality (12) as β, it then holds that

$$\begin{aligned}\beta &\geq x((1-2x)+x)\delta'(x) + a(2\delta(x) - x\delta'(x))\\ &= x(1-x)\delta'(x) + a(2\delta(x) - x\delta'(x)) \geq 0.\end{aligned}$$

□

Example 5. Consider the diagonal function $\delta(x) = \frac{x}{2-x}$. Clearly, the conditions of Proposition 3 are fulfilled. The corresponding focal function is a focal copula.

Note that the conditions of Proposition 3 are not necessary in general. This fact is illustrated in the following example.

Example 6. Consider the diagonal function of Example 2. Note that the function λ_δ is not convex, while the conditions of Theorem 1 are fulfilled.

3. Class 2: Focal Points $(0, 1+c)$ and $(1+c, 0)$

In this section, we introduce functions that are constructed by linear interpolation on segments connecting points on the diagonal of the unit square to the focal points $(0, 1+c)$ and $(1+c, 0)$, with $c \in [0, \infty]$. This interpolation scheme is depicted in Fig. 2 for some segments.

Let us introduce the notations

$$p = \frac{(1+c)x}{1+c+x-y}, \tag{13}$$

$$q = \frac{(1+c)y}{1+c+y-x}, \tag{14}$$

with $c \in [0, \infty]$. Let $\delta \in \mathcal{D}_S$ and $c \in [0, \infty]$. The function $C_{\delta,c} \colon [0,1]^2 \to [0,1]$ given by

$$C_{\delta,c}(x,y) = \begin{cases} y - (1-x)\dfrac{q-\delta(q)}{1-q}, & \text{if } (x,y) \in I_1,\\ x - (1-y)\dfrac{p-\delta(p)}{1-p}, & \text{if } (x,y) \in I_2, \end{cases} \tag{15}$$

is well defined. This function is called a *focal function* since it is linear on segments connecting the points (x, x) and $(0, 1+c)$ as well as on segments connecting the points (x, x) and $(1+c, 0)$. Since the points $(0, 1+c)$ and $(1+c, 0)$ are symmetric, any focal function $C_{\delta,c}$ is symmetric. For any focal function $C_{\delta,c}$, condition (i) of the definition of a semi-copula always holds.

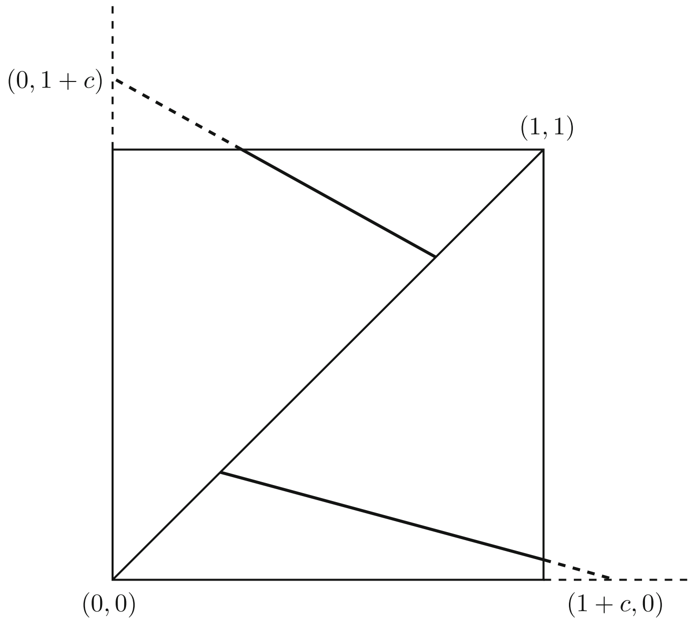

Figure 2. Some segments on which a focal function of class 2 is linear

Note that a focal function $C_{\delta,c}$ is uniquely determined by its diagonal section. Clearly, a focal function $C_{\delta,c}$ is continuous if and only if δ is continuous.

Remark 2. (i) If $c = 0$, then the definition of a biconic function is retrieved [20].

(ii) If $c = \infty$, then the definition of an upper semilinear function is retrieved [10].

3.1. Focal Semi-Copulas of Class 2

In this subsection, we characterize the elements of $\mathcal{D}_{\mathrm{S}}^{\mathrm{ac}}$ for which the corresponding focal function $C_{\delta,c}$ is a semi-copula.

Proposition 4. *Let $\delta \in \mathcal{D}_{\mathrm{S}}^{\mathrm{ac}}$ and $c \in [0, \infty]$. Then, the focal function $C_{\delta,c}$ defined in (15) is a semi-copula if and only if the following conditions are fulfilled:*

(i) *the function $\tau_{\delta,c} \colon]0,1[\to \mathbb{R}$ defined by*

$$\tau_{\delta,c}(x) = \left(\frac{x}{1-x}\right)^{1+c} \mu_\delta(x)$$

is decreasing;

(ii) *the inequality*

$$1 - \frac{(1-x)(1+c-x)}{1+c}\mu_\delta'(x) \geq 0$$

holds for any $x \in]0,1[$ where the derivative exists.

Proof. Suppose that conditions (i) and (ii) are fulfilled. To prove that $C_{\delta,c}$ is a semi-copula, it suffices to prove its increasingness in each variable. Since $C_{\delta,c}$ is symmetric, it suffices to prove its increasingness in each variable on I_2. We prove that $C_{\delta,c}$ is increasing in the second variable (the proof of the increasingness in the first variable is similar). Let $(x, y'), (x, y) \in I_2$ such that $y \leq y'$. Let us introduce the notation $p_1 = \frac{(1+c)x}{1+c+x-y'}$. The increasingness of $C_{\delta,c}$ is then equivalent to

$$x - (1-y')\frac{p_1 - \delta(p_1)}{1-p_1} - x + (1-y)\frac{p-\delta(p)}{1-p}$$
$$= (1-y)\left(\frac{1-p}{p}\right)^{1+c} \tau_{\delta,c}(p) - (1-y')\left(\frac{1-p_1}{p_1}\right)^{1+c} \tau_{\delta,c}(p_1)) \geq 0.$$

Since $p \leq p_1$ and $\tau_{\delta,c}$ is decreasing, it follows that

$$(1-y)\left(\frac{1-p}{p}\right)^{1+c} \tau_{\delta,c}(p) - (1-y')\left(\frac{1-p_1}{p_1}\right)^{1+c} \tau_{\delta,c}(p_1)$$
$$\geq (1-y)\left(\frac{1-p}{p}\right)^{1+c} (\tau_{\delta,c}(p) - \tau_{\delta,c}(p_1)) \geq 0.$$

Conversely, suppose that $C_{\delta,c}$ is a semi-copula. Let $y, y' \in]0,1[$ such that $y < y'$ and $x \in [0,1]$ such that $x \leq y$. The increasingness of $C_{\delta,c}$ in the second variable implies

$$(1-y)\frac{p-\delta(p)}{1-p} - (1-y')\frac{p_1-\delta(p_1)}{1-p_1} \geq 0. \tag{16}$$

Dividing by $y' - y$ and taking the limit $y' \to y$, inequality (16) becomes

$$\mu_\delta(p) - \frac{(1+c)x(1-y)}{(1+c+x-y)^2}\mu'_\delta(p) \geq 0,$$

where the derivative exists. Setting $x = y$, the last inequality reduces to

$$(1+c)\mu(x) - x(1-x)\mu'_\delta(x) \geq 0,$$

or, equivalently, $\tau'_{\delta,c}(x) \leq 0$, where the derivative exists. Since δ is absolutely continuous, it holds that $\tau_{\delta,c}$ is absolutely continuous. The fact that $\tau'_{\delta,c}(x) \leq 0$ on the interval $]0,1[$, where the derivative exists, then implies that $\tau_{\delta,c}$ is decreasing. Condition (ii) can be proved similarly using the increasingness of $C_{\delta,c}$ in the first variable. □

Example 7. Consider the diagonal function of Example 2. Clearly, the conditions of Proposition 4 are fulfilled. The corresponding family of focal functions is a family of focal semi-copulas.

3.2. Focal (Quasi-)Copulas of Class 2

Consider a diagonal function δ and let δ_1 be the diagonal function defined by $\delta_1(x) = 2x - 1 + \delta(1-x)$. Let $A_{\delta_1,a}$, with $a \in [0,\infty]$, be the focal function defined in (4), and $C_{\delta,a}$ be the focal function defined in (15). One easily verifies that

$$C_{\delta,a}(x,y) = x + y - 1 + A_{\delta_1,a}(1-x, 1-y).$$

If $A_{\delta_1,a}$ is a (quasi-)copula, then $C_{\delta,a}$ is a (quasi-)copula as well [15]. This transformation allows us to characterize focal (quasi-)copulas in a straightforward manner.

Proposition 5. *Let $\delta \in \mathcal{D}$ and $c \in [0,\infty]$. Then, the focal function $C_{\delta,c}$ defined in (15) is a quasi-copula if and only if the following conditions are fulfilled:*

(i) *the conditions of Proposition 4;*
(ii) *the function μ_δ defined in (5) is increasing.*

Proposition 6. *Let $\delta \in \mathcal{D}^{\mathrm{d}}$ and $c \in [0,\infty]$. Then the focal function $C_{\delta,c}$ defined in (15) is a copula if and only if the following conditions are fulfilled:*

(i) *the conditions of Proposition 4;*
(ii) *the function $k_{\delta,c}\colon]0,1[\to \mathbb{R}$ defined by*

$$k_{\delta,c}(x) = \frac{1-x}{x}(x(1+c-x))^{1+c}\mu'_\delta(x)$$

is decreasing;
(iii) *the inequality*

$$c((1-x)\delta'(x) - 2(x-\delta(x))) + (1-x)((1-2x)\delta'(x) + 2\delta(x)) \geq 0$$

holds for any $x \in]0,1[$.

4. Class 3: Focal Points $(-b, 1+b)$ and $(1+b, -b)$

In this section, we introduce functions that are constructed by linear interpolation on segments connecting points on the diagonal of the unit square to the points $(-b, 1+b)$ and $(1+b, -b)$, with $b \in [0,\infty]$. This interpolation scheme is depicted in Fig. 3 for some segments.

Let us introduce the notations

$$u = \frac{x + b(x+y)}{1 + 2b + x - y}, \tag{17}$$

$$v = \frac{y + b(x+y)}{1 + 2b + x - y}, \tag{18}$$

with $b \in [0,\infty]$, and (see Fig. 4 for an illustration)

$$\begin{aligned}
T_1 &:= \{(x,y) \in [0,1]^2 | 0 \leq y \leq 1/2 \text{ and } y \leq x \leq 1-y\},\\
T_2 &:= \{(x,y) \in [0,1]^2 | 0 \leq x \leq 1/2 \text{ and } x \leq y \leq 1-x\},\\
T_3 &:= \{(x,y) \in [0,1]^2 | 1/2 \leq y \leq 1 \text{ and } 1-y \leq x \leq y\},\\
T_4 &:= \{(x,y) \in [0,1]^2 | 1/2 \leq x \leq 1 \text{ and } 1-x \leq y \leq x\},\\
O &:= (T_2 \cup T_3) \cap (T_1 \cup T_4).
\end{aligned}$$

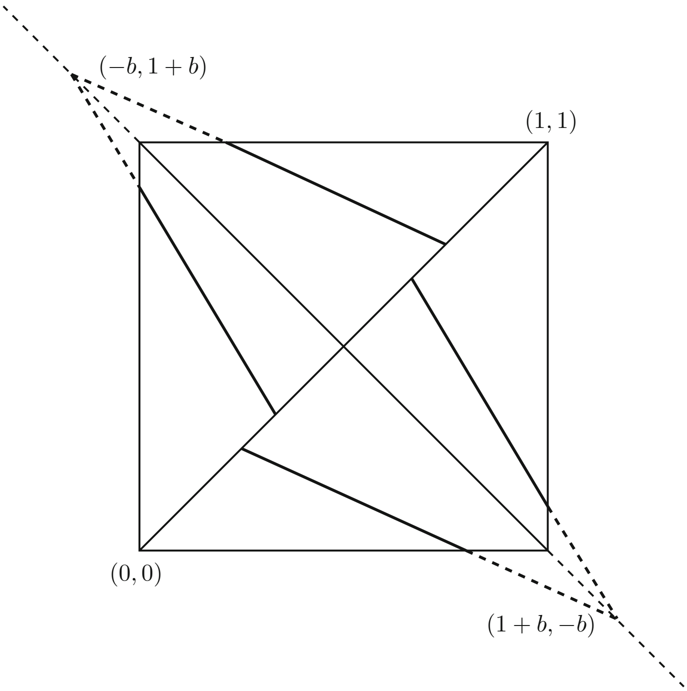

Figure 3. Some segments on which a focal function of class 3 is linear

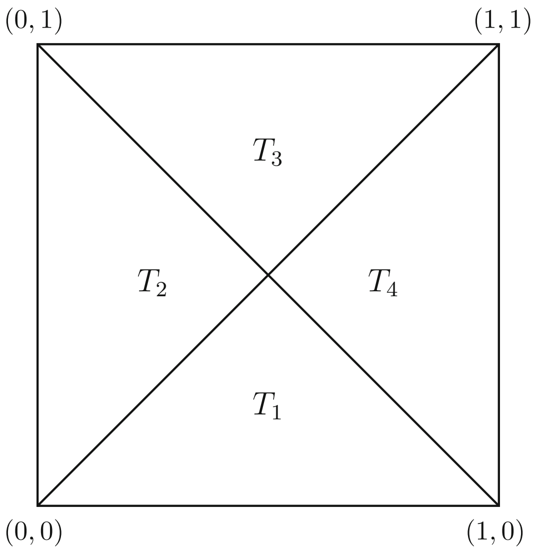

Figure 4. Illustration for the triangles T_1, T_2, T_3 and T_4

Let $\delta \in \mathcal{D}_{\mathrm{S}}$ and $b \in [0, \infty]$. The function $B_{\delta,b} \colon [0,1]^2 \to [0,1]$ given by

$$B_{\delta,b}(x,y) = \begin{cases} y\dfrac{\delta(v)}{v}, & \text{if } (x,y) \in T_1, \\ x\dfrac{\delta(u)}{u}, & \text{if } (x,y) \in T_2, \\ x - (1-y)\dfrac{u-\delta(u)}{1-u}, & \text{if } (x,y) \in T_3, \\ y - (1-x)\dfrac{v-\delta(v)}{1-v}, & \text{if } (x,y) \in T_4, \end{cases} \tag{19}$$

is well defined. This function is called a *focal function* since it is linear on segments connecting the points (x, x) and $(-b, 1 + b)$ as well as on segments connecting the points (x, x) and $(1 + b, -b)$. Since the points $(-b, 1 + b)$ and $(1+b, -b)$ are symmetric, any focal function $B_{\delta,b}$ is symmetric. For any focal function, condition (i) of the definition of a semi-copula always holds. Note that a focal function $B_{\delta,b}$ is uniquely determined by its diagonal section. Clearly, a focal function $B_{\delta,b}$ is continuous if and only if δ is continuous.

Remark 3. (i) If $b = 0$, then the definition of a biconic function is retrieved [20].
(ii) If $b = \infty$, then the definition of an ortholinear function is retrieved [21].

4.1. Focal Semi-Copulas of Class 3

In this subsection, we characterize the elements of $\mathcal{D}_{\mathrm{S}}^{\mathrm{ac}}$ for which the corresponding focal function $B_{\delta,b}$ is a semi-copula.

Proposition 7. *Let $\delta \in \mathcal{D}_{\mathrm{S}}^{\mathrm{ac}}$ and $b \in [0, \infty]$. Then, the focal function $B_{\delta,b}$ defined in* (19) *is a semi-copula if and only if the following conditions are fulfilled*:

(i) *the function λ_δ defined in* (5) *is increasing on the interval* $]0, 1/2]$;
(ii) *the function $\xi_{\delta,b} \colon]0, 1[\to \mathbb{R}$ defined by*

$$\xi_{\delta,b}(x) = \left(\frac{1-x}{b+x}\right)^{\frac{2b+1}{1+b}} \mu_\delta(x)$$

is decreasing on the interval $[1/2, 1[$.

Proof. Suppose that conditions (i) and (ii) are fulfilled. To prove that $B_{\delta,b}$ is a semi-copula, it suffices to prove its increasingness in each variable. Since $B_{\delta,b}$ is symmetric, it suffices to prove its increasingness in each variable on $T_2 \cup T_3$. We prove that $B_{\delta,b}$ is increasing in the second variable (the proof of the increasingness in the first variable is similar). Let $(x, y), (x, y') \in T_2 \cup T_3$ such that $y \le y'$. Let us introduce the notation $u_1 = \frac{x+b(x+y')}{1+2b+x-y'}$. If $(x, y), (x, y') \in T_2$, then the increasingness of $B_{\delta,b}$ is equivalent to

$$x\frac{\delta(u_1)}{u_1} - x\frac{\delta(u)}{u} = x(\lambda_\delta(u_1) - \lambda_\delta(u)) \ge 0. \tag{20}$$

Since $u \le u_1$ and λ_δ is increasing on the interval $]0, 1/2]$, inequality (20) immediately follows. If $(x, y), (x, y') \in T_3$, then the increasingness of $B_{\delta,b}$ is equivalent to

$$-(1-y')\frac{u_1-\delta(u_1)}{1-u_1}+(1-y)\frac{u-\delta(u)}{1-u}\geq 0,$$

or, equivalently,

$$(1-y)\left(\frac{b+u}{1-u}\right)^{\frac{1+2b}{1+b}}\xi_{\delta,b}(u)-(1-y')\left(\frac{b+u_1}{1-u_1}\right)^{\frac{1+2b}{1+b}}\xi_{\delta}(u_1)\geq 0.$$

Since $y\leq y'$, $u\leq u_1$ and $\xi_{\delta,b}$ is decreasing on the interval $[1/2,1[$, it holds that

$$\frac{(1-y)(b+u)}{(1-u)}\left(\frac{b+u}{1-u}\right)^{\frac{b}{1+b}}\xi_{\delta,b}(u)-\frac{(1-y')(b+u_1)}{(1-u_1)}\left(\frac{b+u_1}{1-u_1}\right)^{\frac{b}{1+b}}\xi_{\delta}(u_1)$$
$$\geq\frac{(1-y')(b+u_1)}{(1-u_1)}\left(\frac{b+u_1}{1-u_1}\right)^{\frac{b}{1+b}}(\xi_{\delta,b}(u)-\xi_{\delta,b}(u_1))\geq 0.$$

If $(x,y)\in T_2$ and $(x,y')\in T_3$, then the preceding cases imply that $B_{\delta,b}(x,y')-B_{\delta,b}(x,y)=$

$$(B_{\delta,b}(x,y')-B_{\delta,b}(x,1-x))+(B_{\delta,b}(x,1-x)-B_{\delta,b}(x,y))\geq 0.$$

Conversely, suppose that $B_{\delta,b}$ is a semi-copula. Let $y,y'\in]0,1/2]$ such that $y\leq y'$ and $x\in[0,1]$ such that $x\leq y$ and $x+y'\leq 1$. Clearly, the points $(x,\frac{(x+2b+1)y-(1+b)x}{y+b})$ and $(x,\frac{(x+2b+1)y'-(1+b)x}{y'+b})$ are located in T_2. The increasingness of $B_{\delta,b}$ in the second variable implies

$$B_{\delta,b}\left(x,\frac{(x+2b+1)y'-(1+b)x}{y'+b}\right)-B_{\delta,b}\left(x,\frac{(x+2b+1)y-(1+b)x}{y+b}\right)\geq 0, \tag{21}$$

or, equivalently,

$$x(\lambda_\delta(y')-\lambda_\delta(y))\geq 0.$$

Hence, the increasingness of λ_δ on the interval $]0,1/2]$ follows. Let $y,y'\in[1/2,1[$ such that $y<y'$ and $x\in[0,1]$ such that $x\leq y$ and $x+y\geq 1$. The increasingness of $B_{\delta,b}$ in the second variable implies

$$(1-y)\frac{u-\delta(u)}{1-u}-(1-y')\frac{u_1-\delta(u_1)}{1-u_1}\geq 0. \tag{22}$$

Dividing by $y'-y$ and taking the limit $y'\to y$, inequality (22) becomes

$$\frac{u-\delta(u)}{1-u}-\frac{(1-y)(y+b)}{2b+1}\left(\frac{u-\delta(u)}{1-u}\right)'\geq 0,$$

where the derivative exists. Setting $y=x$, the last inequality reduces to

$$(2b+1)x-(1+b-x)\delta(x)+(1-x)(x+b)\delta'(x)\geq 0,$$

or, equivalently, $\xi'_{\delta,b}(x)\leq 0$, where the derivative exists. Since δ is absolutely continuous, it holds that $\xi_{\delta,b}$ is absolutely continuous. The fact that $\xi'_{\delta,b}(x)\leq 0$ on the interval $[1/2,1[$, where the derivative exists, then implies that $\xi_{\delta,b}$ is decreasing on the interval $[1/2,1[$. □

Example 8. Consider the diagonal functions $\delta_{\mathbf{M}}$ and $\delta_{\mathbf{W}}$. Clearly, $\delta_{\mathbf{M}}$ and $\delta_{\mathbf{W}}$ belong to $\mathcal{D}_{\mathrm{S}}^{\mathrm{ac}}$. One easily verifies that the functions $\lambda_{\delta_{\mathbf{M}}}$ and $\lambda_{\delta_{\mathbf{W}}}$ are increasing on the interval $]0,1/2]$, and that the functions $\xi_{\delta_{\mathbf{M}},b}$ and $\xi_{\delta_{\mathbf{W}},b}$ are decreasing on the interval $[1/2,1[$. The corresponding focal semi-copulas are $T_{\mathbf{M}}$ and $T_{\mathbf{L}}$, respectively.

Example 9. Consider the diagonal function of Example 2. Clearly, $\delta \in \mathcal{D}_{\mathrm{S}}^{\mathrm{ac}}$. As mentioned before in Example 2, λ_δ is increasing. One easily verifies that $\xi_{\delta,b}$ is decreasing on the interval $[1/2,1[$. The corresponding family of focal semi-copulas is given by

$$B_{\delta,b}(x,y) = \begin{cases} yv^\theta, & \text{if } (x,y) \in T_1, \\ xu^\theta, & \text{if } (x,y) \in T_1, \\ x-(1-y)\dfrac{u(1-u^\theta)}{1-u}, & \text{if } (x,y) \in T_3, \\ y-(1-x)\dfrac{v(1-v^\theta)}{1-v}, & \text{if } (x,y) \in T_4, \end{cases}$$

where u and v are given in (17) and (18).

4.2. Focal Quasi-Copulas of Class 3

In this subsection, we characterize the elements of $\mathcal{D}$ for which the corresponding focal function $B_{\delta,b}$ is a quasi-copula.

Proposition 8. *Let $\delta \in \mathcal{D}$ and $b \in [0,\infty]$. Then, the focal function $B_{\delta,b}$ defined in* (19) *is a quasi-copula if and only if the following conditions are fulfilled*:

(i) *the conditions of Proposition* 7;

(ii) *the function $\nu_{\delta,b}\colon]0,1[\to \mathbb{R}$ defined by*

$$\nu_{\delta,b}(x) = \left(\frac{x}{1+b-x}\right)^{\frac{2b+1}{1+b}} (1-\lambda_\delta(x))$$

is increasing on the interval $]0,1/2]$;

(iii) *the function μ_δ defined in* (5) *is increasing on the interval* $[1/2,1[$.

Proof. Suppose that conditions (i)–(iii) are fulfilled. Due to Proposition 7, the function $B_{\delta,b}$ is increasing. Therefore, to prove that $B_{\delta,b}$ is a quasi-copula, we need to show that it is 1-Lipschitz continuous. Recall that the 1-Lipschitz continuity is equivalent to the 1-Lipschitz continuity in each variable. Since $B_{\delta,b}$ is symmetric, it is sufficient to show that $B_{\delta,b}$ is 1-Lipschitz continuous in each variable on $T_2 \cup T_3$. We prove that $B_{\delta,b}$ is 1-Lipschitz continuous in the first variable on $T_2 \cup T_3$ (the proof of the 1-Lipschitz continuity in the second variable is similar). Let $(x,y),(x',y) \in T_2 \cup T_3$ such that $x \le x'$. Let us introduce the notation $u_2 = \frac{x'+b(x'+y)}{1+2b+x'-y}$. If $(x,y),(x',y) \in T_2$, then the 1-Lipschitz continuity of $B_{\delta,b}$ is equivalent to

$$x'\frac{\delta(u_2)}{u_2} - x\frac{\delta(u)}{u} \le x'-x,$$

or, equivalently,

$$x'\left(\frac{x'}{1+b-x}\right)^{\frac{-2b-1}{b+1}} \nu_{\delta,b}(u_2) - x\left(\frac{x}{1+b-x}\right)^{\frac{-2b-1}{b+1}} \nu_{\delta,b}(u) \ge 0.$$

Since $x \le x'$, $u \le u_2$ and $\nu_{\delta,a}$ is increasing on the interval $]0, 1/2]$, it holds that

$$x'\left(\frac{x'}{1+b-x}\right)^{\frac{-2b-1}{b+1}}\nu_{\delta,b}(u_2) - x\left(\frac{x}{1+b-x}\right)^{\frac{-2b-1}{b+1}}\nu_{\delta,b}(u)$$
$$\ge x\left(\frac{x}{1+b-x}\right)^{\frac{-2b-1}{b+1}}(\nu_{\delta,b}(u_2) - \nu_{\delta,b}(u)) \ge 0.$$

If $(x, y), (x', y) \in T_3$, then the 1-Lipschitz continuity of $B_{\delta,b}$ is equivalent to

$$(1-y)(\mu_\delta(u_2) - \mu_\delta(u)) \ge 0. \tag{23}$$

Since $u \le u_2$ and μ_δ is increasing on the interval $[1/2, 1[$, inequality (23) immediately follows. If $(x, y) \in T_2$ and $(x', y) \in T_3$, then the preceding cases imply that

$$B_{\delta,b}(x', y) - B_{\delta,b}(x, y)$$
$$= (B_{\delta,b}(x', y) - B_{\delta,b}(1-y, y)) + (B_{\delta,b}(1-y, y) - B_{\delta,b}(x, y)) \le x' - x.$$

Conversely, suppose that $B_{\delta,b}$ is a quasi-copula. Proposition 7 implies the increasingness of λ_δ on the interval $]0, 1/2]$ and the decreasingness of $\xi_{\delta,b}$ on the interval $[1/2, 1[$. Let $x, x' \in [1/2, 1[$ such that $x \le x'$ and $y \in [0, 1]$ such that $x' \le y$ and $x + y \ge 1$. Clearly, the points $(\frac{(2b+1)x-(x+b)y}{1+b-x}, y)$ and $(\frac{(2b+1)x'-(x'+b)y}{1+b-x'}, y)$ are located in T_3. The 1-Lipschitz continuity of $B_{\delta,b}$ in the first variable implies that

$$B_{\delta,b}\left(\frac{(2b+1)x'-(x'+b)y}{1+b-x'}, y\right) - B_{\delta,b}\left(\frac{(2b+1)x-(x+b)y}{1+b-x}, y\right)$$
$$\le \frac{(2b+1)x'-(x'+b)y}{1+b-x'} - \frac{(2b+1)x-(x+b)y}{1+b-x},$$

or, equivalently,

$$(1-y)(\mu_\delta(x') - \mu_\delta(x)) \ge 0.$$

Hence, the increasingness of μ_δ on the interval $[1/2, 1[$ follows.

Let $x, x' \in]0, 1/2[$ such that $x < x'$ and $y \in [0, 1]$ such that $x' \le y$ and $x' + y \le 1$. The 1-Lipschitz continuity of $B_{\delta,b}$ in the first variable implies that

$$x'\frac{\delta(u_2)}{u_2} - x\frac{\delta(u)}{u} \le x' - x. \tag{24}$$

Dividing by $x' - x$ and taking the limit $x' \to x$, inequality (24) becomes

$$\frac{\delta(u)}{u} + x\frac{(2b+1)(1+b-y)}{(x-y+2b+1)^2}\left(\frac{\delta(u)}{u}\right)' \le 1,$$

where the derivative exists. Setting $y = x$, the last inequality reduces to

$$(2b+1)x - (x+b)\delta(x) - x(1+b-x)\delta'(x) \ge 0,$$

or, equivalently, $\nu'_{\delta,b}(x) \ge 0$, where the derivative exists. Since δ is absolutely continuous, it holds that $\nu_{\delta,b}$ is absolutely continuous. The fact that $\nu'_{\delta,b}(x) \ge 0$ on the interval $]0, 1/2]$, where the derivative exists, then implies that $\nu_{\delta,b}$ is increasing on the interval $]0, 1/2]$. □

Example 10. Consider the diagonal function of Example 2. Clearly, the conditions of Proposition 8 are fulfilled. The corresponding family of focal functions is a family of focal quasi-copulas.

4.3. Focal Copulas of Class 3

In this subsection, we characterize the elements of $\delta \in \mathcal{D}^{\mathrm{d}}$ for which the corresponding focal function $B_{\delta,b}$ is a copula.

Theorem 2. *Let $\delta \in \mathcal{D}^{\mathrm{d}}$ and $b \in [0,\infty]$. Then, the focal function $B_{\delta,b}$ defined in (19) is a copula if and only if the following conditions are fulfilled:*

(i) *the conditions of Proposition 7;*
(ii) *the function $f_{\delta,b}\colon]0,1[\to \mathbb{R}$ defined by*

$$f_{\delta,b}(x) = x(b+x)\left(\frac{x}{1+b-x}\right)^{\frac{b}{1+b}} \lambda'_\delta(x)$$

is increasing on the interval $]0,1/2]$;
(iii) *the function $g_{\delta,b}\colon]0,1[\to \mathbb{R}$ defined by*

$$g_{\delta,b}(x) = (1-x)(1+b-x)\left(\frac{1-x}{b+x}\right)^{\frac{b}{1+b}} \mu'_\delta(x)$$

is decreasing on the interval $[1/2,1[$.

Proof. Suppose that $B_{\delta,b}$ is a copula. The conditions of Proposition 7 immediately follow. For any $(x,y) \in T_2\backslash(D \cup O)$, the 2-increasingness of $B_{\delta,b}$ implies that

$$\frac{\partial^2 B_{\delta,b}(x,y)}{\partial x \partial y} \geq 0.$$

Substituting the expression of $B_{\delta,b}$, the latter inequality is equivalent to

$$\left(\frac{1+b-y}{(1+2b+x-y)^2} + \frac{x(1-x-y)}{(1+2b+x-y)^3}\right)\lambda'_\delta(u) + \frac{(1+2b)x(b+x))(1+b-y)}{(1+2b+x-y)^4}\lambda''_\delta(u) \geq 0, \tag{25}$$

where the derivatives exist. Setting $y = x$, the latter inequality reduces to

$$((1+b-x)(1+2b) + x(1-2x))\lambda'_\delta(x) + x(b+x)(1+b-x)\lambda''_\delta(x) \geq 0, \tag{26}$$

where the derivatives exist. Some elementary manipulations yield $f'_{\delta,b}(x) \geq 0$, where the derivative exists. Hence, condition (ii) follows.

Similarly, for any $(x,y) \in T_3\backslash(D \cup O)$, the 2-increasingness of $B_{\delta,b}$ implies that

$$\left(\frac{1+b-y}{(1+2b+x-y)^2} - \frac{(1-y)(1-x-y)}{(1+2b+x-y)^3}\right)\mu'_\delta(u) - \frac{(1+2b)(1-y)(b+x))(1+b-y)}{(1+2b+x-y)^4}\mu''_\delta(u) \geq 0$$

where the derivatives exist. Setting $y = x$, the latter inequality reduces to

$$((1+b-x)(1+2b)-(1-x)(1-2x))\mu'_\delta(x)$$
$$-(1-x)(b+x)(1+b-x)\mu''_\delta(x) \geq 0,$$

where the derivatives exist. Some elementary manipulations yield $g'_{\delta,b}(x) \leq 0$, where the derivative exists. Hence, condition (iii) follows.

Conversely, suppose that conditions (i)–(iii) are fulfilled. Due to Proposition 7, $B_{\delta,b}$ is a semi-copula. To prove that $B_{\delta,b}$ is a copula, we need to show its 2-increasingness. We distinguish the following cases:

(a) If $(x,y) \in T_2\backslash(D \cup O)$, then the 2-increasingness of $B_{\delta,b}$ is equivalent to inequality (25). Since $f_{\delta,b}$ is increasing, inequality (26) holds and the left-hand side of inequality (25) is greater than or equal to

$$\left(\frac{1+b-y}{(1+2b+x-y)^2} + \frac{x(1-x-y)}{(1+2b+x-y)^3}\right)\lambda'_\delta(u)$$
$$-\frac{(1+2b)x(b+x))(1+b-y)}{(1+2b+x-y)^4}\frac{(1+b-u)(1+2b)+u(1-2u)}{u(1+b-u)(a+u)}\lambda'_\delta(u).$$

Some elementary manipulations show that the positivity of the latter expression side is equivalent to

$$(1+2b)(y-x)\lambda'_\delta(u) \geq 0.$$

Since λ_δ is increasing, the latter inequality clearly holds.

(b) The proof of the 2-increasingness of $B_{\delta,b}$ for any $(x,y) \in T_3\backslash(D \cup O)$ is similar to the previous one.

(c) Let $S = [x,x'] \times [1-x', 1-x]$, with $x' \leq 1/2$, be a square centered around the opposite diagonal of the unit square (the proof when $x \geq 1/2$ is identical due to the symmetry of $B_{\delta,b}$). Its volume is given by

$$V_{B_{\delta,a}}(S) = -\omega(x) - \omega(x') + B_{\delta,b}(x, 1-x') + B_{\delta,b}(x', 1-x).$$

Due to (a), it holds that

$$\frac{\partial^2 B_{\delta,b}(s,t)}{\partial s \partial t} \geq 0,$$

for any $(s,t) \in T_2\backslash(D \cup O)$, which implies that

$$V_1 = \int_{1-x'}^{1-x} \mathrm{d}t \int_x^{1-t} \frac{\partial^2 B_{\delta,b}(s,t)}{\partial s \partial t}\,\mathrm{d}s \geq 0.$$

Due to (b), it holds that

$$\frac{\partial^2 B_{\delta,b}(s,t)}{\partial s \partial t} \geq 0,$$

for any $(s,t) \in T_3\backslash(D \cup O)$, which implies that

$$V_2 = \int_{1-x'}^{1-x} \mathrm{d}t \int_{1-t}^{x'} \frac{\partial^2 B_{\delta,b}(s,t)}{\partial s \partial t}\,\mathrm{d}s \geq 0.$$

Since δ is a diagonal function, it holds that

$$(1-2\delta(1/2))\frac{b(1-2s)}{s+b} \geq 0,$$

for any $s \in]0, 1/2]$, which implies that

$$V_3 = (1 - 2\delta(1/2)) \int_{1-x'}^{1-x} \frac{b(1-2s)}{s+b} \, \mathrm{d}s \geq 0.$$

It now follows that

$$\begin{aligned} V_1 + V_2 + V_3 &= -\omega(x) - \omega(x') + B_{\delta,b}(x, 1-x') + B_{\delta,b}(x', 1-x) \\ &= V_{B_{\delta,a}}(S) \geq 0. \end{aligned}$$

(d) Finally, let $S = [x, x'] \times [x, x']$, with $x' \leq 1/2$, be a square centered around the main diagonal (the proof when $x \geq 1/2$ is identical). Its volume is given by

$$V_{B_{\delta,b}}(S) = \delta(x) + \delta(x') - 2B_{\delta,b}(x, x').$$

Due to (a), it holds that

$$\frac{\partial^2 B_{\delta,b}(s,t)}{\partial s \partial t} \geq 0,$$

for any $(s, t) \in T_2 \backslash (D \cup O)$, which implies that

$$V_1 = \int_x^{x'} \mathrm{d}s \int_s^{x'} \frac{\partial^2 B_{\delta,b}(s,t)}{\partial s \partial t} \, \mathrm{d}t \geq 0.$$

Since δ is a diagonal function, it holds that

$$\frac{1}{1+2b} \frac{s(1-2s)\delta'(s) + 2(b+s)\delta(s)}{s} \geq 0,$$

for any $s \in]0, 1/2]$, which implies that

$$V_2 = \frac{1}{1+2b} \int_x^{x'} \frac{s(1-2s)\delta'(s) + 2(b+s)\delta(s)}{s} \, \mathrm{d}s \geq 0.$$

It now follows that

$$2V_1 + V_2 = \delta(x) + \delta(x') - 2B_{\delta,b}(x, x') = V_{B_{\delta,b}}(S) \geq 0.$$

□

Example 11. Consider the diagonal functions $\delta_{\mathbf{M}}$ and $\delta_{\mathbf{W}}$. Clearly, the conditions of Theorem 2 are fulfilled. The corresponding focal copulas are $T_{\mathbf{M}}$ and $T_{\mathbf{L}}$, respectively.

Example 12. Consider the diagonal function $\delta_\theta(x) = \frac{x^2}{1-\lambda(1-x)^2}$ with $\theta \in [-1, 1]$. Clearly, the conditions of Theorem 2 are fulfilled. The corresponding family of focal functions is a family of focal copulas.

In the following proposition, we provide sufficient conditions for the conditions of Theorem 2.

Proposition 9. *Let $\delta \in \mathcal{D}^{\mathrm{d}}$ and $b \in [0, \infty]$. If the following conditions*

(i) *the functions λ_δ and μ_δ defined in (5) are increasing;*
(ii) *the function λ_δ is convex on the interval $]0, 1/2]$;*
(iii) *the function μ_δ is concave on the interval $[1/2, 1[$,*

are fulfilled, then the focal function $B_{\delta,b}$ is a copula.

Proof. The proof is similar to the proof of Proposition 3. □

Example 13. Consider the diagonal function $\delta(x) = \frac{x^2}{1+(1-x)^2}$. Clearly, the conditions of Proposition 9 are fulfilled. The corresponding focal function is a focal copula.

Note that the conditions of Proposition 9 are not necessary in general. This fact is illustrated in the following example.

Example 14. Consider the diagonal function $\delta(x) = \frac{x}{2-x}$. Note that the function μ_δ is not concave on the interval $[1/2, 1[$, while the conditions of Theorem 2 are fulfilled.

5. Conclusion

We have introduced the notion of a focal function with a given diagonal section. We have constructed three classes of focal functions with a given diagonal section. For each class, we have also characterized the corresponding classes of focal semi-copulas, quasi-copulas and copulas. Biconic functions with a given diagonal section, ortholinear functions, lower semilinear functions and upper semilinear functions turn out to be special cases of these focal functions.

Several commonly used operations allow us further to identify other classes of semilinear (quasi-)copulas. The diagonal splice operation amounts to gluing parts of two (quasi-)copulas that share the same diagonal section [26]. For quasi-copulas (resp. symmetric copulas), this method always leads to a quasi-copula (resp. copula). The resulting (quasi-)copulas are non-symmetric in general. Since the focal (quasi-)copulas of classes 1–3 are symmetric, applying the diagonal splice operation on these focal (quasi-)copulas will always lead to a (quasi-)copula. For instance, a horizontal (resp. vertical) semilinear (quasi-)copula with diagonal section δ is obtained by applying the diagonal splice operation on the (quasi-)copula $A_{\delta,\infty}$ (resp. $C_{\delta,\infty}$) from class 1 (resp. class 2) and the (quasi-)copula $C_{\delta,\infty}$ (resp. $A_{\delta,\infty}$) from class 2 (resp. class 1).

It is well known that appropriate transformations of (quasi-)copulas result in new (quasi-)copulas. For instance, consider a (quasi-)copula C with diagonal section δ. The function $F\colon [0,1] \to [0,1]$ defined by $F(x,y) = x - C(x, 1-y)$ is a (quasi-)copula with opposite diagonal section $\omega(x) = F(x, 1-x) = x - \delta(x)$ [24]. Applying this transformation on focal (quasi-)copulas of classes 1–3 enables us to obtain similar classes of focal (quasi-)copulas with a given opposite diagonal section.

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