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Differentiability and Weak Differentiability

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Abstract. We show that although the differentiability and the weak differentiability are different "locally", these notions coincide almost everywhere "globally". It is proved that a Banach space valued function $F : [0, 1] \to X$ is differentiable almost everywhere on $[0, 1]$, if and only if F is weakly differentiable almost everywhere on $[0, 1]$.

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1. Introduction and Preliminary

We continue the investigation of characterizations of the differentiability of Banach space-valued functions defined on $[0, 1]$, started in [5]. At first, the "local" relationships between the notions of weak differentiability, the Lipschitz points and the convex average range are studied. Then, we investigate the "global" relationships between differentiability and weak differentiability. We give new necessary and sufficient conditions for a function $F : [0, 1] \to X$ to be differentiable almost everywhere on $[0, 1]$, see Theorem 3.2.

Throughout this paper, X denotes a real Banach space with its norm $||\cdot||$ and the topological dual X^*. By $B(x, \varepsilon)$, the open ball with center x and radius $\varepsilon > 0$ is denoted. Assume that a function $F : [0, 1] \to X$ and a point $t \in [0, 1]$ are given. We set

$$\Delta F(t,h) = \frac{F(t+h) - F(t)}{h} \;(h \neq \emptyset), \quad A_F(t,\delta) = \{\Delta F(t,h) : 0 < |h| < \delta\}$$
$$\widetilde{A}_F(t,\delta) = \overline{conv}\,(A_F(t,\delta))$$

and

$$A_F(t) = \bigcap_{\delta>0} \overline{A}_F(t,\delta), \quad \widetilde{A}_F(t) = \bigcap_{\delta>0} \widetilde{A}_F(t,\delta),$$

where $\overline{conv}\,(A_F(t,\delta))$ is the closure of the convex hull of $A_F(t,\delta)$; $A_F(t)$ and $\widetilde{A}_F(t)$ are said to be *the average range* and *the convex average range* of F

at the point t, respectively. Clearly, $A_F(t) \neq \emptyset$ implies $\widetilde{A}_F(t) \neq \emptyset$, but the converse is not valid, as the following example shows.

Example 1.1. Let $F : [-1, 1] \to \ell_\infty$ be a function defined as follows

$$F(t) = (\sqrt{|t|}, \dots, \sqrt{|t|}, \dots), \quad \text{for all} \quad t \in [-1, 1].$$

Then $A_F(0) = \emptyset$ and $\widetilde{A}_F(0) \neq \emptyset$.

Proof. Fix a real number $0 < h < 1$. Since

$$\Delta F(0, h) = \left(\frac{1}{\sqrt{h}}\right) \quad \text{and} \quad \Delta F(1, -h) = \left(\frac{-1}{\sqrt{h}}\right),$$

we obtain

$$\frac{1}{2}\Delta F(0, h) + \left(1 - \frac{1}{2}\right) \Delta F(0, -h) = (0, \dots, 0, \dots) \in \widetilde{A}_F(t, \delta)$$

for each $\delta > 0$. Hence

$$(0, \dots, 0, \dots) \in \widetilde{A}_F(0).$$

Suppose that there exists $w \in A_F(0)$. Then, since

$$A_F(0) = \bigcap_{n=1}^{+\infty} \overline{A}_F\left(0, \frac{1}{n}\right)$$

there exists a sequence $(\Delta F(0, h_n))$, such that $0 < |h_n| < \frac{1}{n}$, for all $n \in \mathbb{N}$, and

$$\lim_{n\to\infty} \Delta F(0, h_n) = w.$$

On the other hand, we have

$$\lim_{n\to\infty} ||\Delta F(0, h_n)||_{\ell_\infty} = \lim_{n\to\infty} \frac{1}{\sqrt{|h_n|}} = +\infty.$$

This contradicts the fact that every convergent sequence is bounded. Thus, we have $A_F(0) = \emptyset$ and the proof is finished. □

The function F is said to be *differentiable* at the point t, if there exists a vector $x \in X$ such that

$$\lim_{h\to 0} ||\Delta F(t, h) - x|| = 0.$$

We say that F is *weakly differentiable* at the point t, if there exists a vector $x_w \in X$ such that

$$\lim_{h\to 0} \Delta(x^* \circ F)(t, h) = x^*(x_w), \quad \text{for each} \quad x^* \in X^*.$$

We denote by $F'(t) = x$, the derivative of F and by $F'_w(t) = x_w$, the weak-derivative of F at t.

By $\mathcal{I}$, the family of all non-degenerate closed subintervals of $[0, 1]$ is denoted and by λ the Lebesgue measure on the family $\mathcal{L}$ of all Lebesgue measurable subset of $[0, 1]$. The intervals $I, J \in \mathcal{I}$ are said to be *nonoverlapping* if $\text{int}(I) \cap \text{int}(J) = \emptyset$, where $\text{int}(I)$ denotes the interior of I. If $F : [0, 1] \to X$

is a function, then we denote by $\widetilde{F}$ the interval function $\widetilde{F} : \mathcal{I} \to X$ defined by $\widetilde{F}([u, v]) = F(v) - F(u)$, for all $[u, v] \in \mathcal{I}$.

A function $F : [0, 1] \to X$ is said to be *absolutely continuous (AC)* if for each $\varepsilon > 0$ there exists $\eta > 0$ such that $|| \sum_{j=1}^{p} \widetilde{F}(I_j)|| < \varepsilon$, whenever $\{I_1, \ldots, I_p\}$ is a finite collection of pairwise non-overlapping subintervals in $\mathcal{I}$ with $\sum_{i=1}^{p} \lambda(I_j) < \eta$.

A function $F : [0, 1] \to X$ is said to be *Lipschitz*, if there exists $M > 0$ such that

$$||F(t') - F(t'')|| \leq M \cdot |t' - t''|, \quad \text{for all} \quad t', t'' \in [0, 1].$$

We say that F is *pointwise Lipschitz at* $t \in [0, 1]$, if there exist $c_t > 0$ and $\delta_t > 0$ such that

$$|h| < \delta_t \text{ and } t + h \in [0, 1] \Rightarrow ||F(t + h) - F(t)|| \leq c_t \cdot |h|.$$

If $F : [0, 1] \to X$ is a function, then we denote by $S(F)$, the set of all points $t \in [0, 1]$ at which F is pointwise Lipschitz and by $S_n(F)$ the set of all points $t \in S(F)$ such that $||F(t + h) - F(t)|| \leq n \cdot |h|$, whenever $|h| < 1/n$ and $t + h \in [0, 1]$. Then, $S_n(F)$ is a closed set, see Lemma 1 in [1], and $S(F) = \cup_{n=1}^{\infty} S_n(F)$. Fix an arbitrary $n \in \mathbb{N}$, and let

$$(0, 1)\backslash S_n(F) = \bigcup_{k=1}^{\infty} (a_k^{(n)}, b_k^{(n)}).$$

Define the function $F_n : [0, 1] \to X$ by $F_n(t) = F(t)$ for all $t \in S_n(F)$, $F_n(0) = F(0)$, $F_n(1) = F(1)$ and

$$F_n(t) = F(a_k^{(n)}) + \frac{F(b_k^{(n)}) - F(a_k^{(n)})}{b_k^{(n)} - a_k^{(n)}} \cdot (t - a_k^{(n)}), \tag{1.1}$$

for all $t \in [a_k^{(n)}, b_k^{(n)}]$ and $k \in \mathbb{N}$.

2. The "Local" Relationships Between the Weak Differentiability, the Lipschitz Points and the Convex Average Range

In this section, we study the "local" relationships between weak differentiability, the Lipschitz points and the convex average range. Let $\mathscr{F}$ be the family of all functions defined on $[0, 1]$ and taking values in the Banach space X, and let $t \in (0, 1)$. We set

$$\mathscr{D}_d(t) = \{F \in \mathscr{F} : F \text{ is differentiable at } t\},$$
$$\mathscr{D}_w(t) = \{F \in \mathscr{F} : F \text{ is weakly differentiable at } t\}$$
$$\mathscr{D}_{lip}(t) = \{F \in \mathscr{F} : F \text{ is pointwise Lipschitz at } t\}$$

It is known that

$$\mathscr{D}_d(t) \subset \mathscr{D}_w(t) \; (\mathscr{D}_w(t) \not\subset \mathscr{D}_d(t)) \quad \text{and} \quad \mathscr{D}_d(t) \subset \mathscr{D}_{lip}(t) \; (\mathscr{D}_{lip}(t) \not\subset \mathscr{D}_d(t)).$$

At first, we will show

- $\mathscr{D}_w(t) \subset \mathscr{D}_{lip}(t)$ (Theorem 2.1),
- $\mathscr{D}_{lip}(t) \not\subset \mathscr{D}_w(t)$ (Example 2.2).

Theorem 2.1. *Let $F : [0,1] \to X$ be a function and let $t \in (0,1)$. If F is weakly differentiable at t, then F is pointwise Lipschitz at t.*

Proof. Let (h_n) be an arbitrary sequence of real numbers such that

$$\lim_{n\to\infty} h_n = 0.$$

Then, there exists a vector $x_w \in X$ such that

$$\lim_{n\to\infty} x^*(x_n) = x^*(x_w), \quad \text{for each} \quad x^* \in X^*, \tag{2.1}$$

where

$$x_n = \Delta F(t, h_n), \quad \text{for all} \quad n \in \mathbb{N}.$$

For each $n \in \mathbb{N}$, define a function $\hat{x}_n : X^* \to \mathbb{R}$ as follows

$$\hat{x}_n(x^*) = x^*(x_n), \quad \text{for all} \quad x^* \in X^*.$$

By (2.1) we obtain that the set

$$\{\hat{x}_n(x^*) : n \in \mathbb{N}\}$$

is bounded for each $x^* \in X^*$. Therefore, by Corollary II.1.1 in [8], we have

$$\{||x_n|| : n \in \mathbb{N}\} = \{||\Delta F(t, h_n)|| : n \in \mathbb{N}\} \quad \text{is a bounded set.} \tag{2.2}$$

Since the sequence (h_n) is arbitrary, the last result holds for each sequence (h_n) which converges to zero.

Now suppose that F is not pointwise Lipschitz at the point t. Then, for each $n \in \mathbb{N}$ there exists $h_n \in \mathbb{N}$ such that

$$0 < |h_n| < \frac{1}{n} \quad \text{and} \quad ||\Delta F(t, h_n)|| > n.$$

This contradicts (2.2). Hence, F is pointwise Lipschitz at t, and this ends the proof. □

Example 2.2. Let $F : [-1,1] \to \ell_1$ be a function given as follows

$$F(t) = \begin{cases} (0,\ldots,0,\ldots) & \text{if} \quad t \neq \frac{1}{n} \\ (0,\ldots,0,\frac{1}{n},0,\ldots) & \text{if} \quad t = \frac{1}{n} \end{cases}, \quad t \in [-1,1] \quad n = 1,2,3,\ldots$$

Then F is Lipschitz at the point $t = 0$ but not weakly differentiable.

Proof. Note that

$$\varDelta F(0,h) = \begin{cases} (0,\ldots,0,\ldots) & \text{if} \quad h \neq \frac{1}{n} \\ (0,\ldots,0,1,0,\ldots) & \text{if} \quad h = \frac{1}{n} \end{cases}$$

and

$$||F(t) - F(0)||_{\ell_1} \leq |t|, \quad \text{for all} \quad t \in [-1,1].$$

Thus, F is Lipschitz at the point $t = 0$.

Taking $x_0^* = (1,1,1,\ldots) \in \ell_1^* = \ell_\infty$, we have

$$x_0^*\left(\frac{F(\frac{1}{n}) - F(0)}{\frac{1}{n}}\right) = 1, \quad \text{for all} \quad n \in \mathbb{N}$$

and

$$x_0^*\left(\frac{F(h)-F(0)}{h}\right)=0, \quad \text{for} \quad h\neq\frac{1}{n}.$$

Therefore, the weak derivative of F cannot exist at $t=0$. □

We now investigate the relation between the weak differentiability and the convex average range at a point.

Theorem 2.3. *Let $F:[0,1]\to X$ be a function and let $t\in(0,1)$. If F is weakly differentiable at t with $F'_w(t)=x_w$, then*

$$\widetilde{A}_F(t)=\{x_w\}.$$

Proof. By hypothesis, we have

$$\lim_{h\to 0}\Delta(x^*\circ F)(t,h)=x^*(x_w), \quad \text{for each} \quad x^*\in X^*. \tag{2.3}$$

Fix an arbitrary $x^*\in X^*$. We claim that

$$\widetilde{A}_{x^*\circ F}(t)=\{x^*(x_w)\}. \tag{2.4}$$

Indeed, by (2.3), we obtain

$$x^*(x_w)\in\widetilde{A}_{x^*\circ F}(t)$$

and

$$\lim_{\delta\to 0}\operatorname{diam}\left(A_{x^*\circ F}(t,\delta)\right)=0,$$

where $\operatorname{diam}(R)=\sup\{|r'-r''|:r',r''\in R\}$, $R\subset\mathbb{R}$. The last equality together with

$$\operatorname{diam}\left(A_{x^*\circ F}(t,\delta)\right)=\operatorname{diam}\left(\widetilde{A}_{x^*\circ F}(t,\delta)\right),$$

yields

$$\lim_{\delta\to 0}\operatorname{diam}\left(\widetilde{A}_{x^*\circ F}(t,\delta)\right)=0,$$

Therefore,

$$\operatorname{diam}\left(\widetilde{A}_{x^*\circ F}(t)\right)=0,$$

and since $x^*(x_w)\in\widetilde{A}_{x^*\circ F}(t)$, we obtain that (2.4) holds true. Since x^* is arbitrary the equality (2.4) holds for each $x^*\in X^*$.

We now claim that $x_w\in\widetilde{A}_F(t)$. Indeed, if we suppose that $x_w\notin\widetilde{A}_F(t)$, then there exists $\delta_w>0$ such that $x_w\notin\widetilde{A}_F(t,\delta_w)$. Therefore, by Theorem V.2.10 in [3], there exist $x_w^*\in X^*$, $c\in\mathbb{R}$ and $\alpha>0$ such that

$$x_w^*(x_w)\le c<c+\alpha\le x_w^*\left(\widetilde{A}_F(t,\delta_w)\right). \tag{2.5}$$

On the other hand, it is easy to see that

$$\overline{x_w^*\left[\ \operatorname{conv}\left(A_F(t,\delta_w)\right)\right]}=\widetilde{A}_{x_w^*\circ F}(t,\delta_w),$$

and since $\overline{x_w^*(\overline{Y})}=\overline{x_w^*(Y)}$, $Y\subset X$, it follows that

$$\overline{x_w^*\left(\widetilde{A}_F(t,\delta_w)\right)}=\widetilde{A}_{x_w^*\circ F}(t,\delta_w).$$

The last equality together with (2.5) contradicts (2.4). Thus, $x_w \in \widetilde{A}_F(t)$.

It remains to prove that if $x_1 \in \widetilde{A}_F(t)$, then $x_1 = x_w$. By the inclusion

$$x^*\left(\widetilde{A}_F(t)\right) \subset \widetilde{A}_{x^*\circ F}(t),$$

we obtain $x^*(x_1) = x^*(x_w)$, for all $x^* \in X^*$. Hence, by Hahn–Banach theorem, $x_1 = x_w$. Thus, $\widetilde{A}_F(t) = \{x_w\}$, and this ends the proof. □

Corollary 2.4. *Let $F : [0,1] \to X$ be a function and let $t \in (0,1)$. If F is differentiable at t with $F'(t) = x_0$, then*

$$\widetilde{A}_F(t) = \{x_0\}.$$

The following example shows that the converse to Theorem 2.3 is false.

Example 2.5. Let $F : [-\frac{1}{2}, \frac{1}{2}] \to \ell_2$ be a function defined as follows

$$F(t) = \begin{cases} (0,\dots,0,\dots) & \text{if } \ t \in [-\frac{1}{2}, 0] \\ (0,\dots,\frac{k}{2^{k+1}}, \frac{k+1}{2^{k+2}}\dots) & \text{if } \ t \in \left(\frac{1}{2^{k+1}}, \frac{1}{2^k}\right] \end{cases} \qquad k = 1,2,\dots .$$

Then $\widetilde{A}_F(0) = \{(0,\dots,0,\dots)\}$ and F is not weakly differentiable at $t = 0$.

Proof. First, we will prove that

$$\widetilde{A}_F(0) = \{(0,\dots,0,\dots)\}. \tag{2.6}$$

We have

$$\Delta F(0,h) = \begin{cases} (0,\dots,0,\dots) & \text{if } \ -\frac{1}{2} < h < 0 \\ \frac{1}{h}\cdot(0,\dots,\frac{k}{2^{k+1}}, \frac{k+1}{2^{k+2}}\dots) & \text{if } \ h \in \left(\frac{1}{2^{k+1}}, \frac{1}{2^k}\right] \end{cases}$$

and

$$(0,\dots,0,\dots) \in \widetilde{A}_F\left(0, \frac{1}{2^k}\right), \quad \text{for all} \quad k \in \mathbb{N},$$

and since

$$\widetilde{A}_F(0) = \bigcap_{k=1}^{+\infty} \widetilde{A}_F\left(0, \frac{1}{2^k}\right)$$

it follows that

$$(0,\dots,0,\dots) \in \widetilde{A}_F(0).$$

Assume that

$$w = (w_1,\dots,w_n,\dots) \in \widetilde{A}_F(0).$$

Then, there exists a sequence $(z_k) \subset \ell_2$ such that

$$z_k \in \operatorname{conv}\left(A_F\left(0, \frac{1}{2^k}\right)\right), \quad \text{for each} \quad k \in \mathbb{N}$$

and

$$\lim_{k\to\infty} ||z_k - w||_{\ell_2} = 0. \tag{2.7}$$

Since

$$z_k = (0,\dots,0,\dots) \quad \text{or} \quad z_k = (0,\dots,0,z_{k+1}^{(k)}, z_{k+2}^{(k)}\dots)$$

for each $k \in \mathbb{N}$, we have

$$||z_k - w||_{\ell_2} = \left(\sum_{i=k+1}^{+\infty} |z_i^{(k)} - w_i|^2 + \sum_{i=1}^{k} |w_i|^2 \right)^{\frac{1}{2}}.$$

The last equality together with (2.7) yields

$$w = (0, \dots, 0, \dots)$$

and, therefore, (2.6) holds true.

We now prove that F is not weakly differentiable at $t = 0$. To see this, let us consider the vector

$$x^* = \left(1, \frac{1}{2}, \dots, \frac{1}{k}, \dots\right) \in \ell_2^* = \ell_2.$$

Since

$$x^*(x) = \sum_{k=1}^{+\infty} \frac{1}{k} \cdot x_k, \quad \text{for each} \quad x = (x_1, \dots, x_k, \dots) \in \ell_2,$$

we obtain

$$x^*\left(\Delta F\left(0, \frac{1}{2^k}\right)\right) = \sum_{m=1}^{+\infty} \frac{1}{2^m} > \frac{1}{2}.$$

This means that F is not weakly differentiable at $t = 0$. □

3. The "Global" Relationships Between the Differentiability and the Weak Differentiability

We now investigate the relationships between the differentiability and the weak differentiability almost everywhere on $[0, 1]$ of a function $F : [0, 1] \to X$. Let us now set

$$\begin{aligned}
\mathscr{D}_d &= \{F \in \mathscr{F} : F \text{ is differentiable a.e. on } [0,1]\},\\
\mathscr{D}_w &= \{F \in \mathscr{F} : F \text{ is weakly differentiable a.e. on } [0,1]\},\\
\mathscr{D}_{lip} &= \{F \in \mathscr{F} : F \text{ is pointwise Lipschitz a.e. on } [0,1]\},\\
\mathscr{D}_{car} &= \{F \in \mathscr{F} : \widetilde{A}_F(t) \neq \emptyset \text{ a.e. on } [0,1]\},\\
\mathscr{D}_{ar} &= \{F \in \mathscr{F} : A_F(t) \neq \emptyset \text{ a.e. on } [0,1]\}.
\end{aligned}$$

We will show

$$\mathscr{D}_d = \mathscr{D}_w = \mathscr{D}_{lip} \cap \mathscr{D}_{car} = \mathscr{D}_{lip} \cap \mathscr{D}_{ar} \text{ (Theorem 3.2).}$$

Let us start with the following auxiliary lemma.

Lemma 3.1. *If $F : [0,1] \to X$ is a Lipschitz function and $\widetilde{A}_F(t) \neq \emptyset$ for almost all $t \in [0,1]$, then $A_F(t) \neq \emptyset$ for almost all $t \in [0,1]$.*

Proof. By hypothesis, there exists $Z \subset [0,1]$ with $\lambda(Z) = 0$, such that $\widetilde{A}_F(t) \neq \emptyset$ for all $t \in [0,1]\backslash Z$. Then, we can choose a vector $x_t \in \widetilde{A}_F(t)$, for all $t \in [0,1]\backslash Z$, and define the function $f : [0,1] \to X$ as follows

$$f(t) = \begin{cases} x_t & \text{if } t \in [0,1]\backslash Z \\ 0 & \text{if } t \in Z \end{cases}.$$

Since $x^* \circ F$ is Lipschitz, $x^* \circ F$ is differentiable almost everywhere on $[0,1]$. Thus, for each $x^* \in X^*$, there exists $Z^{(x^*)} \subset [0,1]$ with $\lambda(Z^{(x^*)}) = 0$ such that $(x^* \circ F)'(t)$ exists for all $t \in [0,1]\backslash Z^{(x^*)}$. Further, by Corollary 2.4, we obtain

$$\widetilde{A}_{x^* \circ F}(t) = \{(x^* \circ F)'(t)\}, \quad \text{for all} \quad t \in [0,1]\backslash Z^{(x^*)},$$

and since

$$x^* \left(\widetilde{A}_F(t)\right) \subset \widetilde{A}_{x^* \circ F}(t),$$

it follows that

$$(x^* \circ F)'(t) = (x^* \circ f)(t), \quad \text{for all} \quad t \in [0,1]\backslash (Z \cup Z^{(x^*)}).$$

We have also that F is AC. Therefore, by Theorem 5.1 in [7], f is Pettis integrable on $[0,1]$ with the primitive F, i.e., $\widetilde{F}(I) = (P)\int_I f(t)d\lambda$, for all $I \in \mathcal{I}$.

In the same manner as in the proof of Theorem 2.3 (iv) ($\Rightarrow$) (i), [4], we can prove that f is Bochner integrable with the primitive F, i.e., $\widetilde{F}(I) = (B)\int_I f(t)d\lambda$, for all $I \in \mathcal{I}$. Hence, by Theorem II.2.9 in [2], F is differentiable almost everywhere on $[0,1]$. Thus, there exists $Z_B \subset [0,1]$ with $\lambda(Z_B) = 0$ such that $F'(t)$ exists at all $t \in [0,1]\backslash Z_B$, and since

$$F'(t) \in A_F(t), \quad \text{for each} \quad t \in [0,1]\backslash Z_B,$$

the proof is finished. □

We are now ready to present the main result of this paper. The equivalence (i) $\Leftrightarrow$ (iv) in the following theorem has been proved in [5]. Here similar techniques and ideas are developed.

Theorem 3.2. *Let $F : [0,1] \to X$ be a function. Then, the following are equivalent:*

(i) *F is differentiable almost everywhere on $[0,1]$,*
(ii) *F is weakly differentiable almost everywhere on $[0,1]$,*
(iii) *F is pointwise Lipschitz at t and $\widetilde{A}_F(t) \neq \emptyset$ for almost all $t \in [0,1]$.*
(iv) *F is pointwise Lipschitz at t and $A_F(t) \neq \emptyset$ for almost all $t \in [0,1]$.*

Proof. Clearly, (i) $\Rightarrow$ (ii). Theorem 2.1 together with Theorem 2.3 yields (ii) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (iv). Assume that there exists $Z \subset [0,1]$ with $\lambda(Z) = 0$ such that $\widetilde{A}_F(t) \neq \emptyset$ and F is pointwise Lipschitz for all $t \in [0,1]\backslash Z$.

It is enough to show that $A_F(t) \neq \emptyset$ for almost all $t \in S(F)$. To see this, fix an arbitrary $n \in \mathbb{N}$. In the same manner as in the proof of Theorem 2.2

in [5], we can prove that there exists $Z_n \subset [0,1]$ with $\lambda(Z_n) = 0$ such that the equality

$$\lim_{h\to 0} \frac{F(t+h) - F_n(t+h)}{h} = 0, \tag{3.1}$$

holds for all $t \in S_n(F)\backslash Z_n$, where F_n is defined by (1.1).

We now prove that the inclusion

$$\widetilde{A}_F(t) \subset \widetilde{A}_{F_n}(t), \tag{3.2}$$

holds for each $t \in S_n(F)\backslash(Z \cup Z_n)$. Assume that an arbitrary point $t \in S_n(F)\backslash(Z \cup Z_n)$, a vector $x \in \widetilde{A}_F(t)$ and a natural number $k \in \mathbb{N}$ are given. By virtue of (3.1), given $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that for each $h \in \mathbb{R}$, we have

$$0 < |h| < \delta_\varepsilon \Rightarrow \left\| \frac{F(t+h) - F_n(t+h)}{h} \right\| < \frac{\varepsilon}{2}. \tag{3.3}$$

Fix $k_\varepsilon \in \mathbb{N}$ such that

$$\frac{1}{k_\varepsilon} < \min\left\{\frac{1}{k}, \delta_\varepsilon\right\}. \tag{3.4}$$

Since

$$x \in \widetilde{A}_F(t) = \bigcap_{m=1}^{+\infty} \widetilde{A}_F\left(t, \frac{1}{m}\right) \subset \widetilde{A}_F\left(t, \frac{1}{k_\varepsilon}\right)$$

there exist $r_1, \ldots, r_s \in [0,1]$ with $\sum_{i=1}^{s} r_1 = 1$, and

$$\Delta F(t, h_1), \ldots, \Delta F(t, h_s) \in A_F\left(t, \frac{1}{k_\varepsilon}\right),$$

such that

$$\sum_{i=1}^{s} r_i \cdot \Delta F(t, h_i) \in B\left(x, \frac{\varepsilon}{2}\right) \quad \text{or} \quad \left\| \sum_{i=1}^{s} r_i \cdot \Delta F(t, h_i) - x \right\| < \frac{\varepsilon}{2}. \tag{3.5}$$

Since $t \in S_n(F)$, we have also

$$\Delta F_n(t, h) = \frac{F_n(t+h) - F(t+h)}{h} + \Delta F(t, h). \tag{3.6}$$

The last equality together with (3.5), (3.3) and (3.4) yields

$$\begin{aligned}
&\left\| \sum_{i=1}^{s} r_i \cdot \Delta F_n(t, h_i) - x \right\| \\
&= \left\| \left(\sum_{i=1}^{s} r_i \cdot \Delta F(t, h_i) - x \right) + \sum_{i=1}^{s} r_i \cdot \frac{F_n(t+h_i) - F(t+h_i)}{h_i} \right\| \\
&\le \left\| \sum_{i=1}^{s} r_i \cdot \Delta F(t, h_i) - x \right\| + \left\| \sum_{i=1}^{s} r_i \cdot \frac{F_n(t+h_i) - F(t+h_i)}{h_i} \right\| \\
&< \frac{\varepsilon}{2} + \sum_{i=1}^{s} r_i \cdot \left\| \frac{F_n(t+h_i) - F(t+h_i)}{h_i} \right\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

This means that $B(x,\varepsilon)\cap conv\left(A_{F_n}(t,\frac{1}{k_\varepsilon})\right)\neq\emptyset$. Hence, by (3.4), we obtain

$$B(x,\varepsilon)\cap conv\left(A_{F_n}\left(t,\frac{1}{k}\right)\right)\neq\emptyset.$$

Thus, $x\in\widetilde{A}_{F_n}\left(t,\frac{1}{k}\right)$, and since k is arbitrary, it follows that

$$x\in\widetilde{A}_{F_n}(t)=\bigcap_{m=1}^{+\infty}\widetilde{A}_{F_n}\left(t,\frac{1}{m}\right).$$

Since t and x have been taken arbitrarily, (3.2) holds for all $t\in S_n(F)\backslash(Z\cup Z_n)$. Hence

$$\widetilde{A}_{F_n}(t)\neq\emptyset,\quad \text{for almost all}\quad t\in S_n(F),$$

and by (1.1), we have also $\widetilde{A}_{F_n}(t)\neq\emptyset$ at all $t\in(a_k^{(n)},b_k^{(n)})$, for each $k\in\mathbb{N}$. Thus

$$\widetilde{A}_{F_n}(t)\neq\emptyset,\quad \text{for almost all}\quad t\in[0,1]. \tag{3.7}$$

Since $\lim_{k\to\infty}(b_k^{(n)}-a_k^{(n)})=0$, there is a real number $M_n\geq 1$ such that

$$\frac{||F(b_k^{(n)})-F(a_k^{(n)})||}{(b_k^{(n)}-a_k^{(n)})}\leq M_n,\quad \text{for all}\quad k\in\mathbb{N}.$$

It follows that

$$||\widetilde{F}_n(I)||\leq\lambda(I)\cdot\max\{n,M_n\},\quad \text{for all}\quad I\in\mathcal{I}.$$

The last result together with (3.7) and Lemma 3.1 yields

$$A_{F_n}(t)\neq\emptyset,\quad \text{for almost all}\quad t\in[0,1],$$

and since

$$A_{F_n}(t)=A_F(t),\quad \text{for almost all}\quad t\in S_n(F),$$

(see [Claim 5] in Theorem 2.6, [6]), we obtain

$$A_F(t)\neq\emptyset,\quad \text{for almost all}\quad t\in S_n(F).$$

Since n is arbitrary, $A_F(t)\neq\emptyset$ for almost all $t\in S(F)=\cup_{n=1}^{\infty}S_n(F)$.

Finally, by Theorem 2.2 in [5], we obtain immediately (iv) ⇒ (i), and this ends the proof. □

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