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Rectifying Curves in the Three-Dimensional Hyperbolic Space

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Abstract. B. Y. Chen introduced rectifying curves in $\mathbb{R}^3$ as space curves whose position vector always lies in its rectifying plane. Recently, the authors have extended this definition (as well as several results about rectifying curves) to curves in the three-dimensional sphere. In this paper, we study rectifying curves in the three-dimensional hyperbolic space, and obtain some results of characterization and classification for such kind of curves. Our results give interesting and significant differences between hyperbolic, spherical and Euclidean geometries.

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1. Introduction

Given a unit speed space curve x in $\mathbb{R}^3$, its Frenet frame $\{T = x', N, B = T \times N\}$ satisfies the well-known Frenet–Serret equations: $T' = \kappa N$, $N' = -\kappa T + \tau B$, and $B' = -\tau N$, where the functions $\kappa > 0$ and τ are called the curvature and torsion of the curve, respectively. At each point of the curve, the planes spanned by $\{T, N\}$, $\{T, B\}$ and $\{N, B\}$ are called the osculating plane, the rectifying plane, and the normal plane, respectively. Chen proposed in [2] the geometric question of what are the space curves whose position vector always lies in its rectifying plane; for simplicity, he called such a curve a rectifying curve. So, in general, a space curve x is said to be a rectifying curve if there exists a point p in $\mathbb{R}^3$ such that

$$x(s) - p = \lambda(s)T(s) + \mu(s)B(s), \tag{1}$$

for some functions λ and μ. The double interpretation of the space $\mathbb{R}^3$ (on one hand, as the differentiable manifold where the curve lies, and, on the other hand, as the tangent vector space at any point of the curve) is precisely what makes the above equation make sense. Since the paper by Chen, many authors have extended the notion of rectifying curve to other ambient spaces (of dimension $n \geq 3$), endowed with a Riemannian or pseudo-Riemannian metric (see, e.g., [4–10,14]). In all cases, the tangent vector spaces can be identified with the manifold, and this identification turns out to be crucial to make a study analogous to that made by Chen. To extend this concept to other ambient spaces, it is necessary, however, to distinguish between the manifold and its tangent vector spaces. The key property of Eq. (1) is that the straight line that connects $x(s)$ with the point p is orthogonal to the principal normal line [i.e., the line starting at $x(s)$ in the direction of $N(s)$]. This idea was used by the authors to define rectifying curves in the three-dimensional sphere $\mathbb{S}^3(r)$ [13]. In this paper, we study rectifying curves in the three-dimensional hyperbolic space $\mathbb{H}^3(-r)$, and obtain several results of characterization and classification for that family of curves.

The paper is organized as follows. After a section devoted to some basic preliminaries about the geometry of the hyperbolic space $\mathbb{H}^3(-r) \subset \mathbb{R}^4_1$, we introduce the concept of rectifying curve (Definition 1). In Sect. 3, we prove that a curve γ in $\mathbb{H}^3(-r)$ is a rectifying curve if and only if γ is a geodesic of a conical surface. In Sect. 4, we present a nice characterization for rectifying curves. In fact, we prove that a unit speed curve $\gamma = \gamma(s)$ is a rectifying curve if and only if the ratio of torsion and curvature is given by $(\tau/\kappa)(s) = c_1 \sinh((s+s_0)/r) + c_2 \cosh((s+s_0)/r)$, for some constants c_1, c_2 and s_0, with $1 - c_1^2 + c_2^2 < 0$. In Sect. 5, we study the rectifying developable of a curve and show that the rectifying developable of γ is a conical surface in $\mathbb{H}^3(-r)$ if and only if γ is a rectifying curve in $\mathbb{H}^3(-r)$. In Sect. 6, we present a classification result for rectifying curves in the hyperbolic space. In fact, we prove that a curve $\gamma(t) = \exp_p(\rho(t)V(t))$ in $\mathbb{H}^3(-r)$ is a rectifying curve if and only if $\rho(t) = r \operatorname{arg}\tanh(a \sec(t+t_0))$, for some constants a and t_0, with $0 < a^2 < 1$. Finally, in Sect. 7, we show that a rectifying curve $\gamma = \exp_p(\rho V)$ in $\mathbb{H}^3(-r)$ is characterized by the property that a certain function (depending on its speed v, κ and ρ) takes its minimum value, equal to k_V^2, among the hyperbolic curves with the same spherical projection V (k_V denotes the geodesic curvature of V). A similar result, which does not appear in [13], is stated for rectifying curves in the three-dimensional sphere $\mathbb{S}^3(r)$.

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2. Setup and Basic Preliminaries

Let $\mathbb{R}^4_1$ be the four-dimensional Lorentz–Minkowski space with the standard flat metric g given by

$$g = -\mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2 + \mathrm{d}x_4^2,$$

where $x = (x_1, x_2, x_3, x_4)$ is a rectangular coordinate system of $\mathbb{R}^4_1$. For a positive number r and a point $c \in \mathbb{R}^4_1$, we denote by $\mathbb{H}^3(c, -r)$ the (connected) hyperbolic space centered at c with radius r, which is standardly embedded in $\mathbb{R}^4_1$ by

$$\mathbb{H}^3(c, -r) = \{x \in \mathbb{R}^4_1 | \, \langle x - c, x - c\rangle = -r^2, \text{ and } x_1 > 0\},$$

where $\langle , \rangle$ denotes the Lorentzian inner product on $\mathbb{R}^4_1$. To simplify the notation, we write $\mathbb{H}^3(-r) \equiv \mathbb{H}^3(0,-r)$ and $\mathbb{H}^3 \equiv \mathbb{H}^3(0,-1)$. We will also use $\langle , \rangle$ to denote the flat metric g. Without loss of generality, we assume that $c = 0$.

Let $\overline{\nabla}$ and ∇^0 denote the Levi-Civita connections on $\mathbb{H}^3(-r)$ and $\mathbb{R}^4_1$, respectively. If X and Y are vector fields tangent to $\mathbb{H}^3(-r)$, then $\overline{\nabla}$ and ∇^0 are related by the Gauss formula as follows

$$\nabla^0_X Y = \overline{\nabla}_X Y + \frac{1}{r^2} \langle X, Y \rangle \phi,$$

where $\phi \colon \mathbb{H}^3(-r) \to \mathbb{R}^4_1$ denotes the position vector.

In the hyperbolic space $\mathbb{H}^3(-r)$, we can define a cross product as follows. Consider a point $q \in \mathbb{H}^3(-r)$ and take two tangent vectors $v_1, v_2 \in T_q\mathbb{H}^3(-r)$. The cross product of v_1 and v_2 is the unique tangent vector $v_1 \times v_2$ in $T_q\mathbb{H}^3(-r)$ such that

$$\langle v_1 \times v_2, w \rangle = \frac{1}{r} \det(v_1, v_2, w, q), \quad \forall w \in T_q\mathbb{H}^3(-r),$$

where $\langle , \rangle$ denotes the induced metric on $\mathbb{H}^3(-r)$ and the vectors are considered as column vectors in $\mathbb{R}^4_1$.

Let us consider a unit speed curve $\gamma \colon I \to \mathbb{H}^3(-r)$, where I is a real open interval, and assume that γ is not a geodesic curve. It is well known that there exists a moving frame $\{T_\gamma = \gamma', N_\gamma, B_\gamma = T_\gamma \times N_\gamma\}$ (called the Frenet frame), and two functions $\kappa_\gamma > 0$ and τ_γ, such that

$$\nabla^0_{T_\gamma} T_\gamma = \kappa_\gamma N_\gamma + \frac{1}{r^2}\gamma, \quad \nabla^0_{T_\gamma} N_\gamma = -\kappa_\gamma T_\gamma + \tau_\gamma B_\gamma, \quad \nabla^0_{T_\gamma} B_\gamma = -\tau_\gamma N_\gamma. \tag{2}$$

For any point $\gamma(s)$ in the curve γ, the *principal normal geodesic* in $\mathbb{H}^3(-r)$ starting at $\gamma(s)$ is defined as the geodesic curve parametrized by

$$t \to \exp_{\gamma(s)}(tN_\gamma(s)) = \cosh\left(\frac{t}{r}\right)\gamma(s) + r\sinh\left(\frac{t}{r}\right)N_\gamma(s), \quad t \in \mathbb{R}.$$

Now, we introduce the notion of rectifying curve in the three-dimensional hyperbolic space.

Definition 1. A unit speed curve $\gamma = \gamma(s)$ $(s \in I)$ in $\mathbb{H}^3(-r)$, with $\kappa_\gamma > 0$, is said to be a *rectifying curve* if there exists a point $p \in \mathbb{H}^3(-r)$ such that $p \notin \mathrm{Im}(\gamma) \equiv \gamma(I)$ and the geodesics connecting p with $\gamma(s)$ are orthogonal to the principal normal geodesics at $\gamma(s)$, for all s.

Note that this definition is equivalent to saying that the geodesics connecting p with $\gamma(s)$ are tangent to the rectifying planes of γ, i.e., the planes generated by $\{T_\gamma(s), B_\gamma(s)\}$.

3. Rectifying Curves and Conical Surfaces

The surface in $\mathbb{H}^3(-r)$ formed by the union of all the geodesics that connect a fixed point $p \in \mathbb{H}^3(-r)$ (the *vertex*) and any point of some curve in $\mathbb{H}^3(-r)$ that does not contain the vertex (a *directrix*) is called a *conical surface* in $\mathbb{H}^3(-r)$. Each of those geodesics is called a *ruling* of the surface. Hence, a conical surface $M \subset \mathbb{H}^3(-r)$, with vertex at a point p, can be parametrized as

$$\Psi(t,z) = \exp_p(zV(t)) = \cosh\left(\frac{z}{r}\right) p + r\sinh\left(\frac{z}{r}\right) V(t), \quad z > 0, \tag{3}$$

where $V = V(t)$ is a unit speed curve in $\mathbb{S}^2(1) \subset T_p\mathbb{H}^3(-r) \subset \mathbb{R}^4_1$ called the *director curve*. The tangent plane at a point $\Psi(t,z)$ is the plane spanned by the vectors $\{\Psi_t(t,z), \Psi_z(t,z)\}$, given by

$$\Psi_t(t,z) = r\sinh\left(\frac{z}{r}\right) V'(t), \tag{4}$$

$$\Psi_z(t,z) = \frac{1}{r}\sinh\left(\frac{z}{r}\right) p + \cosh\left(\frac{z}{r}\right) V(t). \tag{5}$$

The unit normal vector field $N(t,z)$ can be computed as

$$N(t,z) = \frac{\Psi_t \times \Psi_z}{||\Psi_t \times \Psi_z||}(t,z) = -N_V(t), \tag{6}$$

where $N_V(t) = V(t) \times V'(t)$ is a unit vector field tangent to $\mathbb{S}^2(1) \subset T_p\mathbb{H}^3(-r)$.

A straightforward computation, bearing in mind Eqs. (4) and (5), yields the following equations:

$$\overline{\nabla}_{\Psi_t}\Psi_t = -r\sinh\left(\frac{z}{r}\right)\cosh\left(\frac{z}{r}\right)\Psi_z - rk_V(t)\sinh\left(\frac{z}{r}\right)N, \tag{7}$$

$$\overline{\nabla}_{\Psi_z}\Psi_t = \overline{\nabla}_{\Psi_t}\Psi_z = \frac{1}{r}\coth\left(\frac{z}{r}\right)\Psi_t, \tag{8}$$

$$\overline{\nabla}_{\Psi_z}\Psi_z = 0, \tag{9}$$

where k_V is the geodesic curvature of V as a curve in $\mathbb{S}^2(1)$.

From (6), we can compute the shape operator S of the conical surface M as follows:

$$S(\Psi_t) = -\overline{\nabla}_{\Psi_t}N = \frac{-k_V(t)}{r\sinh(\frac{z}{r})}\Psi_t, \quad S(\Psi_z) = -\overline{\nabla}_{\Psi_z}N = 0.$$

Hence, the Gaussian and mean curvatures of the conical surface are given, respectively, by

$$K = -\frac{1}{r^2} + \det(S) = -\frac{1}{r^2} \quad \text{and} \quad H = \frac{1}{2}\mathrm{tr}(S) = \frac{-k_V(t)}{2r\sinh(\frac{z}{r})}.$$

Let ∇ denote the Levi-Civita connection on the conical surface $M_{p,V}$. A unit speed curve $\beta = \beta(t)$ in $M_{p,V}$ is called a *geodesic* if $\nabla_{\beta'}\beta' = 0$. This condition is equivalent to the property that the acceleration of β in $\mathbb{H}^3(-r)$, $\overline{\nabla}_{\beta'}\beta'$, is a vector field orthogonal to $M_{p,V}$ along β.

Our first characterization of rectifying curves in the hyperbolic space $\mathbb{H}^3(-r)$ is given in the next result. It can be proved similarly to [13, Theorem 1].

Theorem 1. *Let $\gamma = \gamma(s)$ be a unit speed curve in $\mathbb{H}^3(-r)$. Then, γ is a rectifying curve if and only if there exists a point $p \in \mathbb{H}^3(-r)$, with $p \notin \mathrm{Im}(\gamma)$, and a unit speed curve $V = V(t)$ in the two-dimensional unit sphere $\mathbb{S}^2(1) \subset T_p\mathbb{H}^3(-r)$ such that γ is a geodesic of the conical surface $M_{p,V}$ with vertex p and director curve V.*

4. Helices Versus Rectifying Curves

A nice relation between generalized helices (or Lancret curves) and rectifying curves in $\mathbb{R}^3$ is found by Chen [2]. In fact, both families of curves are characterized by the condition that the ratio of torsion and curvature is a linear function in arclength s, i.e., $(\tau/\kappa)(s) = c_1 s + c_2$, for some constants c_1 and c_2. If $c_1 = 0$, we obtain generalized helices; otherwise, we obtain rectifying curves.

For curves in the three-dimensional hyperbolic space $\mathbb{H}^3(-r)$, a classical result by Barros [1] states that ordinary helices are the only generalized helices. Hence, the following geometric question arises naturally: Is there any relation between helices and rectifying curves in $\mathbb{H}^3(-r)$? More precisely:

Question. *What shape are the rectifying curves in the hyperbolic space?*

The goal of this section is to find a very simple and nice characterization for rectifying curves in $\mathbb{H}^3(-r)$ in terms of the ratio of torsion and curvature.

Let γ be a unit speed curve in a conical surface M parametrized by (3). Then, we have $\gamma(s) = \Psi(t(s), z(s))$, for some functions $t(s)$ and $z(s)$, and so $T_\gamma = t'\Psi_t + z'\Psi_z$. From (4) and (5), it is easy to see that

$$1 = \langle T_\gamma, T_\gamma \rangle = r^2 (t')^2 \sinh^2\left(\frac{z}{r}\right) + (z')^2. \tag{10}$$

If γ is a geodesic in M (and so a rectifying curve in $\mathbb{H}^3(-r)$), then the functions t and z satisfy the following system of ordinary differential equations:

$$t'' + \frac{2}{r} t' z' \coth\left(\frac{z}{r}\right) = 0, \tag{11}$$

$$z'' - r(t')^2 \sinh\left(\frac{z}{r}\right) \cosh\left(\frac{z}{r}\right) = 0, \tag{12}$$

$$-r(t')^2 k_V \sinh\left(\frac{z}{r}\right) = \pm\kappa_\gamma \neq 0. \tag{13}$$

Write $y(s) = r\cosh\left(\frac{z(s)}{r}\right)$. Then, using Eqs. (10) and (12), it is not difficult to see that

$$y''(s) - \frac{1}{r^2}\, y(s) = 0,$$

and then we deduce

$$z(s) = r \operatorname{arg\,cosh}\left(\frac{A}{r} \sinh\left(\frac{s+s_0}{r}\right) + \frac{B}{r}\cosh\left(\frac{s+s_0}{r}\right)\right), \tag{14}$$

for some constants A, B and s_0. Since $N_\gamma(s)$ is parallel to $N(t(s), z(s))$, we can write without loss of generality that

$$B_\gamma(s) = a(s)\Psi_t(t(s), z(s)) + b(s)\Psi_z(t(s), z(s)),$$

where the functions a and b are given by $-z'/\sqrt{E}$ and $t'\sqrt{E}$, respectively. A direct computation yields

$$\overline{\nabla}_{T_\gamma} B_\gamma = a'\Psi_t + b'\Psi_z + at'\overline{\nabla}_{\Psi_t}\Psi_t + (az' + bt')\overline{\nabla}_{\Psi_t}\Psi_z.$$

Now, using the Frenet–Serret equations (2), jointly with (7) and (8), we get that, up to the sign, the torsion is given by

$$\tau_\gamma(s) = -t'(s)z'(s)k_V(t(s)).$$

This equation and (13) lead to

$$\frac{\tau_\gamma}{\kappa_\gamma}(s) = \frac{z'(s)}{rt'(s)\sinh(\frac{z(s)}{r})}. \tag{15}$$

Now, from (11), we get

$$rt''\sinh\left(\frac{z}{r}\right) + 2t'z'\cosh\left(\frac{z}{r}\right) = 0,$$

and then

$$t'(s)\sinh^2\left(\frac{z(s)}{r}\right) = \lambda, \tag{16}$$

for a nonzero real constant λ. Bearing this equation in mind, jointly with (10) and (14), we obtain the following relation between the constants A, B and λ:

$$A^2 - B^2 + \lambda^2 r^4 = -r^2. \tag{17}$$

Finally, a straightforward computation, bearing in mind (14)–(16), leads to

$$\frac{\tau_\gamma}{\kappa_\gamma}(s) = c_1\sinh\left(\frac{s+s_0}{r}\right) + c_2\cosh\left(\frac{s+s_0}{r}\right), \tag{18}$$

for some constants c_1 and c_2 given by

$$c_1 = \frac{B}{\lambda r^2} \quad \text{and} \quad c_2 = \frac{A}{\lambda r^2}.$$

Note that Eq. (17) implies $1 - c_1^2 + c_2^2 < 0$.

Conversely, let $\gamma = \gamma(s)$ be a unit speed curve in $\mathbb{H}^3(-r)$ with curvature and torsion satisfying (18), for some constants c_1 and c_2 with $1 - c_1^2 + c_2^2 < 0$. Let λ be a nonzero constant such that

$$\lambda^2 = \frac{-1}{r^2(1 - c_1^2 + c_2^2)},$$

and define two constants $A = \lambda r^2 c_2$ and $B = \lambda r^2 c_1$. Let $z(s)$ be the function given by (14) and consider a solution $t(s)$ of Eq. (16), which is given by

$$t(s) = \arctan\left[\frac{1}{\lambda r^3}\left(AB + (A^2 + r^2)\tanh\left(\frac{s+s_0}{r}\right)\right)\right].$$

Now, define the function k_V by the Eq. (13). Let p be a point in $\mathbb{H}^3(-r)$ such that $p \notin \mathrm{Im}(\gamma)$, and take $V = V(t)$ a unit speed curve in the unit sphere $\mathbb{S}^2(1) \subset T_p\mathbb{H}^3(-r)$ whose geodesic curvature is given by k_V. Let M be the conical surface determined by p and V, which is parametrized by (3), and consider the curve $\widetilde{\gamma}$ defined by $\widetilde{\gamma}(s) = \Psi(t(s), z(s))$. It is a straightforward computation to show that $\widetilde{\gamma}$ is a geodesic of M, with curvature $\kappa_{\widetilde{\gamma}} = \kappa_\gamma$ and torsion $\tau_{\widetilde{\gamma}} = \tau_\gamma$. Hence, γ is congruent to a geodesic in a conical surface.

Summarizing, we have proved that Eq. (18) characterizes the curves in $\mathbb{H}^3(-r)$ that are geodesics in a conical surface, which are parametrized by

$\Psi(t(s), z(s))$. Therefore, bearing Theorem 1 in mind, we have the following characterization for rectifying curves.

Theorem 2. *Let $\gamma = \gamma(s)$ be a unit speed curve in $\mathbb{H}^3(-r)$. Then, γ is a rectifying curve if and only if the ratio of torsion and curvature of the curve is given by*

$$\frac{\tau_\gamma}{\kappa_\gamma}(s) = c_1 \sinh\left(\frac{s+s_0}{r}\right) + c_2 \cosh\left(\frac{s+s_0}{r}\right),$$

for some constants c_1, c_2 and s_0, with $1 - c_1^2 + c_2^2 < 0$.

Let us compare this result with [2, Theorem 2] for curves in $\mathbb{R}^3$ and [13, Theorem 2] for curves in the three-dimensional sphere. Denote by $\bar{M}^3(c)$, $c \in \mathbb{R}$, the Euclidean space $\mathbb{R}^3$ if $c = 0$, the sphere $\mathbb{S}^3(r)$ if $c = 1/r^2 > 0$, or the hyperbolic space $\mathbb{H}^3(-r)$ if $c = -1/r^2 < 0$. Let f and g be the following functions:

	$c = 0$	$c = \frac{1}{r^2}$	$c = -\frac{1}{r^2}$
$f(s)$	s	$r\sin\left(\frac{s}{r}\right)$	$r\sinh\left(\frac{s}{r}\right)$
$g(s)$	1	$\cos\left(\frac{s}{r}\right)$	$\cosh\left(\frac{s}{r}\right)$

The functions f and g determine the geodesic flow of the manifold: given a point $q \in \bar{M}^3(c)$ and a unit vector $v \in T_q\bar{M}^3(c)$, the unit speed geodesic $\gamma_{(q,v)}(s)$ with initial condition (q, v) is given by $\gamma_{(q,v)}(s) = g(s)q + f(s)v$. In the following result, we put together our Theorem 2 [2, Theorem 2] and [13, Theorem 2].

Theorem 3. *Let $\gamma = \gamma(s)$ be a unit speed twisted curve in $\bar{M}^3(c)$. Then, γ is a rectifying curve if and only if the ratio of torsion and curvature of the curve is given by*

$$\frac{\tau_\gamma}{\kappa_\gamma}(s) = c_1\, f(s+s_0) + c_2\, g(s+s_0),$$

for some constants c_1, c_2 and s_0, with $c_1^2 + c\,(1 + c_2^2) > 0$.

5. Rectifying Curves and Developable Surfaces

For any unit speed curve x in $\mathbb{R}^3$ with curvature $\kappa \neq 0$, the vector field defined by $D = \tau T + \kappa B$ is called the *Darboux vector field* of x. The direction of the Darboux vector is that of the instantaneous axis of rotation, and its length $\sqrt{\kappa^2 + \tau^2}$ is called the angular speed (see, e.g., [12, p. 12]). In terms of the Darboux vector, the Frenet–Serret equations can be expressed as follows

$$T' = D \times T, \quad N' = D \times N, \quad B' = D \times B.$$

In [11], the authors define a vector field $\widetilde{D} = (\tau/\kappa)T + B = (1/\kappa)D$ and call it the *modified Darboux vector field* along x. The ruled surface with base curve x and director curve $\widetilde{D}$ is called the *rectifying developable* of x, and it is parametrized as $F_{(x,\widetilde{D})}(s, u) = x(s) + u\widetilde{D}(s)$; $F_{(x,\widetilde{D})}$ is the envelope of rectifying planes of the curve x. The authors show (see [11, Proposition 3.1]) that $F_{(x,\widetilde{D})}$ is a conical surface if and only if x is a conical geodesic curve (that we know is a rectifying curve).

The goal of this section is to extend, if possible, the above result by Izumiya and Takeuchi to curves in the three-dimensional hyperbolic space $\mathbb{H}^3(-r)$.

Let $\gamma = \gamma(s)$ be a unit speed curve in $\mathbb{H}^3(-r)$, with $\kappa_\gamma > 0$, and consider the ruled surface generated by γ and $\widetilde{D}_\gamma = (\tau_\gamma/\kappa_\gamma)T_\gamma + B_\gamma$:

$$\Psi(s,u) = \exp_{\gamma(s)}(u\widetilde{D}_\gamma(s)) = \cosh\left(\frac{u|\widetilde{D}_\gamma(s)|}{r}\right)\gamma(s) + \frac{r}{|\widetilde{D}_\gamma(s)|}\sinh\left(\frac{u|\widetilde{D}_\gamma(s)|}{r}\right)\widetilde{D}_\gamma(s).$$

This surface will be called as in $\mathbb{R}^3$, the rectifying developable of the curve γ.

To simplify the following computations, write $\lambda = \tau_\gamma/\kappa_\gamma$ and $\rho = |\widetilde{D}_\gamma| = \sqrt{\lambda^2+1}$. The vector fields tangent to the surface are given by

$$\Psi_s = \frac{1}{r}\left[u\rho' + \frac{\lambda}{\rho}\right]\sinh\left(\frac{u\rho}{r}\right)\gamma + \left[\cosh\left(\frac{u\rho}{r}\right) + \frac{r\lambda'}{\rho}\sinh\left(\frac{u\rho}{r}\right)\right]T_\gamma + \frac{\rho'}{\rho}\left[u\cosh\left(\frac{u\rho}{r}\right) - \frac{r}{\rho}\sinh\left(\frac{u\rho}{r}\right)\right]\widetilde{D}_\gamma, \tag{19}$$

$$\Psi_u = \frac{\rho}{r}\sinh\left(\frac{u\rho}{r}\right)\gamma + \cosh\left(\frac{u\rho}{r}\right)\widetilde{D}_\gamma. \tag{20}$$

From here, and after a long and straightforward computation, we obtain the following expression for the determinant Δ of the induced metric:

$$\Delta(s,u) = \left[\cosh\left(\frac{u\rho}{r}\right) + \frac{r\lambda'}{\rho}\sinh\left(\frac{u\rho}{r}\right)\right]^2.$$

As a consequence, we have the following result, which can be proved in a similar way as [13, Proposition 6].

Proposition 4. *Let $\gamma = \gamma(s)$ be a unit speed curve in $\mathbb{H}^3(-r)$, $\kappa_\gamma > 0$. Then, (s_0, u_0) is a singular point of the rectifying developable of γ if and only if*

$$\lambda'(s_0) \neq 0 \quad \textit{and} \quad u_0 = \frac{r}{\rho(s_0)}\operatorname{arg\,tanh}\left(-\frac{\rho(s_0)}{r\lambda'(s_0)}\right).$$

Let σ be the singular locus of the rectifying developable of γ, which is given by $\sigma(s) = \Psi(s,u(s)) = \exp_{\gamma(s)}(u(s)\widetilde{D}(s))$, where the function $u(s)$ satisfies the following equation:

$$\cosh\left(\frac{u\rho}{r}\right) + \frac{r\lambda'}{\rho}\sinh\left(\frac{u\rho}{r}\right) = 0. \tag{21}$$

Note that $\sinh(\frac{u\rho}{r}) \neq 0$; otherwise, the above equation implies that $\cosh(\frac{u\rho}{r}) = 0$, which is a contradiction.

The rectifying developable is a conical surface if there exists a point p that belongs to every geodesic with initial condition $(\gamma(s), \widetilde{D}_\gamma(s))$. Hence, $\sigma(s) \equiv p$. In this case, define $W(s) = \exp_p^{-1}(u(s)\widetilde{D}_\gamma(s))$, $z(s) = |W(s)|$ and $V(s) = W(s)/z(s)$. Then, we have $\gamma(s) = \exp_p(z(s)V(s))$, and the rectifying developable can be parametrized as $F(s, z) = \exp_p(zV(s))$, so it is a conical surface with vertex at p.

By taking derivative in (21) and using that $\sinh(\frac{u\rho}{r}) \neq 0$, and (21) again, we deduce

$$(u\rho)' = \frac{-r^2\rho^2}{\rho^2 - r^2(\lambda')^2}\left(\frac{\lambda'}{\rho}\right)'. \tag{22}$$

Observe that (21) implies $\rho^2 - r^2(\lambda')^2 < 0$.

On the other hand, from (21), we can rewrite Eqs. (19) and (20) as follows

$$\Psi_s = \frac{1}{r}\left(u\rho' + \frac{\lambda}{\rho}\right)\sinh\left(\frac{u\rho}{r}\right)\gamma - \frac{r\rho'}{\rho^2}(1 + u\lambda')\sinh\left(\frac{u\rho}{r}\right)\widetilde{D}_\gamma,$$

$$\Psi_u = \frac{\rho}{r}\sinh\left(\frac{u\rho}{r}\right)\gamma - \frac{r\lambda'}{\rho}\sinh\left(\frac{u\rho}{r}\right)\widetilde{D}_\gamma.$$

Bearing this in mind, we deduce that $\sigma'(s) = \Psi_s(s, u(s)) + u'(s)\Psi_u(s, u(s)) \equiv 0$ if and only if $(u\rho)' = -\lambda/\rho$. This equation, jointly with (22), yields $r^2\lambda''(s) - \lambda(s) = 0$, and then we deduce that

$$\lambda(s) = \frac{\tau_\gamma}{\kappa_\gamma}(s) = c_1 \sinh\left(\frac{s + s_0}{r}\right) + c_2 \cosh\left(\frac{s + s_0}{r}\right),$$

for some constants c_1, c_2 and s_0. Observe that $1 - c_1^2 + c_2^2 = \rho^2 - r^2(\lambda')^2 < 0$. Hence, bearing Theorem 2 in mind, we have proved the following result.

Theorem 5. *Let $\gamma = \gamma(s)$ be a unit speed curve in $\mathbb{H}^3(-r)$, $\kappa_\gamma > 0$. Then the rectifying developable of γ is a conical surface if and only if γ is a rectifying curve.*

6. Classification of Rectifying Curves

The following result provides some simple characterizations for twisted rectifying curves in the three-dimensional hyperbolic space. It can be proved similarly to [13, Theorem 4].

Theorem 6. *Let $\gamma = \gamma(s)$ be a unit speed twisted curve in $\mathbb{H}^3(-r)$. The following conditions are equivalent*:

(i) *γ is a rectifying curve.*
(ii) *There exists a point $p \in \mathbb{H}^3(-r)$, with $p \notin Im(\gamma)$, such that*

$$\langle p, T_\gamma(s)\rangle = b_1 \sinh\left(\frac{s + s_0}{r}\right) + b_2 \cosh\left(\frac{s + s_0}{r}\right) \quad \textit{and} \quad |p^\perp|^2 = b^2,$$

for some constants b_1, b_2, b and s_0, with $-b_1^2+b_2^2+b^2=-r^2$. Here, $p^\perp$ denotes the component of p orthogonal to T_γ in $\mathbb{H}^3(-r)$.

(iii) *There exists a point $p\in\mathbb{H}^3(-r)$, with $p\notin Im(\gamma)$, such that $\langle p,N_\gamma(s)\rangle=0$.*

(iv) *There exists a point $p\in\mathbb{H}^3(-r)$, with $p\notin Im(\gamma)$, such that $\langle p,B_\gamma(s)\rangle=a$, for some constant a.*

(v) *There exists a point $p\in\mathbb{H}^3(-r)$, with $p\notin Im(\gamma)$, such that*

$$\langle p,\gamma(s)\rangle=a_1\sinh\left(\frac{s+s_0}{r}\right)+a_2\cosh\left(\frac{s+s_0}{r}\right),$$

for some constants a_1, a_2 and s_0, with $-a_1^2+a_2^2\geq r^4$.

(vi) *There exists a point $p\in\mathbb{H}^3(-r)$, with $p\notin Im(\gamma)$, such that the distance function in $\mathbb{H}^3(-r)$ between p and $\gamma(s)$, $\rho(s)=d(p,\gamma(s))$, satisfies*

$$\cosh\left(\frac{\rho(s)}{r}\right)=d_1\sinh\left(\frac{s+s_0}{r}\right)+d_2\cosh\left(\frac{s+s_0}{r}\right),$$

for some constants d_1, d_2 and s_0, with $-d_1^2+d_2^2\geq 1$.

Now, we present the main result of this section, which determines all the rectifying curves in the three-dimensional hyperbolic space.

Theorem 7. *Let γ be a twisted curve in $\mathbb{H}^3(-r)$. Then, γ is a rectifying curve if and only if, up to reparametrization, it is given by*

$$\gamma(t)=\exp_p(\rho(t)V(t))=\cosh\left(\frac{\rho(t)}{r}\right)p+r\sinh\left(\frac{\rho(t)}{r}\right)V(t), \tag{23}$$

where $p\in\mathbb{H}^3(-r)$ is a point such that $p\notin Im(\gamma)$, $V=V(t)$ is a unit speed curve in $\mathbb{S}^2(1)\subset T_p\mathbb{H}^3(-r)$, and $\rho(t)=r\arg\tanh(a\sec(t+t_0))$, for some constants a and t_0, with $0<a^2<1$.

Proof. Take a point $p\in\mathbb{H}^3(-r)$, and consider a unit speed curve $V=V(t)$ in $\mathbb{S}^2(1)\subset T_p\mathbb{H}^3(-r)$ and a positive function $\rho=\rho(t)$. Put $\gamma(t)=\exp_p(\rho(t)V(t))$. Then, we have

$$\gamma'=\frac{\rho'}{r}\sinh\left(\frac{\rho}{r}\right)p+\rho'\cosh\left(\frac{\rho}{r}\right)V+r\sinh\left(\frac{\rho}{r}\right)V',$$

and

$$v^2=\langle\gamma',\gamma'\rangle=(\rho')^2+r^2\sinh^2\left(\frac{\rho}{r}\right). \tag{24}$$

Then, the unit tangent vector field T_γ is given by

$$T_\gamma=\frac{1}{v}\gamma'=\frac{\rho'}{rv}\sinh\left(\frac{\rho}{r}\right)p+\frac{\rho'}{v}\cosh\left(\frac{\rho}{r}\right)V+\frac{r}{v}\sinh\left(\frac{\rho}{r}\right)V'. \tag{25}$$

Let $s=s(t)$ denote the arclength parameter of γ, so $v(t)=s'(t)$. From the Frenet–Serret equations, we have $(\kappa_\gamma N_\gamma)(s)=(\frac{1}{v}T_\gamma'-\frac{1}{r^2}\gamma)(t)$ and so the principal normal vector field N_γ of γ is parallel to the vector field given by

$\frac{1}{v}T'_\gamma - \frac{1}{r^2}\gamma$. Here, the prime ($'$) on a vector field X along the curve γ denotes the covariant derivative in $\mathbb{R}^4_1$, i.e., $X' = \nabla^0_{T_\gamma} X$.

On the other hand, for the unit speed curve V, we have

$$V'' = -V + k_V N_V, \tag{26}$$

where $N_V = V \times V'$ is tangent to $\mathbb{S}^2(1)$, but normal to V and p, and k_V is the geodesic curvature of V.

From (25) and (26), we have

$$\left\langle p, \frac{1}{v}T'_\gamma - \frac{1}{r^2}\gamma \right\rangle = -\frac{r}{v}\left[\frac{\rho'}{v}\sinh\left(\frac{\rho}{r}\right)\right]' + \cosh\left(\frac{\rho}{r}\right).$$

From Theorem 6, we know that γ is a rectifying curve if and only if $\langle p, N_\gamma\rangle = 0$, which is equivalent, since N_γ is parallel to $\frac{1}{v}T'_\gamma - \frac{1}{r^2}\gamma$, to the following equation:

$$-\frac{r}{v}\left[\frac{\rho'}{v}\sinh\left(\frac{\rho}{r}\right)\right]' + \cosh\left(\frac{\rho}{r}\right) = 0.$$

A straightforward computation shows that this equation is equivalent to

$$r\sinh\left(\frac{\rho}{r}\right)\rho'' - 2\cosh\left(\frac{\rho}{r}\right)(\rho')^2 - r^2\cosh\left(\frac{\rho}{r}\right)\sinh^2\left(\frac{\rho}{r}\right) = 0.$$

Similar to [13, Theorem 7], we deduce that the nontrivial solutions of the above ODE are given by $\rho(t) = r \arg\tanh(a\sec(t+t_0))$. Hence, we conclude that $\gamma(t) = \exp_p(\rho(t)V(t))$ is a rectifying curve if and only if $\rho(t) = r \arg\tanh(a\sec(t+t_0))$, for some constants a and t_0, with $0 < a^2 < 1$. □

7. Rectifying Curves as Extremal Curves

If a curve γ in $\mathbb{H}^3(-r)$ is given by $\gamma(t) = \exp_p(\rho(t)V(t))$, $p \in \mathbb{H}^3(-r)$, where $\rho(t) \neq 0$ is an arbitrary function and $V(t)$ is a curve lying in $\mathbb{S}^2(1) \in T_p\mathbb{H}^3(-r)$, then V is called the *spherical projection* of γ (see [3]). In the following result, we show that a rectifying curve γ in $\mathbb{H}^3(-r)$ is characterized by the property that a certain function (depending on v, κ_γ and ρ) takes its minimum value, equal to k_V^2, among the curves with the same spherical projection V. A similar result for curves in $\mathbb{R}^3$ was obtained by Chen and Dillen [3].

Theorem 8. *Let $p \in \mathbb{H}^3(-r)$ and consider a unit speed curve $V(t)$ in $\mathbb{S}^2(1) \subset T_p\mathbb{H}^3(-r)$. Then, for any nonzero function $\rho(t)$, the curvature κ_γ and the speed v of the curve $\gamma(t) = \exp_p(\rho(t)V(t))$, and the geodesic curvature k_V of V satisfy the inequality*

$$k_V^2 \leq \frac{v^4\kappa_\gamma^2}{r^2\sinh^2(\frac{\rho}{r})}, \tag{27}$$

with the equality sign holding identically if and only if γ is a rectifying curve.

Proof. Suppose that a curve γ is given by $\gamma(t) = \exp_p(\rho(t)V(t))$, where ρ is a nonzero function and V is a unit speed curve in $\mathbb{S}^2(1) \subset T_p\mathbb{H}^3(-r)$. From the definition of $\exp_p$ and (25), and bearing in mind that N_V is a vector orthogonal to span$\{p, V, V'\}$, we deduce that N_V is orthogonal to both γ and T_γ, and then

$$N_V = V \times V' = \cos\alpha\, N_\gamma + \sin\alpha\, B_\gamma, \tag{28}$$

where $\alpha = \alpha(t)$ is an arbitrary function.

By differentiating (28) with respect to t, and by applying (26) and the Frenet equations (2), we obtain

$$k_V\, V' = v\kappa_\gamma \cos\alpha\, T_\gamma + (\alpha' + v\tau_\gamma)[\sin\alpha\, N_\gamma - \cos\alpha\, B_\gamma], \tag{29}$$

where $v = ||\gamma'||$ is the speed of γ, which is given by (24). Then, using that $\langle V', V'\rangle = 1$, we have

$$k_V^2 = (v\kappa_\gamma \cos\alpha)^2 + (\alpha' + v\tau_\gamma)^2. \tag{30}$$

Now, we are going to express the point p in terms of the orthogonal frame $\{\gamma, T_\gamma, N_\gamma, B_\gamma\}$. Some computations are needed. From (23), we get

$$\langle p, \gamma\rangle = -r^2 \cosh\left(\frac{\rho}{r}\right), \tag{31}$$

and by taking derivative here we obtain

$$\langle p, T_\gamma\rangle = -\frac{r\rho'}{v} \sinh\left(\frac{\rho}{r}\right). \tag{32}$$

To simplify next computations, write

$$V' = a\, T_\gamma + b[\sin\alpha\, N_\gamma - \cos\alpha\, B_\gamma],$$

where

$$a = \frac{v\kappa_\gamma \cos\alpha}{k_V} \quad \text{and} \quad b = \frac{\alpha' + v\tau_\gamma}{k_V}.$$

From the conditions $\langle p, N_V\rangle = 0$ and $\langle p, V'\rangle = 0$, we get a system of two linear equations in $\langle p, N_\gamma\rangle$ and $\langle p, B_\gamma\rangle$ whose solution, bearing (32) in mind, is given by

$$\langle p, N_\gamma\rangle = \frac{ar\rho' \sin\alpha}{bv} \sinh\left(\frac{\rho}{r}\right), \tag{33}$$

$$\langle p, B_\gamma\rangle = -\frac{ar\rho' \cos\alpha}{bv} \sinh\left(\frac{\rho}{r}\right). \tag{34}$$

Putting together (31)–(34), we obtain

$$p = \cosh\left(\frac{\rho}{r}\right)\gamma - \frac{r\rho'}{v} \sinh\left(\frac{\rho}{r}\right) T_\gamma + \frac{ar\rho'}{bv} \sinh\left(\frac{\rho}{r}\right)[\sin\alpha\, N_\gamma - \cos\alpha\, B_\gamma],$$

and then

$$-r^2 = \langle p, p\rangle = -r^2\cosh^2\left(\frac{\rho}{r}\right) + r^2\frac{(\rho')^2}{v^2}\left(1+\left(\frac{a}{b}\right)^2\right)\sinh^2\left(\frac{\rho}{r}\right).$$

But this equation implies necessarily that

$$\frac{(\rho')^2}{v^2}\left(1+\left(\frac{a}{b}\right)^2\right) = 1,$$

which leads to

$$(\alpha' + v\tau_\gamma)^2 = \frac{(\rho')^2}{v^2-(\rho')^2}(v\kappa_\gamma\cos\alpha)^2. \tag{35}$$

By substituting (35) into (30), and bearing (24) in mind, we obtain

$$k_V^2 = \frac{v^4\kappa_\gamma^2\cos^2\alpha}{r^2\sinh^2(\frac{\rho}{r})}, \tag{36}$$

which implies inequality (27). Obviously, the equality sign of (27) holds if and only if $\sin\alpha = 0$. This condition, bearing (28) in mind, is equivalent to the condition $N_V = \pm N_\gamma$. But N_V is parallel to N_γ if and only if γ is a geodesic of the conical surface $\Psi(t,z) = \exp_p(zV(t))$, i.e., γ is a rectifying curve (see Theorem 1). Consequently, the equality sign of (27) holds identically if and only if γ is a rectifying curve. □

As a consequence of Theorem 8, we obtain the following classification of hyperbolic curves with nonzero constant curvature and linear hyperbolic trigonometric torsion in terms of spiral type rectifying curves. This result extends [3, Corollary 1].

Corollary 9. *A curve $\gamma(s) = \exp_p(\rho(s)V(s))$ in $\mathbb{H}^3(-r)$ has nonzero constant curvature k_0 and torsion $\tau(s) = d_1\sinh((s+s_0)/r) + d_2\cosh((s+s_0)/r)$, for some constants d_1, d_2 and s_0, with $k_0^2 - d_1^2 + d_2^2 < 0$, if and only if it is congruent to a rectifying curve over a unit speed spiral type curve $V(t)$ in $\mathbb{S}^2(1) \subset T_p\mathbb{H}^3(-r)$ with geodesic curvature $k_V(t) = c\,(\cos^2(t+t_0) - a^2)^{-3/2}$ for some constants $c \neq 0$, $0 < a^2 < 1$, and t_0.*

Proof. If a curve $\gamma = \exp_p(\rho V)$ in $\mathbb{H}^3(-r)$ has nonzero constant curvature k_0 and linear hyperbolic trigonometric torsion in arclength s, with $k_0^2 - d_1^2 + d_2^2 < 0$, then Theorem 2 implies that γ is (congruent to) a rectifying curve. Then, by Theorem 7, we can assume that $\rho(t) = r\,\mathrm{arg}\tanh(a\sec(t+t_0))$, for some constants a and t_0, with $0 < a^2 < 1$. From Theorem 8, we have $k_V(t) = c\,(\cos^2(t+t_0) - a^2)^{-3/2}$, where the nonzero constant c is either $c = a\,(1-a^2)\,k_0\,r$ or $c = -a\,(1-a^2)\,k_0\,r$.

Conversely, if $\gamma = \exp_p(\rho V)$ is a rectifying curve in $\mathbb{H}^3(-r)$ over a unit speed curve $V = V(t)$ in $\mathbb{S}^2(1) \subset T_p\mathbb{H}^3(-r)$ with geodesic curvature $k_V(t) = c\,(\cos^2(t+t_0) - a^2)^{-3/2}$, $c \neq 0$, then by Theorem 7 we have $\rho(t) =$

$r \arg\tanh(a \sec(t + t_0))$, for some constants a and t_0, with $0 < a^2 < 1$. From this, we find

$$\frac{v^4}{r^2 \sinh^2(\frac{\rho}{r})} = a^2(1-a^2)^2 r^2 (\cos^2(t+t_0) - a^2)^{-3}.$$

Since γ is a rectifying curve, Theorem 8 implies

$$k_V^2 = \frac{v^4 \kappa_\gamma^2}{r^2 \sinh^2(\frac{\rho}{r})}.$$

Hence, we get

$$\kappa_\gamma^2 = \frac{c^2}{a^2(1-a^2)^2 r^2},$$

which is a nonzero constant. Therefore, bearing Theorem 2 in mind, the proof is finished. □

In the case of rectifying curves in the three-dimensional sphere $\mathbb{S}^3(r)$, see [13], we can obtain a similar result. The proof is analogous to the one of Theorem 8, and it is left to the reader.

Theorem 10. *Let $p \in \mathbb{S}^3(r)$ and consider a unit speed curve $V(t)$ in $\mathbb{S}^2(1) \subset T_p\mathbb{S}^3(r)$. Then, for any nonzero function $\rho(t)$, the curvature κ_γ and the speed v of the curve $\gamma(t) = \exp_p(\rho(t)V(t))$, and the geodesic curvature k_V of V satisfy the inequality*

$$k_V^2 \leq \frac{v^4 \kappa_\gamma^2}{r^2 \sin^2(\frac{\rho}{r})}, \tag{37}$$

with the equality sign holding identically if and only if γ is a rectifying curve.

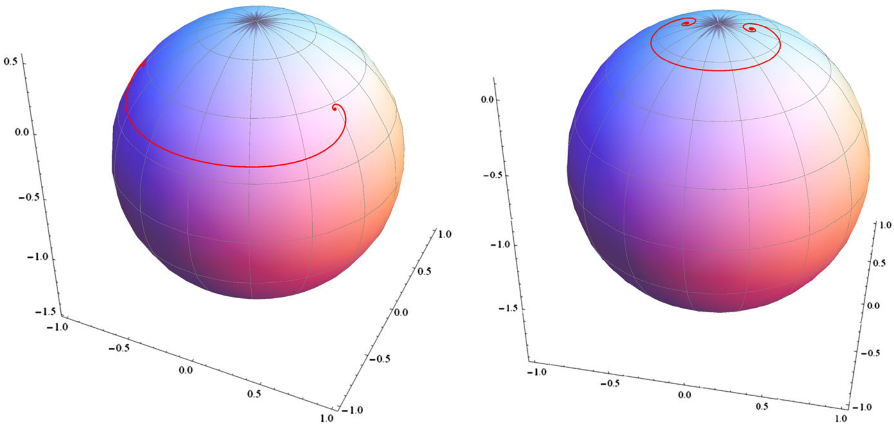
As in the hyperbolic case, as a consequence of Theorem 10, we obtain the following classification of spherical curves with nonzero constant curvature and linear trigonometric torsion in terms of spiral type rectifying curves. The proof uses Theorem 10 and several results of [13].

Corollary 11. *A curve $\gamma(s) = \exp_p(\rho(s)V(s))$ in $\mathbb{S}^3(r)$ has nonzero constant curvature and torsion $\tau(s) = d_1 \sin((s+s_0)/r) + d_2 \cos((s+s_0)/r)$, for some constants d_1, d_2 and s_0, if and only if it is congruent to a rectifying curve over a unit speed curve $V(t)$ in $\mathbb{S}^2(1) \subset T_p\mathbb{S}^3(r)$ with geodesic curvature $k_V(t) = c\,(\cos^2(t+t_0) + a^2)^{-3/2}$ for some constants $c \neq 0, a \neq 0$ and t_0.*

To illustrate Corollaries 9 and 11, we put in Figs. 1 and 2 the graphs of several spherical projections in $\mathbb{S}^2(1)$ of rectifying curves in $\mathbb{H}^3(-1)$ and $\mathbb{S}^3(1)$, respectively.

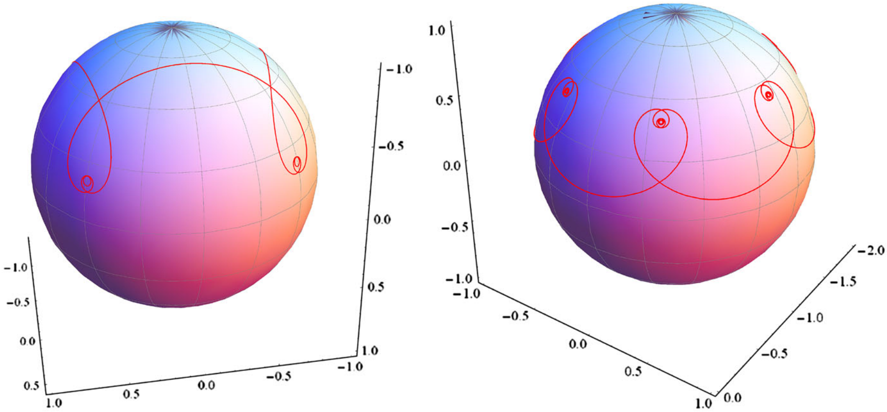
Finally, using the function f defined in Sect. 4, we can put together Theorems 8 and 10, and Theorem 3 of [3], as follows.

Theorem 12. *Let $p \in \bar{M}^3(c)$ and consider a unit speed curve $V(t)$ in $\mathbb{S}^2(1) \subset T_p\bar{M}^3(c)$ (we can take $p = 0$ in the case $c = 0$). Then, for any nonzero function $\rho(t)$, the curvature κ_γ and the speed v of the curve $\gamma(t) = \exp_p(\rho(t)V(t))$, and the geodesic curvature k_V of V satisfy the inequality*



$a = \frac{1}{4},\ c = \frac{15}{32}$ $a = \frac{1}{5},\ c = \frac{48}{25}$

Figure 1. Spherical projections V of rectifying curves in $\mathbb{H}^3(-1)$ with geodesic curvature $k_V(t) = c\,(\cos^2(t) - a^2)^{-3/2}$



$a = \frac{1}{4},\ c = \frac{17}{32}$ $a = \frac{1}{4},\ c = \frac{85}{64}$

Figure 2. Spherical projections V of rectifying curves in $\mathbb{S}^3(1)$ with geodesic curvature $k_V(t) = c\,(\cos^2(t) + a^2)^{-3/2}$

$$k_V^2 \leq \frac{v^4 \kappa_\gamma^2}{f^2(\rho)}, \tag{38}$$

with the equality sign holding identically if and only if γ is a rectifying curve.

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