Mediterr. J. Math. 13 (2016), 1807–1821
DOI 10.1007/s00009-015-0617-0
1660-5446/16/041807-15
published online July 30, 2015

Mediterranean Journal of Mathematics

Bi-space Global Attractors for a Class of Nonclassical Parabolic Equations with Arbitrary Polynomial Growth in Unbounded Domain

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Abstract. In this article, we consider the dynamical behavior of the nonclassical diffusion equation in unbounded domain while the nonlinearity satisfy the arbitrary order polynomial growth conditions. Using the tail-estimated method and the asymptotic a priori estimate method, we obtain the existence of $(H^1(\Omega) \cap L^p(\Omega), L^2(\Omega))$-global attractor, $(H^1(\Omega) \cap L^p(\Omega), L^p(\Omega))$-global attractor, $(H^1(\Omega) \cap L^p(\Omega), H^1(\Omega))$-global attractor and $(H^1(\Omega) \cap L^p(\Omega), H^1(\Omega) \cap L^p(\Omega))$-global attractor.

Mathematics Subject Classification. 35B41, 35Q35.

Keywords. Nonclassical diffusion equations, global attractor, absorbing set, unbounded domain.

1. Introduction

The study of global attractor for infinite-dimensional dynamical systems has attracted much attention and has made fast progress in recent decades, see, for instance [4,7–9,16,18,19,26,32] and the references therein. Meanwhile, the asymptotical behavior of infinite-dimensional dynamical systems for partial differential equations in Poincaré domain has attracted much attention in mathematical literature, see, for example [3,20–22] and the references therein.

Let $\Omega \subset \mathbb{R}^N$ be an open set, not necessarily bounded, and suppose that Ω satisfies the Poincaré inequality, i.e., there exists a constant $\lambda_1 > 0$ such that

$$\int_\Omega |u(x)|^2 \mathrm{d}x \leq \lambda_1^{-1} \int_\Omega |\nabla u(x)|^2 \mathrm{d}x, \quad \forall x \in H_0^1(\Omega).$$

We investigate the long-time behavior of the solutions for the following nonclassical parabolic equations

$$u_t - \Delta u_t - \Delta u + \varphi(u) = g(x), \quad x \in \Omega, \tag{1.1}$$

Birkhäuser

with the initial data

$$u(x,0) = u_0, \quad x \in \Omega, \tag{1.2}$$

and zero Dirichlet boundary condition

$$u = 0, \quad x \in \partial\Omega, \tag{1.3}$$

where $g(x) \in L^2(\Omega)$, and the nonlinearity $\varphi(u)$ satisfies:

$$\varphi(u)u \geq c_0|u|^p - \psi_1(x), \tag{1.4}$$

$$|\varphi(u)| \leq c_1|u|^{p-1} + \psi_2(x), \tag{1.5}$$

$$\varphi'(u) \geq -c_2, \tag{1.6}$$

and

$$c_3|u|^p - \psi_3(x) \leq \Phi(u) \leq c_4|u|^p + \psi_4(x), \tag{1.7}$$

where $\Phi(s) = \int_0^s \varphi(r)dr$, $p \geq 2$, $\psi_1(x), \psi_3(x), \psi_4(x) \in L^1(\Omega)$, $\psi_2(x) \in L^{\frac{p}{p-1}}(\Omega)$ are nonnegative functions, and $c_i (i = 0, 1, 2, 3, 4)$ are all positive constants.

This equation is a special form of the nonclassical diffusion equations used in fluid mechanics, solid mechanics and heat conduction theory (see [1,2,11–14]) for details.

In recent decades, on bounded domains, the long-time behavior for problem (1.1), especially the global attractor, exponential attractors and pullback attractor, has been discussed by many authors in [15,23,24,27,29]. On unbounded domain, using the tail-estimate method introduced in [28], the pullback attractor was obtained in [31] in $H^1(\mathbb{R}^N)$, and in [17], the authors proved the existence of global attractor in $H^1(\mathbb{R}^N)$ when $f(u) = f_1(u) + a(x)f_2(u)$. To the best of our knowledge, the existence of bi-space global attractor for Eq. (1.1) in unbounded domains has not been considered by predecessors.

Since the nonclassical diffusion equations contain the term $-\Delta u_t$, it is essentially different from the classical reaction diffusion equation. For example, the reaction diffusion equation has some kind of "regularity"; e.g., although the initial data only belong to a weaker topology space, the solution will belong to a stronger topology space with higher regularity. However, for problem (1.1), because of $-\Delta u_t$, the solution has no higher regularity, which is similar to hyperbolic equations. This brings some difficulty in establishing the existence of bi-space global attractors for nonclassical diffusion equations.

To prove the existence of bi-space global attractors, we need to show the existence of a family of compact sets. This can be done by using the standard compact Sobolev embedding of several functional spaces, when we consider the systems in some bounded domains. However, when we consider the asymptotic behavior of solutions, particularly, the existence of attractor in some unbounded domains, the Sobolev embedding is no longer compact.

To overcome these difficulties, using the ideas of Ball [5], and the tail-estimate method of Wang [28] for reaction diffusion equations, and the asymptotic a priori estimate method of Zhong et al. [32], we prove the existence of bi-space global attractors for Eq. (1.1) in unbounded domains.

This paper is organized as following: in Sect. 2, we recall some basic definitions and related theorems that will be used later. In Sect. 3, we obtain the existence of weak solution and bounded absorbing set. The main result is stated and proved in Sect. 4.

2. Preliminaries

In this section, we recall some notations and abstract results.

Definition 2.1 [32]. Let M be a metric space and A be a bounded subsets of M. The Kuratowski measure of noncompactness of A defined by

$$\text{mes}(A) = \inf\{\delta > 0 | \text{A admits a finite cover by sets whose diameter} \leq \delta\}.$$

Definition 2.2 [32]. Let X be a Banach space and $\{S(t)\}_{t\geq 0}$ be a family of operators on X. We say that $\{S(t)\}_{t\geq 0}$ is a continuous semigroup (C_0 semigroup) (or norm-to-weak continuous semigroup) on X, if $\{S(t)\}_{t\geq 0}$ satisfies

(i) $S(0) = Id$ (the identity);
(ii) $S(t)S(s) = S(t+s), \ \forall t, s \geq 0$;
(iii) $S(t_n)x_n \to S(t)x$, if $t_n \to t, x_n \to x$ in X [or (iii) $S(t_n)x_n \rightharpoonup S(t)x$, if $t_n \to t, x_n \to x$ in X].

Definition 2.3 [32]. A C_0 semigroup (or norm-to-weak continuous semigroup) $\{S(t)\}_{t\geq 0}$ in a complete metric space M is called ω-limit compact if for every bounded subset B of M and for every $\varepsilon > 0$, there is a $t(B) > 0$, such that

$$\text{mes}\left(\bigcup_{t\geq t(B)} S(t)B\right) \leq \varepsilon.$$

Condition C [32]. For any bounded set B of a Banach space X, there exists a $t(B) > 0$ and a finite dimensional subspace X_1 of X such that $\{\|P_m S(t)B\|\}$ is bounded and

$$\|(I - P_m)S(t)x\| < \varepsilon \quad \text{for } t \geq t(B),\ x \in B.$$

where $P_m\colon X \to X_1$ is a bounded projector.

Lemma 2.1 [32]. *Let X be a Banach space and $\{S(t)\}_{t\geq 0}$ be a C_0 semigroup (or norm-to-weak continuous semigroup) in X.*

(1) *If Condition C holds, the $\{S(t)\}_{t\geq 0}$ is ω-limit compact.*
(2) *Let X be a uniformly convex Banach space. Then $\{S(t)\}_{t\geq 0}$ is ω-limit compact if and only if Condition C holds.*

Theorem 2.1 [32]. *Let X be a Banach space. Then, the C_0 semigroup (or norm-to-weak continuous semigroup) $\{S(t)\}_{t\geq 0}$ has a global attractor in X if and only if*

(1) *there is a bounded absorbing set $B \subset X$.*
(2) *$\{S(t)\}_{t\geq 0}$ is ω-limit compact.*

Next, we iterate some definitions and abstract results concerning the global attractor, which are necessary to obtain our main results, we refer to see [4–6,19,25,26,30] for more details.

Definition 2.4. A set $\mathcal{A} \subset X$, which is invariant, closed in X, compact in Z and attracts the bounded subsets of X in the topology of Z, is called an (X, Z)-global attractor.

Definition 2.5. Let $\{S(t)\}_{t\geq 0}$ be a semigroup on Banach space X. A set $B_0 \subset Z$, satisfying that, for any bounded subset $B \subset X$, there is a $T = T(B)$, such that $S(t)B \subset B_0$, for any $t \geq T$, is called an (X, Z)-bounded absorbing set.

Definition 2.6. Let $\{S(t)\}_{t\geq 0}$ be a semigroup on Banach space X. $\{S(t)\}_{t\geq 0}$ is called (X, Z)-asymptotically compact, if for any bounded (in X) sequence $\{x_n\}_{n=1}^{\infty} \subset X$ and $t_n \geq 0$, $t_n \to \infty$ as $n \to \infty$, $\{S(t_n)x_n\}_{n=1}^{\infty}$ has a convergence subsequence with respect to the topology of Z.

Lemma 2.2. *Let X be a Sobolev space and $\{S(t)\}_{t\geq 0}$ be a continuous semigroup on X. Furthermore, we also assume that $S(t)X \subset L^p(\mathbb{R}^N)$ for some $1 \leq p < \infty$ [the nested relation between X and $L^p(\mathbb{R}^N)$ is unknown]. Then $\{S(t)\}_{t\geq 0}$ has a $(X, L^p(\mathbb{R}^N))$-global attractor provided that the following conditions hold:*

(i) $\{S(t)\}_{t\geq 0}$ *has a $(X, L^p(\mathbb{R}^N))$-bounded absorbing set $B_0 \subset L^p(\mathbb{R}^N)$*;
(ii) *there is a $q(1 \leq q \leq p)$ such that $\{S(t)\}_{t\geq 0}$ is $(X, L^q(\mathbb{R}^N))$-asymptotically compact*;
(iii) *for any $\varepsilon > 0$ and any bounded subset $B \subset X$, there exist positive constants $M = (\varepsilon, B)$ and $T = T(\varepsilon, B)$, such that*

$$\int_{\Omega(|S(t)u_0|\geq M)} |S(t)u_0|^p < \varepsilon \quad \textit{for any } u_0 \in B \textit{ and } t \geq T. \tag{2.1}$$

With the usual notation, hereafter let $|u|$ be the norm of $L^2(\Omega)$, $|\cdot|_p$ be the norm of $L^p(\Omega)$. Let C the arbitrary positive constant, which may be different from line to line and even in the same line.

3. Bounded Absorbing Set

3.1. Well-Posedness

Using the Galerkin approximation method [4,19,26], and similar to the proof of Theorem 3.1 in [17], we can get the following result easily.

Theorem 3.1. *Under the assumptions of (1.4)–(1.7), then for any $T > 0$ and $u_0 \in H^1(\Omega) \cap L^p(\Omega)$, there is a unique solution u of (1.1)–(1.3) such that*

$$u \in \mathcal{C}^1([0, T]; H^1(\Omega)) \cap L^p(0, T; L^p(\Omega)).$$

Moreover, the solution continuously depends on the initial data.

According to Theorem 3.1 above, and let $S(t)u_0 = u(t)$, $S(t)\colon H^1(\Omega) \cap L^p(\Omega) \to H^1(\Omega) \cap L^p(\Omega)$ is a $\mathcal{C}^0$ semigroup.

3.2. Bounded Absorbing Set

Now, we construct the existence of the absorbing set in $H^1(\mathbb{R}^N)\cap L^{p+1}(\mathbb{R}^N)$ for the semigroup generated by the Eq. (1.1).

Lemma 3.1. *Under the assumptions of* (1.4)–(1.7), *there is a positive constant* ρ *such that for any bounded subset* $B\in\mathcal{B}(H^1(\mathbb{R}^N)\cap L^{p+1}(\mathbb{R}^N))$, *there exists* $T_1>0$ *such that*

$$|u(t)|^2+|\nabla u(t)|^2+|u(t)|_p^p\le\rho^2,\quad \textit{for all } t\ge T_1 \textit{ and } u_0\in B.$$

Proof. Multiplying (1.1) by u, using (1.4), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(|\nabla u|^2+|u|^2)+|\nabla u|^2+C|u(t)|_p^p\le(g(x),u)+C. \tag{3.1}$$

By the Poincaré inequality, for some $\nu>0$, there holds

$$\frac{\mathrm{d}}{\mathrm{d}t}(|\nabla u|^2+|u|^2)+\nu(|\nabla u|^2+|u|^2)+C(|\nabla u|^2+|u|^2+|u(t)|_p^p)\le C. \tag{3.2}$$

In particular, we infer

$$\frac{\mathrm{d}}{\mathrm{d}t}(|\nabla u|^2+|u|^2)+\nu(|\nabla u|^2+|u|^2)\le C. \tag{3.3}$$

By the Gronwall lemma, we get

$$|\nabla u(t)|^2+|u(t)|^2\le e^{-\nu t}(|\nabla u(0)|^2+|u(0)|^2)+C. \tag{3.4}$$

Now, integrating (3.2) from s to $s+1$, by virtue of (3.4), we obtain

$$\int_s^{s+1}(|\nabla u(t)|^2+|u(t)|^2+|u(t)|_p^p)\le C. \tag{3.5}$$

According to (1.7), we get

$$C_1|u|_p^p-C_2\le\int\Phi(u)\mathrm{d}x\le C_3|u|_p^p+C_4. \tag{3.6}$$

Combining (3.5) and (3.6), we obtain

$$\int_s^{s+1}\left(|\nabla u(t)|^2+|u(t)|^2+2\int\Phi(u)\mathrm{d}x\right)\le C. \tag{3.7}$$

On the other hand, multiplying (1.1) by u_t, we infer

$$2|u_t|^2+2|\nabla u_t|^2+\frac{\mathrm{d}}{\mathrm{d}t}\left(|\nabla u|^2+2\int\Phi(u)\mathrm{d}x\right)=2(g(x),u_t). \tag{3.8}$$

Noting that $2|(g(x),u_t)|\le|g(x)|^2+|u_t|^2$, we infer

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(|\nabla u|^2+2\int\Phi(u)\mathrm{d}x\right)\le C. \tag{3.9}$$

By virtue of (3.5), using the Poincaré inequality and the uniform Gronwall inequality, we obtain

$$|\nabla u|^2+|u|^2+2\int\Phi(u)\mathrm{d}x\le C. \tag{3.10}$$

By (3.6), there exists $T_1=T_1(B)$, such that for all $t\ge T_1$, we infer

$$|u(t)|^2+|\nabla u(t)|^2+|u(t)|_p^p\le\rho^2. \tag{3.11}$$

This completes the proof. □

According to *Lemma* 3.1, we know that the semigroup of operators $\{S(t)\}_{t\geq 0}$ has an $(H^1(\Omega)\cap L^p(\Omega), H^1(\Omega)\cap L^p(\Omega))$-bounded absorbing set $\mathcal{B}$.

Remark 3.2. The family of $(H^1(\Omega)\cap L^p(\Omega), H^1(\Omega)\cap L^p(\Omega))$-bounded absorbing set $\mathcal{B}$ is also $(H^1(\Omega)\cap L^p(\Omega), L^2(\Omega))$, $(H^1(\Omega)\cap L^p(\Omega), L^p(\Omega))$, $(H^1(\Omega)\cap L^p(\Omega), H^1(\Omega))$-bounded absorbing set for the semigroup of operators $\{S(t)\}_{t\geq 0}$.

4. Global Attractors

4.1. Norm-to-Weak Continuous Semigroup

Lemma 4.1 [32]. *Let X, Y be two Banach spaces, and X^*, Y^* be their dual spaces, respectively. We also assume that X is a dense subspace of Y, the injection $i\colon X\to Y$ is continuous and its adjoint $i^*\colon Y\to X$ is densely injective. $\{S(t)\}_{t\geq 0}$ be a semigroup on X and Y, respectively, and assume furthermore that $\{S(t)\}_{t\geq 0}$ is continuous or weak continuous on Y. Then $\{S(t)\}_{t\geq 0}$ is a norm-to-weak continuous semigroup on X if and only if $\{S(t)\}_{t\geq 0}$ maps compact subsets of $X\times\mathbb{R}^+$ into bounded sets of X.*

According to the fact that $\{S(t)\}_{t\geq 0}$ is continuous in $H^1(\Omega)$ and $L^2(\Omega)$, by the above lemma, we deduce that $\{S(t)\}_{t\geq 0}$ is norm-to-weak continuous in $L^p(\Omega)$. It is well known that the continuity of the semigroup can guarantee the invariance of the global attractor, e.g., see [16,32].

4.2. $(H^1(\Omega)\cap L^p(\Omega), L^2(\Omega))$-Global Attractor

The aim of this section is to establish the existence of the $(H^1(\Omega)\cap L^p(\Omega)$, $L^2(\Omega))$-global attractor.

Lemma 4.2. *For any $\varepsilon>0$, and any $B\in\mathcal{B}(H^1(\Omega)\cap L^p(\Omega))$, there exist $T_2>0$ such that*

$$|u_t(t)|^2+|\nabla u_t(t)|^2\leq C, \tag{4.1}$$

for all $t\geq T_2$, $u_0\in B$, where $u_t(s)=\frac{\mathrm{d}}{\mathrm{d}t}(S(t)u_0)|_{t=s}$.

Proof. By differentiating Eq. (1.1) with respect to t, we have

$$u_{tt}-\Delta u_{tt}-\Delta u_t+\varphi'(u)u_t=0. \tag{4.2}$$

Multiplying (4.2) with u_t and using (1.6), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(|\nabla u_t|^2+|u_t|^2)+2|\nabla u_t|^2\leq 2c_2|u_t|. \tag{4.3}$$

By the Young inequality, we infer

$$\frac{\mathrm{d}}{\mathrm{d}t}(|\nabla u_t|^2+|u_t|^2)\leq C(|\nabla u_t|^2+|u_t|^2). \tag{4.4}$$

In view of (3.8) and (3.10), we have

$$\int_s^{s+1}(|\nabla u_t|^2+|u_t|^2)\leq C. \tag{4.5}$$

Using the uniform Gronwall inequality, we complete the proof.

We now establish the following skillful estimates, and these estimates are crucial for proving the asymptotic compactness. □

Lemma 4.3. *Under the assumptions of* (1.4)–(1.7)*, for any* $u_0 \in H^1(\Omega) \cap L^p(\Omega)$ *and* $\varepsilon > 0$*, there are some* $T_3 > 0$ *and* $K > 0$ *such that*

$$\int_{|x|\geq 2k} (|u(t)|^2 + |\nabla u(t)|^2) \leq C\varepsilon, \tag{4.6}$$

whenever $k \geq K$ *and* $t \geq T_3$.

Proof. Choose a smooth function $\varrho(x)$ with

$$\varrho(x) = \begin{cases} 0, & 0 \leq s \leq 1, \\ 1, & s \geq 2, \end{cases} \tag{4.7}$$

where $0 \leq \varrho(s) \leq 1, 1 \leq s \leq 2$, and there is a constant c such that $|\varrho'(s)| \leq c$.

Multiplying (1.1) with $\varrho^2(\frac{|x|^2}{k^2})u$ and integrating on Ω, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \varrho^2\left(\frac{|x|^2}{k^2}\right)(|\nabla u|^2 + |u|^2)\mathrm{d}x - \int \varrho^2\left(\frac{|x|^2}{k^2}\right) u\Delta u \mathrm{d}x \\ &= -\int \varrho^2\left(\frac{|x|^2}{k^2}\right)\varphi(u)u\mathrm{d}x + \int \frac{4x}{k^2}\varrho\left(\frac{|x|^2}{k^2}\right)\varrho'\left(\frac{|x|^2}{k^2}\right)u\nabla u_t \mathrm{d}x \\ &\quad + \int \varrho^2\left(\frac{|x|^2}{k^2}\right) ug\mathrm{d}x. \end{aligned} \tag{4.8}$$

Noting that

$$\begin{aligned} \int_\Omega \varrho^2\left(\frac{|x|^2}{k^2}\right) u\Delta u\mathrm{d}x = &-\int_\Omega \varrho^2\left(\frac{|x|^2}{k^2}\right)|\nabla u|^2\mathrm{d}x \\ &- \int_\Omega \frac{4x}{k^2}\varrho\left(\frac{|x|^2}{k^2}\right)\varrho'\left(\frac{|x|^2}{k^2}\right)u\nabla u\mathrm{d}x. \end{aligned} \tag{4.9}$$

According to the condition $|\varrho'(s)| \leq c$ and the existence of a bounded absorbing set in $H^1(\Omega) \cap L^p(\Omega)$ for $t \geq t_*$, it follows that

$$\begin{aligned} &\left|\int_\Omega \frac{4x}{k^2}\varrho\left(\frac{|x|^2}{k^2}\right)\varrho'\left(\frac{|x|^2}{k^2}\right)u\nabla u\mathrm{d}x\right| \\ &= \left|\int_{k\leq|x|\leq\sqrt{2}k} \frac{4x}{k^2}\varrho\left(\frac{|x|^2}{k^2}\right)\varrho'\left(\frac{|x|^2}{k^2}\right)u\nabla u\mathrm{d}x\right| \\ &\leq \frac{4\sqrt{2}}{k}\int_{k\leq|x|\leq\sqrt{2}k}\varrho^2\left(\frac{|x|^2}{k^2}\right)|u||\nabla u|\mathrm{d}x \\ &\leq \frac{C}{k}(|u|^2 + |\nabla u|^2) \\ &\leq \frac{C_\rho}{k}, \end{aligned} \tag{4.10}$$

where C_ρ is independent of k.

Similarly to (4.10), using Lemma 4.2, we infer

$$\left|\int_\Omega \frac{4x}{k^2}\varrho\left(\frac{|x|^2}{k^2}\right)\varrho'\left(\frac{|x|^2}{k^2}\right)u\nabla u_t \mathrm{d}x\right| \leq \frac{C}{k}(|u|^2+|\nabla u_t|^2) \leq \frac{C}{k}, \quad (4.11)$$

where C is independent of k.

Using (1.4), and noting that $\psi_1(x)\in L^1(\Omega)$, there exits $k_2(\varepsilon)\geq k_1(\varepsilon)$ such that

$$\begin{aligned} -\int_\Omega \varrho^2\left(\frac{|x|^2}{k^2}\right)\varphi(u)u\mathrm{d}x &\leq -\int_\Omega \varrho^2\left(\frac{|x|^2}{k^2}\right)|u|^p\mathrm{d}x + \int_\Omega \varrho^2\left(\frac{|x|^2}{k^2}\right)\psi_1(x)\mathrm{d}x \\ &\leq \int_{|x|\geq k}\psi_1(x)\mathrm{d}x \\ &\leq C\varepsilon. \end{aligned} \quad (4.12)$$

From the assumption $g(x)\in L^2(\Omega)$, provided $k_3(\varepsilon)\geq k_2(\varepsilon)$, such that

$$\int_{|x|\geq k}|g|^2\mathrm{d}x \leq \varepsilon C. \quad (4.13)$$

Thus combining with (4.8)–(4.13), provided $k_4(\varepsilon)>k_3(\varepsilon)$, using the Poincaré inequality, for some $\delta>0$, we infer

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \varrho^2\left(\frac{|x|^2}{k^2}\right)(|\nabla u|^2+|u|^2)\mathrm{d}x + \delta\int_\Omega \varrho^2\left(\frac{|x|^2}{k^2}\right)(|\nabla u|^2+|u|^2)\mathrm{d}x \leq C\varepsilon. \quad (4.14)$$

Thus, using the Gronwall lemma, we get

$$\int_{|x|\geq 2k}(|u(t)|^2+|\nabla u(t)|^2)\mathrm{d}t \leq C\varepsilon, \quad (4.15)$$

provided $T\geq T_3$ and $k\geq K$, this completing the proof. □

Theorem 4.1. *Under the assumptions of* (1.4)–(1.7)*, then the semigroup* $\{S(t)\}_{t\geq 0}$ *associated with the initial value problem* (1.1) *and* (1.3) *has an* $(H^1(\Omega)\cap L^p(\Omega), L^2(\Omega))$*-global attractor* $\mathcal{A}$.

Proof. According to Remark 3.2, the semigroup of operators $\{S(t)\}_{t\geq 0}$ has an $(H^1(\Omega)\cap L^p(\Omega), L^2(\Omega))$-bounded absorbing set. We only need to show that $\{S(t)\}_{t\geq 0}$ is $(H^1(\Omega)\cap L^p(\Omega), L^2(\Omega))$-asymptotic compact.

Let $\{u_{t_k}\}\subset B\in \mathcal{B}(H^1(\Omega)\cap L^p(\Omega))$, and $t_k\to\infty$. For a given $K>0$, denote by

$$\Omega_K=\{x\colon |x|\leq K\} \quad \textit{and} \quad \Omega_K^c=\{x\colon |x|>K\}. \quad (4.16)$$

According to Lemma 4.4, for $\varepsilon>0$, there exist $K>0$,$T_3>0$, such that for $t\geq T_3$,

$$|S(t)u_0|_{L^2(\Omega_K^c)}\leq \varepsilon. \quad (4.17)$$

Noting that $t_k\to\infty$, there exists $N_1\in\mathbb{N}$ such that for all $n\geq N_1$ and $t\geq T_3$, we obtain

$$|S(t_k)u_{t_k}|_{L^2(\Omega_K^c)}\leq \varepsilon. \quad (4.18)$$

On the other hand, by Lemma 3.1, there exist $\rho>0$, and $N_2\in\mathbb{N}$ such that for all $n\geq N_1$, we get

$$|S(t_k)u_{t_k}|_{H^1(\Omega_K^c)}\leq \rho. \quad (4.19)$$

Noting that Ω_K is a bounded domain, by the compactness of embedding $H^1(\Omega_K) \subset L^2(\Omega_K)$, we know that the sequence $\{S(t_k)u_{t_k}\}$ is precompact in $L^2(\Omega_K)$. Hence, for any given $\varepsilon > 0$, $\{S(t_k)u_{t_k}\}$ has a finite covering in $L^2(\Omega_K)$ of balls of radius less than ε. Combining with (4.18), we know that $\{S(t_k)u_{t_k}\}$ has a finite covering in $L^2(\Omega)$ of balls of radius less than ε. This implies that $\{S(t_k)u_{t_k}\}$ is precompact in $L^2(\Omega)$.

This completes the proof. □

4.3. $(H^1(\Omega) \cap L^p(\Omega), L^p(\Omega))$-Global Attractor

In this section, we want to obtain the existence of the $(H^1(\Omega)\cap L^p(\Omega), L^p(\Omega))$-global attractor.

Lemma 4.4. *For any $\varepsilon > 0$, and any $B \in \mathcal{B}(H^1(\Omega) \cap L^p(\Omega))$, there exist $M > 0$, $T_4 > 0$ such that*

$$\mathrm{mes}(\Omega(|S(t)u_0| \geq M)) \leq \varepsilon, \quad \forall u_0 \in B,\ t \geq T_4,$$

where $\Omega(|S(t)u_0| \geq M) = \{x \in \Omega\colon |u(t,x)| \geq M\}$ and mes is the Lebesgue measure.

Proof. Noting that $\mathcal{B}$ is an $H^1(\Omega) \cap L^p(\Omega)$ bounded absorbing set for $\{S(t)\}_{t\geq 0}$, we know that there exists a positive constant M, such that for any bounded subset B of $(H^1(\Omega) \cap L^p(\Omega), L^2(\Omega))$, we can find a constant C, such that $|u|^2 \leq C$. Hence, we have

$$C \geq \int_\Omega |S(t)u_0|^2 \geq \int_{\Omega(|S(t)u_0|\geq M)} |S(t)u_0|^2 \geq M^2 \mathrm{mes}(\Omega(|S(t)u_0| \geq M)). \tag{4.20}$$

Thus, if we choose M large enough, we can obtain

$$\mathrm{mes}(\Omega(|S(t)u_0| \geq M)) \leq \varepsilon.$$

This completes the proof.

Now, we borrow the asymptotic a priori estimate method of Zhong et al. [32] to prove the following the lemma. □

Lemma 4.5. *For any $\varepsilon > 0$, and any $B \in \mathcal{B}(H^1(\Omega) \cap L^p(\Omega))$, there exist $M > 0$, $T_5 > 0$ such that*

$$\int_{\Omega(|u(t)|\geq M)} (|u(t) - M|^2 + |\nabla u(t)|^2) \leq \varepsilon, \tag{4.21}$$

for all $t \geq T_5$, $u_0 \in D$.

Proof. We denote

$$(u - M)^+ = \begin{cases} u - M & u \geq M, \\ 0, & u \leq M. \end{cases} \tag{4.22}$$

According to (1.4) and Lemma 4.4, we can choose M large enough to have $\int_\Omega \varphi(u)(u - M)^+ \geq 0$, and $\mathrm{mes}(\Omega(u \geq M)) < +\infty$.

Multiplying (1.1) with $(u-M)^+$, we infer

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\Omega(u\geq M)}|\nabla u|^2+\int_{\Omega(u\geq M)}|u-M|^2\right)+\int_{\Omega(u\geq M)}|\nabla u|^2$$
$$\leq \int_\Omega g(t)(u-M)^+ \leq \frac{1}{2}\int_{\Omega(u\geq M)}|g(x)|^2+\frac{1}{2}\int_{\Omega(u\geq M)}(u-M)^2,$$

now, for some $\nu>0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\Omega(u\geq M)}|\nabla u|^2+\int_{\Omega(u\geq M)}|u-M|^2\right)$$
$$+\nu\left(\int_{\Omega(u\geq M)}|\nabla u|^2+\int_{\Omega(u\geq M)}|u-M|^2\right)$$
$$\leq \int_{\Omega(u\geq M)}|g(x)|^2. \tag{4.23}$$

By the Gronwall lemma, and noting that $\mathrm{mes}(\Omega(|S(t)u_0|\geq M))\leq\varepsilon$ if we choose M large enough (Lemma 4.4), for $t\geq T_5$, we get

$$\int_{\Omega(u\geq M)}|\nabla u|^2+\int_{\Omega(u\geq M)}|u-M|^2\leq\varepsilon. \tag{4.24}$$

Similarly, replacing $(u-M)^+$ by

$$(u+M)^-=\begin{cases}u+M, & u\leq -M,\\ 0, & u\geq -M.\end{cases} \tag{4.25}$$

we get

$$\int_{\Omega(u\leq -M)}|\nabla u|^2+\int_{\Omega(u\leq -M)}|u-M|^2\leq\varepsilon. \tag{4.26}$$

This completes the proof. □

Theorem 4.2. *Under the assumptions of* (1.4)–(1.7)*, then the semigroup* $\{S(t)\}_{t\geq 0}$ *associated with the initial value problem* (1.1) *and* (1.2) *has an* $(H^1(\Omega)\cap L^p(\Omega), L^p(\Omega))$*-global attractor* $\mathcal{A}$*.*

Proof. According to Lemma 2.2, Remark 3.2 and Theorem 4.1, we only need to show that for any $\varepsilon>0$ and any bounded subset $B\subset H^1(\Omega)\cap L^p(\Omega)$, there exist positive constants $M>0$ and T, such that

$$\int_{\Omega(|S(t)u_0|\geq M)}|S(t)u_0|^p<\varepsilon \quad \textit{for any } u_0\in B \textit{ and } t\geq T. \tag{4.27}$$

Now, multiplying (1.1) with $(u-M)^+$, we infer

$$\int_{\Omega(u\geq M)}u_t(u-M)+\int_{\Omega(u\geq M)}\nabla u_t\nabla u+\int_{\Omega(u\geq M)}|\nabla u|^2+\int_\Omega\varphi(u)(u-M)^+$$
$$=\int_\Omega g(x)(u-M)^+. \tag{4.28}$$

Note that

$$\int_\Omega \varphi(u)(u-M)^+ \geq C\int_{\Omega(u\geq 2M)} |u|^p - C\mathrm{mes}(\Omega(u\geq 2M)),$$

if we choose M large enough. Thus, we get

$$\begin{aligned} C\int_{\Omega(u\geq 2M)} |u|^p &\leq \int_{\Omega(u\geq M)} |u_t||(u-M)| + \int_{\Omega(u\geq M)} |\nabla u_t||\nabla u| \\ &\quad + \int_\Omega g(x)(u-M)^+ + C\mathrm{mes}(\Omega(u\geq 2M)) \\ &\leq |u_t|\left(\int_{\Omega(u\geq M)} |u-M|^2\right)^{\frac{1}{2}} + |\nabla u_t|\left(\int_{\Omega(u\geq M)} |\nabla u|^2\right)^{\frac{1}{2}} \\ &\quad + |g(x)|\left(\int_{\Omega(u\geq M)} |u-M|^2\right)^{\frac{1}{2}} + C\mathrm{mes}(\Omega(u\geq 2M)). \end{aligned} \tag{4.29}$$

Combining Lemmas 4.2 and 4.5, there exist positive constants $M>0$ and T, such that

$$\int_{\Omega(u\geq 2M)} |u|^p < \varepsilon. \tag{4.30}$$

Similarly, replacing $(u-M)^+$ by $(u+M)^-$, we obtain

$$\int_{\Omega(u\leq -2M)} |u|^p < \varepsilon. \tag{4.31}$$

This completes the proof. □

4.4. $(H^1(\Omega)\cap L^p(\Omega), H^1(\Omega))$-Global Attractor

In this section, our aim is to obtain the existence of the $(H^1(\Omega)\cap L^p(\Omega), H^1(\Omega))$-global attractor. The following lemma is the key of this section.

Lemma 4.6. *Assume that the semigroup $\{S(t)\}_{t\geq 0}$ associated with the initial value problem (1.1)–(1.3) has an $(H^1(\Omega)\cap L^p(\Omega), L^r(\Omega))$-global attractor $\mathcal{A}$, where $2\leq r<\infty$. Then for any $\varepsilon>0$, and any $B\in\mathcal{B}(H^1(\Omega)\cap L^p(\Omega))$, there exists $m\in\mathbb{N}$, $T_7>0$ such that*

$$\int_\Omega |(I-P_m)S(t)u_0|^r \leq \varepsilon, \tag{4.32}$$

for all $t\geq T_7$, $u_0\in B$ and $m\geq m_0$, where P_m is the canonical projection of $L^r(\Omega)$ onto an m-dimensional subspace.

Proof. Noting that $\mathcal{A}$ is an $(H^1(\Omega)\cap L^p(\Omega), L^r(\Omega))$-global attractor for the semigroup $\{S(t)\}_{t\geq 0}$, then for any $\varepsilon>0$, and any $B\in\mathcal{B}(H^1(\Omega)\cap L^p(\Omega))$, there exists $T_7>0$ such that

$$\bigcup_{t\geq T_7} S(t)D\subset\mathcal{N}(A,\varepsilon,L^r),$$

where $A \subset \mathcal{A}$, and $\mathcal{N}(A, \varepsilon, L^r)$ is the ε neighborhood of A in $L^r(\Omega)$. Since A is compact in $L^r(\Omega)$, there exist $k \in \mathbb{N}$ and $v_i \in L^r(\Omega)$, $i = 1, \ldots, k$ such that

$$\bigcup_{t \geq T_7} S(t)D \subset \bigcup_{i=1}^{k} \mathcal{N}(A, \varepsilon, v_i).$$

For each v_i, there is an m_i such that

$$\int_\Omega |(I - P_{m_i})v_i|^r \leq \varepsilon.$$

Taking $m_0 = \max\{m_1, \ldots, m_n\}$, for $t \geq T_7$ and $u_0 \in B$, there exist v_i such that

$$\begin{aligned}\int_\Omega |(I - P_{m_0})S(t)u_0|^r &= \int_\Omega |(I - P_{m_0})S(t)u_0 - (I - P_{m_0})v_i + (I - P_{m_0})v_i|^r \\ &\leq C \int_\Omega |(I - P_{m_0})S(t)u_0 - (I - P_{m_0})v_i|^r + C \int_\Omega |(I - P_{m_0})v_i|^r \\ &\leq C\varepsilon. \end{aligned} \tag{4.33}$$

This completes the proof. □

Theorem 4.3. *Under the assumptions of* (1.4)–(1.7)*, then the semigroup* $\{S(t)\}_{t \geq 0}$ *associated with the initial value problem* (1.1) *and* (1.2) *has an* $(H^1(\Omega) \cap L^p(\Omega), H^1(\Omega))$*-global attractor* $\mathcal{A}$.

Proof. Noting that $H^1(\Omega)$ is separable, we can choose $\{\omega_i\}_{i=1}^\infty$ which form the basis in $H^1(\Omega)$. Now, we set $H_m = \text{span}\{\omega_1, \omega_2, \ldots, \omega_m\}$, and P_m is the orthogonal projection onto H_m. For any $u \in H^1(\Omega)$, $u = P_m u + (I - P_m)u \triangleq u_1 + u_2$.

Taking the inner product of (1.1) with u_2, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(|\nabla u_2|^2 + |u_2|^2) + |\nabla u_2|^2 + (\varphi(u), u_2) = (g(x), u_2). \tag{4.34}$$

Hence, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(|\nabla u_2|^2 + |u_2|^2) + 2|\nabla u_2|^2 \leq 2|f(u)|_{\frac{p}{p-1}}|u_2|_p + C|u_2|. \tag{4.35}$$

By (1.5) and Lemma 3.1, we obtain that

$$\begin{aligned}|f(u)|_{\frac{p}{p-1}} &\leq C\left(|u|_p^{p-1} + |\psi_2|_{\frac{p}{p-1}}^{\frac{p}{p-1}}\right) \\ &\leq C_\rho. \end{aligned} \tag{4.36}$$

According to Lemmas 4.2 and 4.6, and using (4.36), we know that, for any $\varepsilon > 0$, there exist $m \in \mathbb{N}$, $T_8 > 0$ such that

$$|f(u)|_{\frac{p}{p-1}}|u_2|_p \leq C\varepsilon. \tag{4.37}$$

for all $t \geq T_8$ and $m \geq m_0$. Now, using the Poincaré inequality, for some $\nu > 0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(|\nabla u_2|^2 + |u_2|^2) + \nu(|\nabla u_2|^2 + |u_2|^2) \leq \varepsilon. \tag{4.38}$$

By the Gronwall inequality, for all $t \geq T_8$ and $m \geq m_0$, we obtain

$$|\nabla u_2(t)|^2 + |u_2(t)|^2 \leq \varepsilon. \tag{4.39}$$

This completes the proof. □

4.5. $(H^1(\Omega) \cap L^p(\Omega), H^1(\Omega) \cap L^p(\Omega))$-Global Attractor

Theorem 4.4. *Under the assumptions of* (1.4)–(1.7)*, then the semigroup* $\{S(t)\}_{t\geq 0}$ *associated with the initial value problem* (1.1) *and* (1.3) *has an* $(H^1(\Omega) \cap L^p(\Omega), H^1(\Omega) \cap L^p(\Omega))$*-global attractor* $\mathcal{A}$*.*

Proof. By Lemma 3.1, Theorems 4.2 and 4.3, we complete the proof. □

Acknowledgements

The authors express their sincere thanks to the anonymous reviewer for his/her careful reading of the paper, providing the idea to write the appendix, and giving helpful linguistic and mathematical comments. The authors also thank the editors for their kind help.

Appendix: Local Solvability of (1.1)

Note that Eq. (1.1) can be written as

$$\begin{aligned} u_t &= (I - \Delta)^{-1}[\Delta u - u + u - \phi(u) + g(x)] \\ &= -u + (I - \Delta)^{-1}[u - \phi(u) + g(x)]. \end{aligned} \tag{4.40}$$

Moreover, the inverse operator $(I-\Delta)^{-1}$ is a sectorial positive operator, and it has nice regularizing properties (e.g., [8,10]). Similar to the arguments in [7,8,10], we know that the solution of (1.1) satisfying the following Cauchy's integral formula:

$$u(t) = e^{-t}u_0 + \int_0^t e^{-t-s}((I - \Delta)^{-1}[u - \phi(u) + g(x)]).$$

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Received: November 3, 2014.
Accepted: July 12, 2015.