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On the Solvability of a Nonlinear Integro-Differential Equation on the Half-Axis

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Abstract. In this work, we combine the iterative techniques with fixed point theory to investigate the existence of absolutely continuous solutions to a class of nonlinear integro-differential equations. Existence results are obtained under fairly simple conditions of Carathéodory type.

Mathematics Subject Classification. 45G10, 47H30.

Keywords. Fixed point theorem, measure of weak noncompactness, nonlinear integro-differential equations, integrable solution, continuous solution.

1. Introduction

A variety of problems in physics, engineering and biology have their mathematical setting as integro-differential equations [2,8,15,16,18,20]. In recent years, there has been a growing interest in integro-differential equations, which have been considered in the literature for theoretical as well as practical purposes.

Recently, in [4], Berenguer et al. used the Banach contraction principle to prove the existence and uniqueness of continuous solutions to nonlinear Volterra integro-differential equations of the form

$$\begin{cases} x'(t) = u(t, x(t)) + \displaystyle\int_0^t K(t, s, x(s))\mathrm{d}s, \quad t \in [0, +\infty) \\ x(0) = x_0; \end{cases} \tag{1.1}$$

where u and K are continuous functions satisfying Lipschitz conditions with respect to the last variable.

The purpose of the present work is to study the existence of an absolutely continuous solution to (1.1) under fairly simple conditions of Carathéodory type.

Our approach is based on the conjunction of the technique of the measure of weak noncompactness with some iterative techniques and an improved version of the Schauder fixed point theorem. It is of interest that our solution is continuous and almost everywhere derivable. However, uniqueness does not follow from our assumptions. The results obtained in this paper generalize their corresponding results in [4,5].

The paper is arranged as follows. In Sect. 2, we recall some definitions and results used in this paper. In Sect. 3, we discuss the existence of absolutely continuous solutions to the integro-differential equation (1.1).

2. Preliminaries

In this section, we recall from the literature some notations, definitions, and auxiliary results which will be used throughout this paper. Let I be an interval of $\mathbb{R}$. We denote by $L^1(I)$ the set of all Lebesgue integrable functions on I, endowed with the standard norm

$$\|x\|_{L^1(I)} = \int_I |x(t)|\mathrm{d}t.$$

$C(I)$ refers to the set of all continuous functions on I. If I is bounded, then $C(I)$ is endowed with the norm

$$\|x\|_{L^\infty(I)} = \sup\{|x(t)|, t \in I\}.$$

Also, denote by $L^1_{\mathrm{Loc}}(I)$ the set of all Lebesgue integrable functions on any compact subset of I. Consider a function $f\colon I \times \mathbb{R} \longrightarrow \mathbb{R}$. We say that f satisfies Carathéodory conditions if, it is measurable in t for any $x \in \mathbb{R}$ and continuous in x for almost all $t \in I$.

The following definitions are frequently used in the subsequent part of this paper.

Definition 2.1 [14,19]. A function $f\colon [a,b] \longrightarrow \mathbb{R}$ is absolutely continuous, if for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$\sum_{i=1}^{n} |f(x_i') - f(x_i)| < \epsilon,$$

for any finite collection $\{]x_i, x_i'[; i = 1, \ldots, n\}$ of pairwise disjoint intervals in $[a,b]$ with $\sum_{i=1}^{n} |x_i' - x_i| < \delta$.

Definition 2.2. A function $f\colon \mathbb{R}^+ \longrightarrow \mathbb{R}$ is absolutely continuous, if it is absolutely continuous on any interval $[a,b]$ in $\mathbb{R}^+$.

Denote by $AC(\mathbb{R}^+)$ the set of all absolutely continuous functions on $\mathbb{R}^+$.

Remark 2.3 [14]. It is clear that any absolutely continuous function is continuous.

Absolutely continuous functions enjoy the following interesting properties.

Theorem 2.4 [14]. *If g is absolutely continuous on $[a,b]$, then g has a derivative almost everywhere on $[a,b]$. Moreover, $g'(t)$ is integrable on $[a,b]$ and*

$$g(t) = g(a) + \int_a^t g'(s)\mathrm{d}s.$$

Theorem 2.5 [19]. *Let g be an integrable function on $[a,b]$, then the function*

$$G(t) = G(a) + \int_a^t g(s)\mathrm{d}s$$

is absolutely continuous. Moreover, G is derivable almost everywhere on $[a,b]$ and

$$G'(t) = g(t) \quad \text{for a.e. } t \in [a,b].$$

From now on, X will always denote a Banach space. Denote by $\mathcal{B}(X)$ the collection of all nonempty bounded subsets of X and by $\mathcal{W}(X)$ the subset of $\mathcal{B}(X)$ consisting of all relatively weakly compact subsets of X. Let B_r denote the closed ball centered at origin with radius r.

Throughout this paper, we will adopt the following axiomatic definition of a measure of weak noncompactness.

Definition 2.6 [3,10]. A function $\mu\colon \mathcal{B}(X) \longrightarrow \mathbb{R}_+$ is said to be a measure of weak noncompactness if it satisfies the following conditions

1. The family $ker(\mu)=\{M \in \mathcal{B}(X)\colon \mu(M) = 0\}$ is nonempty and $ker(\mu)\subset \mathcal{W}(X)$.
2. $M_1 \subset M_2 \Rightarrow \mu(M_1) \leq \mu(M_2)$.
3. $\mu(co(M)) = \mu(M)$, where $co(M)$ is the convex hull of M.
4. $\mu(\lambda M_1 + (1-\lambda)M_2) \leq \lambda\mu(M_1) + (1-\lambda)\mu(M_2)$ for $\lambda \in [0,1]$.
5. If $(M_n)_{n\geq 1}$ is a sequence of nonempty, weakly closed subsets of X with M_1 bounded and $M_1 \supseteq M_2 \supseteq \cdots \supseteq M_n \supseteq \cdots$ such that $\lim_{n\to\infty} \mu(M_n) = 0$, then $M_\infty := \bigcap_{n=1}^{\infty} M_n$ is nonempty.

We refer the reader to [6,7,11] for details concerning measures of weak noncompactness. A handy and useful example of a measure of weak noncompactness in the space $L^1(I)$ (I is a bounded interval) was given by Appel and De Pascale [1] as follows: for a nonempty and bounded subset M of the space $L^1(I)$

$$\mu(M) = \lim_{\epsilon\to 0}\left\{\sup_{x\in M}\left\{\sup\left[\int_D |x(t)|\mathrm{d}t\colon D \subset I, \mathrm{meas}(D) \leq \epsilon\right]\right\}\right\}.$$

The following concept is crucial for our purpose.

Definition 2.7 [13]. Let M be a subset of a Banach space X. A continuous map $A\colon M \longrightarrow X$ is said to be (ws)-compact if for any weakly convergent sequence $(x_n)_{n\in\mathbb{N}}$ in M the sequence $(Ax_n)_{n\in\mathbb{N}}$ has a strongly convergent subsequence in X.

The considerations of this paper are based on the following fixed point result [17].

Theorem 2.8. *Let $\mathcal{M}$ be a nonempty bounded closed convex subset of a Banach space X. Suppose that $A\colon \mathcal{M} \longrightarrow \mathcal{M}$ satisfies*:

(i) *A is (ws)-compact.*
(ii) *$A(\mathcal{M})$ is relatively weakly compact.*

Then there is a $x \in \mathcal{M}$ such that $Ax = x$.

We will also use the following criterion for relatively weakly compact sets.

Theorem 2.9 [9,12]. *A bounded set S is relatively weakly compact in $L^1(I)$ (I is a bounded interval) if and only if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if* $\text{meas}(D) \le \delta$ *then $\int_D |x(t)| \le \epsilon$ for all $x \in S$.*

3. Main Result

Equation (1.1) will be studied under the following assumptions:

(i) $u\colon \mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfies Carathéodory conditions and there exist a constant b_1 and a function $a_1 \in L^1_{\text{Loc}}(\mathbb{R}_+)$ such that $|u(t,x)| \le a_1(t) + b_1|x|$ for all $t \in \mathbb{R}_+$ and for all $x \in \mathbb{R}$.
(ii) The function $h\colon \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ satisfies Carathéodory conditions and the linear Volterra operator

$$Hx(t) = \int_0^t h(t,s)x(s)\text{d}s, \quad t \in \mathbb{R}_+$$

transforms the space $L^1_{\text{Loc}}(\mathbb{R}_+)$ into itself.
(iii) $K\colon \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfies Carathéodory conditions and there exist a constant b_2 and a function $a_2 \in L^1_{\text{Loc}}(\mathbb{R}_+)$ such that $|K(t,s,x)| \le h(t,s)(a_2(s) + b_2|x|)$ for all $t \in \mathbb{R}_+$ and for all $x \in \mathbb{R}$.

Remark 3.1. Under the assumptions above and in view of Theorems 2.4 and 2.5, we have $x \in AC(\mathbb{R}^+)$ is a solution of Eq. (1.1) if and only if x is a solution of the following integral equation

$$x(t) = x_0 + \int_0^t u(s,x(s))\text{d}s + \int_0^t \int_0^s K(s,r,x(r))\text{d}r\text{d}s, \quad t \in [0,+\infty). \tag{3.1}$$

In order to prove an existence theorem for (1.1), we shall first prove the following theorem:

Theorem 3.2. *Let $-\infty < \alpha < \beta < +\infty$ and let $g\colon [\alpha,\beta] \longrightarrow \mathbb{R}$ be an absolutely continuous function. Assume* (i)–(iii) *hold, and in addition suppose*

$$(\beta - \alpha)(b_1 + \|H\|_{L^1(I)} b_2) < 1, \tag{3.2}$$

is satisfied. Then, the nonlinear integral equation

$$x(t) = g(t) + \int_{\alpha}^{t} u(s, x(s))\mathrm{d}s + \int_{\alpha}^{t}\int_{\alpha}^{s} K(s, r, x(r))\mathrm{d}r\mathrm{d}s, \quad t \in I, \tag{3.3}$$

has an absolutely continuous solution on $[\alpha, \beta]$.

Proof. Define the operator A on $L^1([\alpha, \beta])$ by

$$Ax(t) = g(t) + \int_{\alpha}^{t} u(s, x(s))\mathrm{d}s + \int_{\alpha}^{t}\int_{\alpha}^{s} K(s, r, x(r))\mathrm{d}r\mathrm{d}s.$$

We show that A satisfies all the conditions of Theorem 2.8. This will be achieved in three steps.

Step 1. We first show that there exists r_0 such that the operator A transforms the ball B_{r_0} into itself, where B_r is the ball of $L^1([\alpha, \beta])$ centered at origin with radius r. To see this, let $x \in B_r$. Then,

$$\begin{aligned}
\|Ax\| &= \int_{\alpha}^{\beta} \left| g(t) + \int_{\alpha}^{t} u(s, x(s))\mathrm{d}s + \int_{\alpha}^{t}\int_{\alpha}^{s} K(s, r, x(r))\mathrm{d}r\mathrm{d}s \right| \mathrm{d}t \\
&\le \int_{\alpha}^{\beta} |g(t)|\mathrm{d}t + \int_{\alpha}^{\beta}\int_{\alpha}^{t} \left(a_1(s) + b_1|x(s)|\right)\mathrm{d}s\mathrm{d}t \\
&\quad + \int_{\alpha}^{\beta}\int_{\alpha}^{\beta}\int_{\alpha}^{s} h(s, r)(a_2(r) + b_2|x(r)|)\mathrm{d}r\mathrm{d}s\mathrm{d}t \\
&\le \|g\|_{L^1(I)} + \int_{\alpha}^{\beta}\int_{\alpha}^{\beta} \left(a_1(s) + b_1|x(s)|\right)\mathrm{d}s\mathrm{d}t \\
&\quad + \int_{\alpha}^{\beta}\int_{\alpha}^{\beta} \left(Ha_2(s) + b_2H|x|(s)\right)\mathrm{d}s\mathrm{d}t \\
&\le \|g\|_{L^1(I)} + (\beta - \alpha)\left(\|a_1\| + b_1 r + \|H\|_{L^1(I)}(\|a_2\| + b_2 r)\right) \\
&= \|g\|_{L^1(I)} + (\beta - \alpha)\left(\|a_1\| + \|H\|_{L^1(I)}\|a_2\|\right) + (\beta - \alpha)\left(b_1 + \|H\|_{L^1(I)} b_2\right) r.
\end{aligned}$$

Due to assumption (3.2), we have that the operator A transforms the ball B_{r_0} into itself provided that $r_0 > \frac{\|g\|_{L^1(I)} + (\beta-\alpha)\left(\|a_1\| + \|H\|_{L^1(I)}\|a_2\|\right)}{1-(\beta-\alpha)(b_1+\|H\|_{L^1(I)}b_2)}$.

Step 2. We then illustrate that A is ws-compact. To see this, let (y_n) be a weakly convergence sequence in B_{r_0}. Notice first that the set $S = \{y_n, n \in \mathbb{N}\}$ is weakly relatively compact. Now, take $t_1, t_2 \in [\alpha, \beta]$ such that $t_1 \le t_2$. Then, for an arbitrary $n \in \mathbb{N}$, we have

$$\begin{aligned}|Ay_n(t_2) - Ay_n(t_1)| &\le |g(t_2) - g(t_1)| + \left| \int_\alpha^{t_2} u(s, y_n(s))\mathrm{d}s - \int_\alpha^{t_1} u(s, y_n(s))\mathrm{d}s \right| \\ &\quad + \left| \int_\alpha^{t_2}\int_\alpha^{s} K(s, r, y_n(r))\mathrm{d}r\mathrm{d}s - \int_\alpha^{t_1}\int_\alpha^{s} K(s, r, y_n(r))\mathrm{d}r\mathrm{d}s \right| \\ &\le |g(t_2) - g(t_1)| + \int_{t_1}^{t_2} |u(s, y_n(s))|\mathrm{d}s \\ &\quad + \int_{t_1}^{t_2}\int_\alpha^{s} |K(s, r, y_n(r))|\mathrm{d}r\mathrm{d}s \\ &\le |g(t_2) - g(t_1)| + \int_{t_1}^{t_2} (a_1(s) + b_1|y_n(s)|)\,\mathrm{d}s \\ &\quad + \|H\|_{L^1(I)} \int_{t_1}^{t_2} (a_2(s) + b_2 y_n(s))\,\mathrm{d}s.\end{aligned}$$

Taking into account that $\{y_n, n \in \mathbb{N}\}$ is relatively weakly compact and invoking Theorem 2.9 we deduce that the terms $\int_{t_1}^{t_2} |a_1(s)|\mathrm{d}s$, $\int_{t_1}^{t_2} |a_2(s)|\mathrm{d}s$ and $\int_{t_1}^{t_2} |y_n(s)|\mathrm{d}s$ are arbitrarily small provided that the number $t_2 - t_1$ is small enough. This means that the sequence (Ay_n) is equicontinuous on $[\alpha, \beta]$. Notice also that for an arbitrary $t \in [\alpha, \beta]$ and for $n \in \mathbb{N}$, we have

$$\begin{aligned}|Ay_n(t)| &= \left| g(t) + \int_\alpha^{t} u(s, y_n(s))\mathrm{d}s + \int_t^{\alpha}\int_\alpha^{s} K(s, r, y_n(r))\mathrm{d}r\mathrm{d}s \right| \\ &\le \|g\|_{L^1(I)} + \int_\alpha^{t} [a_1(s) + b_1|y_n(s)|)]\mathrm{d}s \\ &\quad + \int_\alpha^{t}\int_\alpha^{s} h(s, r)(a_2(s) + b_2|y_n(s)|)\mathrm{d}s \\ &\le \|g\|_{L^1(I)} + \|a_1\| + b_1 r_0 + \|H\|_{L^1(I)}(\|a_2\| + b_2 r_0).\end{aligned}$$

Hence we conclude that the sequence (Ay_n) is uniformly bounded in $C([\alpha, \beta])$. By applying the Arzela–Ascoli theorem, we obtain that the sequence (Ay_n) has a convergent subsequence (Ay_{n_k}) in $C([\alpha, \beta])$. This subsequence is a Cauchy sequence in $C([\alpha, \beta])$ and therefore it is a Cauchy sequence in $L^1([\alpha, \beta])$. Consequently, A is (ws)-compact.

Step 3. We examine that $A(B_{r_0})$ is relatively weakly compact. To perform this, take an arbitrary number $\epsilon > 0$ and a nonempty subset D of $[\alpha,\beta]$ such that D is measurable and $\text{meas}(D) \le \epsilon$. Then for any $x \in B_{r_0}$ we have,

$$\int_D |Ax(t)|\mathrm{d}t = \int_D \left| g(t) + \int_\alpha^t u(s,x(s))\mathrm{d}s + \int_\alpha^t \int_\alpha^s K(s,r,x(r))\mathrm{d}r\mathrm{d}s \right| \mathrm{d}t$$
$$\le \text{meas}(D) \left(\|g\|_{L^\infty(I)} + \|a_1\| + b_1 r_0 + \|H\|_{L^1(I)}(\|a_1\| + b_2 r_0) \right).$$

This implies that $\mu(A(B_{r_0})) = 0$ and, therefore, $A(B_{r_0})$ is relatively weakly compact.

Thus, the hypotheses of Theorem 2.8 are fulfilled. Consequently, the operator A has a fixed point x^* which is a solution of the integral equation (3.3) in $L^1[\alpha,\beta]$. Now, since the functions $s \longmapsto u(s,x(s)) \in L^1([\alpha,\beta])$ and $s \longmapsto \int_0^s K(s,r,x(r))\mathrm{d}r \in L^1([\alpha,\beta])$, then, in view of Theorem 2.5, the solution x^* is absolutely continuous on $[\alpha,\beta]$. □

Now, we are in a position to state the main result of this section.

Theorem 3.3. *Assume* (i)–(iii) *hold. Then, the nonlinear integro-differential equation* (1.1) *has at least one solution* $x \in AC(\mathbb{R}^+)$*. Moreover,* $x' \in L^1_{\text{Loc}}(\mathbb{R}^+)$*.*

Proof. We first claim that Eq. (3.1) has an absolutely continuous solution on $[0,T]$ for any $T > 0$. To see this, we divide the interval $[0,T]$ into N subintervals $I_n = [t_{n-1}, t_n]$ such that $t_n = n\frac{T}{N}, n = 0,\ldots,N$ and $N > \frac{1}{b_1 + \|H\|_{L^1([0,T])} b_2}$.

Notice that $(t_1 - t_0)(b_1 + \|H\|_{L^1(I_0)} b_2) < 1$, hence by Theorem 3.2 the equation

$$x(t) = x_0 + \int_0^t u(s,x(s))\mathrm{d}s + \int_0^t \int_0^s K(s,r,x(r))\mathrm{d}r\mathrm{d}s, \quad t \in I_0$$

has an absolutely continuous solution x_1 on I_1. By proceeding inductively we infer that Eq. (3.1) takes on I_n the following form

$$x(t) = x_0 + \int_0^{t_{n-1}} u(s,y_n(s))\mathrm{d}s + \int_{t_{n-1}}^t u(s,x(s))\mathrm{d}s$$
$$+ \int_0^{t_{n-1}} \int_0^s K(s,r,y_n(r))\mathrm{d}r\mathrm{d}s + \int_{t_{n-1}}^t \int_0^{t_{n-1}} K(s,r,y_n(r))\mathrm{d}r\mathrm{d}s$$
$$+ \int_{t_{n-1}}^t \int_{t_{n-1}}^s K(s,r,x(r))\mathrm{d}r\mathrm{d}s, \quad t \in I_n,$$

where y_n is defined on $[0, t_{n-1}]$ by $y_n = x_i$ on I_i for $i = 0, \dots, n-1$.
Put

$$g(t) = x_0 + \int_0^{t_{n-1}} u(s, y_n(s))\mathrm{d}s + \int_0^{t_{n-1}} \int_0^s K(s, r, y_n(r))\mathrm{d}r\mathrm{d}s + \int_{t_{n-1}}^t \int_0^{t_{n-1}} K(s, r, y_n(r))\mathrm{d}r\mathrm{d}s, \quad t \in I_n.$$

Since the function $s \longmapsto \int_0^{t_{n-1}} K(s, r, y_n(r))\mathrm{d}r \in L^1(I_n)$, then g is absolutely continuous on I_n. Theorem 3.2 implies that the integral equation

$$x(t) = g(t) + \int_{t_{n-1}}^t u(s, x(s))\mathrm{d}s + \int_{t_{n-1}}^t \int_{t_{n-1}}^s K(s, r, x(r))\mathrm{d}r\mathrm{d}s, \quad t \in I_n$$

has an absolutely continuous solution x_n on I_n.

Accordingly, Eq. (3.1) has an absolutely continuous solution on $[0, T]$ given by

$$x(t) = \begin{cases} x_0(t), & t \in I_0 \\ x_2(t), & t \in I_1 \\ \vdots & \\ x_{N-1}(t), & t \in I_{N-1} \end{cases}$$

for all $T > 0$. Thus, our claim is established.

Now, we show that Eq. (3.1) has a solution $x \in AC(\mathbb{R}_+)$. To see this, notice first that from Claim 1, we know that Eq. (3.1) has an absolutely continuous solution x_n on $[0, n]$ for each $n \in \mathbb{N}^*$. Define $x \colon \mathbb{R}^+ \longrightarrow \mathbb{R}$ as follows: $x = x_1$ on the interval $[0, 1)$, and for $n \geq 2$ and $t \in [n-1, n)$,

$$x(t) = x_0 + \sum_{i=1}^{n-1} \int_{i-1}^{i} u(s, x_i(s))\mathrm{d}s + \sum_{i=1}^{n-1} \int_i^{i-1} \int_0^s K(s, r, x_i(r))\mathrm{d}r\mathrm{d}s + \int_{n-1}^t u(s, x_n(s))\mathrm{d}s + \int_{n-1}^t \int_0^s K(s, r, x_n(r))\mathrm{d}r\mathrm{d}s.$$

Using a simple induction, we can easily prove that the function x is a solution of Eq. (3.1) on $\mathbb{R}^+$. Since $s \longmapsto u(s, x(s)) \in L^1_{\mathrm{Loc}}(\mathbb{R}^+)$ and $s \longmapsto \int_0^s K(s, r, x(r))\mathrm{d}r \in L^1_{\mathrm{Loc}}(\mathbb{R}^+)$, then by Theorem 2.5 we infer that $x \in AC(\mathbb{R}^+)$ and $x' \in L^1_{\mathrm{Loc}}(\mathbb{R}^+)$. This completes the proof. □

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