Mediterr. J. Math. 13 (2016), 1613–1624
DOI 10.1007/s00009-015-0591-6
1660-5446/16/041613-12
published online June 17, 2015

Mediterranean Journal of Mathematics

Existence of Positive Solutions for Second-Order Impulsive Boundary Value Problems on Time Scales

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Abstract. In this paper, by using Leray–Schauder fixed-point theorem, Avery–Henderson fixed-point theorem and Leggett–Williams fixed-point theorem, respectively, we investigate the conditions for the existence of at least one, two and three positive solutions to nonlinear second-order impulsive boundary value problems on time scales.

Mathematics Subject Classification. Primary 34B18; Secondary 34B37, 34N05.

Keywords. Boundary value problems, cone, fixed-point theorems, impulsive dynamic equations, positive solutions, time scales.

1. Introduction

In recent years, much work has been done on the existence and uniqueness of solutions of boundary value problems for differential equations. Some theory and methods of nonlinear functional analysis, for example, fixed-point theorems, the continuation method of topological degree, the upper and lower solutions method, variational method and critical point theory, have been applied to those problems. Impulsive problems describe processes which experience a sudden change in their states at certain moments. Impulsive differential equations have been developed in modeling impulsive problems in physics, chemical technology, population dynamics, ecology, biological systems, biotechnology, industrial robotics, optimal control, economics, and so forth. For the introduction of the theory of impulsive differential equations, we refer to the books [1–3]. Especially, the study of impulsive dynamic equations on time scales has also attracted much attention since it provides an unifying structure for differential equations in the continuous cases and finite difference equations in the discrete cases, see [4–9] and references therein. Some basic definitions and theorems on time scales can be found in the books [10, 11].

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In 2008, Yaslan [12] studied nonlinear second-order three-point boundary value problem on time scales:

$$\begin{cases} u^{\Delta\nabla}(t) + h(t)f(t,u(t)) = 0, \quad t \in [t_1, t_3] \subset \mathbb{T} \\ u^{\Delta}(t_3) = 0, \quad \alpha u(t_1) - \beta u^{\Delta}(t_1) = u^{\Delta}(t_2). \end{cases}$$

By using fixed-point theorems in cones, conditions for the existence of at least one, two and three positive solutions of the problem is obtained.

In 2010, Li et al. [13] studied the existence of at least one and three positive solutions of the following impulsive boundary value problem on time scales

$$\begin{cases} \left[\phi_p\left(y^{\Delta}(t)\right)\right]^{\nabla} + \omega(t)f(t,y(t)) = 0, \quad t \in [0,T]_{\mathbb{T}}, \quad t \neq t_k, \quad k = 1,2,\ldots,m \\ y(t_k^+) - y(t_k^-) = I_k(y(t_k)), \quad k = 1,2,\ldots,m \\ y(0) = \sum\limits_{i=1}^{n-2} a_i y(\xi_i), \quad y^{\Delta}(T) = 0. \end{cases}$$

In 2014, Karaca et al, [14] discussed the following impulsive boundary value problem on time scales

$$\begin{cases} -\left(\phi_p\left(u^{\Delta}\right)\right)^{\nabla}(t) = f(t,u(t)), \quad t \in [0,1]_{\mathbb{T}}, \quad t \neq t_k, \quad k = 1,2,\ldots,m \\ u(t_k^+) - u(t_k^-) = I_k(u(t_k)), \quad k = 1,2,\ldots,m \\ \alpha u(0) - \beta u^{\Delta}(0) = \int\limits_0^1 u(s)\Delta s, \quad u^{\Delta}(1) = 0 \end{cases}$$

and established criteria for the existence of at least two or many positive solutions to the problem.

In 2014, Xu and Wang [15] considered the general second-order nonlinear m-point singular impulsive boundary value problem

$$\begin{cases} u^{\Delta\nabla}(t) + a(t)u^{\Delta}(t) + b(t)u(t) + q(t)f(t,u(t)) = 0, \quad t \in (0,1), \quad t \neq t_k \\ u^{\Delta}(t_k^+) = u^{\Delta}(t_k - I_k(u(t_k)), \quad k = 1,2,\ldots,n \\ u(\rho(0)) = 0, \quad u(\sigma_1) = \sum\limits_{i=1}^{m-2} \alpha_i u(\eta_i). \end{cases}$$

The existence and uniqueness of positive solutions are established by using fixed-point theorems in cones.

In this paper, motivated by the above results, we consider the following boundary value problem (BVP) on time scales:

$$\begin{cases} y^{\Delta\nabla}(t) + h(t)f(t,y(t)) = 0, \quad t \in [a,b] \subset \mathbb{T}^*, \quad t \neq t_k, \quad k = 1,2,\ldots,m \\ y(t_k^+) - y(t_k^-) = I_k(y(t_k)), \quad k = 1,2,\ldots,m \\ y^{\Delta}(b) = 0, \quad \alpha y(a) - \beta y^{\Delta}(a) = \sum\limits_{i=1}^{n-2} y^{\Delta}(\mu_i), \quad n \geq 3, \end{cases} \tag{1.1}$$

where $\mathbb{T}^* := \mathbb{T}^{\kappa} \cap \mathbb{T}_{\kappa}$, $0 \leq a < t_1 < \cdots < t_m < \rho(b)$, $\mu_i \in (a,b) \cap \mathbb{T}$ ($i = 1,2,\ldots,n-2$) with $a < \mu_1 < \cdots < \mu_{n-2} < \rho(b)$. Let us define the set

$$\mathbb{D} = \big\{x : [a,b] \to \mathbb{R}; x^{\Delta} : [a,b] \cap \mathbb{T}^{\kappa} \to \mathbb{R} \,\text{is continuous such that}\, x^{\Delta\nabla} : [a,b] \cap \mathbb{T}^* \to \mathbb{R} \,\text{is ld} - \text{continuous}\big\}.$$

A function $y : [a, b] \to \mathbb{R}$ is said to be a solution of the problem (1.1) provided $y \in \mathbb{D}$ and the BVP (1.1) holds for all $t \in [a, b]$.

We will assume that the following conditions are satisfied.

(H1) $h \in C_{ld}([a, b], [0, \infty))$ and does not vanish identically on any closed subinterval of $[a, b]$;

(H2) $f \in C([a, b] \times [0, \infty), [0, \infty))$;

(H3) $I_k \in C(\mathbb{R}, \mathbb{R}^+)$, $t_k \in [a, b]$ and $y(t_k^+) = \lim_{h\to 0} y(t_k + h)$, $y(t_k^-) = \lim_{h\to 0} y(t_k - h)$ represent the right and left limits of $y(t)$ at $t = t_k$, $k = 1, \ldots, m$.

The rest of paper is arranged as follows. In Sect. 2, we give several lemmas to prove the main results in this paper. In Sect. 3, existence results of at least one positive solutions of the BVP (1.1) are first established as a result of Leray–Schauder fixed-point theorem. Second, we apply the Avery–Henderson fixed-point theorem to prove the existence of at least two positive solutions to the BVP (1.1). Finally, we use Leggett–Williams fixed-point theorem to show that the existence of at least three positive solutions for the BVP (1.1).

2. Preliminaries

We now state and prove several lemmas which are needed later.

Lemma 2.1. *Assume (H3) holds and $\alpha > 0$, $\beta \geq 0$. If $\omega(t) \in C_{ld}[a, b]$ and $\omega(t) \geq 0$ for $t \in [a, b]$, then $y(t)$ is a solution of the following BVP*

$$\begin{cases} y^{\Delta\nabla}(t) + \omega(t) = 0, \quad t \in [a, b] \subset \mathbb{T}^*, \quad t \neq t_k, \quad k = 1, 2, \ldots, m \\ y(t_k^+) - y(t_k^-) = I_k(y(t_k)), \quad k = 1, 2, \ldots, m \\ y^{\Delta}(b) = 0, \quad \alpha y(a) - \beta y^{\Delta}(a) = \sum_{i=1}^{n-2} y^{\Delta}(\mu_i), \quad n \geq 3 \end{cases} \tag{2.1}$$

if and only if $y(t)$ is a solution of the following integral equation

$$y(t) = \int_a^b \left(\frac{\beta}{\alpha} + r - a\right) \omega(r)\nabla r + \int_t^b (t - r)\omega(r)\nabla r + \frac{1}{\alpha} \sum_{i=1}^{n-2} \int_{\mu_i}^b \omega(r)\nabla r + \sum_{a<t_k<t} I_k(y(t_k)) \tag{2.2}$$

and $y(t) \geq 0$ for $t \in [a, b]$.

Proof. Let y be a solution of the problem (2.1). Then

$$y^{\Delta\nabla}(t) = -\omega(t), \quad t \in [a, b].$$

A nabla integration from t to b of both sides of the above equality yields

$$y^{\Delta}(b) - y^{\Delta}(t) = -\int_t^b \omega(r)\nabla r, \quad \text{i.e., } y^{\Delta}(t) = \int_t^b \omega(r)\nabla r.$$

Integrating above equality from a to t, we have

$$y(t)-y(a)=\int_a^t\int_s^b \omega(r)\nabla r\Delta s+\sum_{a<t_k<t} I_k(y(t_k)).$$

Then we get

$$\begin{aligned}y(t)&=y(a)+\int_a^t\int_a^r \omega(r)\Delta s\nabla r+\int_t^b\int_a^t \omega(r)\Delta s\nabla r+\sum_{a<t_k<t} I_k(y(t_k))\\ &=y(a)+\int_a^b (r-a)\omega(r)\nabla r+\int_t^b (t-r)\omega(r)\nabla r+\sum_{a<t_k<t} I_k(y(t_k)).\end{aligned} \tag{2.3}$$

By using the second boundary condition, we obtain

$$\alpha y(a)-\beta\int_a^b \omega(r)\nabla r=\sum_{i=1}^{n-2}\int_{\mu_i}^b \omega(r)\nabla r. \tag{2.4}$$

From (2.3) and (2.4), we find

$$\begin{aligned}y(t)=&\int_a^b\left(\frac{\beta}{\alpha}+r-a\right)\omega(r)\nabla r+\int_t^b (t-r)\omega(r)\nabla r+\frac{1}{\alpha}\sum_{i=1}^{n-2}\int_{\mu_i}^b \omega(r)\nabla r\\ &+\sum_{a<t_k<t} I_k\left(y(t_k)\right).\end{aligned}$$

Since $\alpha>0,\ \beta\geq 0$ and $\omega(t)\geq 0$ for $t\in[a,b]$, $y(t)\geq 0$ for $t\in[a,b]$.

Conversely, it is very easy to show that $y(t)$ in (2.2) satisfies (2.1). □

By Lemma 2.1, the solutions of the BVP (1.1) are the fixed-points of the operator A defined by

$$\begin{aligned}Ay(t)=&\int_a^b\left(\frac{\beta}{\alpha}+r-a\right)h(r)f(r,y(r))\nabla r+\int_t^b (t-r)h(r)f(r,y(r))\nabla r\\ &+\frac{1}{\alpha}\sum_{i=1}^{n-2}\int_{\mu_i}^b h(r)f(r,y(r))\nabla r+\sum_{a<t_k<t} I_k(y(t_k)).\end{aligned}$$

Let

$$\begin{aligned}E=\Big\{&y:[a,b]\to\mathbb{R}\text{ is continuous at } t\neq t_k,\text{ left continuous at the points}\\ &t_k,\ \text{ for which } y(t_k^-)\text{ and } y(t_k^+)\text{ exist with } y(t_k^-)=y(t_k),\\ &k=1,\dots,m.\Big\},\end{aligned}$$

which is a Banach space with the norm $\|y\| = \sup_{t\in[a,b]} |y(t)|$. Define the cone $P \subset E$ by

$$P = \Big\{ y \in E : \ y \text{ is concave, nondecreasing and nonnegative on } [a,b], \\ y^{\Delta}(b) = 0 \Big\}. \tag{2.5}$$

Lemma 2.2. *Let $\alpha > 0$ and $\beta \geq 0$. If $y \in P$, then $y(t)$ in (2.2) satisfies*

$$y(t) \geq \frac{t-a}{b-a}\|y\|, \quad t \in [a,b] \subset \mathbb{T}. \tag{2.6}$$

Proof. Since $y \in P$, $y(t)$ is nondecreasing on $[a,b]$. Then, we have $\|y\| = y(b)$. If

$$g(t) = y(t) - \frac{t-a}{b-a}\|y\|, \quad t \in [a,b] \subset \mathbb{T},$$

we obtain $g(a) = y(a) \geq 0$ and $g(b) = 0$. Since g is concave on $[a,b]$, we get $g(t) \geq 0$ for $t \in [a,b]$. Then we find

$$y(t) \geq \frac{t-a}{b-a}\|y\|, \quad t \in [a,b] \subset \mathbb{T}.$$

□

3. Existence of Solutions

Let us introduce the following hypotheses, which are assumed hereafter:

(H4) There exist constants c_k such that $|I_k(y)| \leq c_k$, for $k = 1, \ldots, m$ and for all $y \in P$.

To prove the existence of at least one positive solution for the BVP (1.1), we will apply the following Leray–Schauder fixed-point theorem.

Theorem 3.1. *Let E be a Banach space, $A : E \to E$ is a completely continuous operator. If the set $\{x \in E : x = \lambda Ax, 0 < \lambda < 1\}$ is bounded, then A has at least one fixed point in the closed $T \subset E$, where*

$$T = \{x \in E : \|x\| \leq R\}, \quad R = sup \{\|x\| : x = \lambda Ax, 0 < \lambda < 1\}.$$

Theorem 3.2. *Assume (H1)–(H4) hold and $\alpha > 0$, $\beta \geq 0$. Then the BVP (1.1) has at least one positive solution.*

Proof. For all $y \in P$, from (H1), (H2), the definition of A and the proof of Lemma 2.1, we know that

$$(Ay)(t) \geq 0, \quad (Ay)^{\Delta}(t) \geq 0, \quad (Ay)^{\Delta\nabla}(t) \leq 0, \quad (Ay)^{\Delta}(b) = 0.$$

So A is an operator from P to P. Standard arguments show that $A : P \to P$ is completely continuous.

We denote

$$N(A) := \{y \in P : y = \lambda Ay, 0 < \lambda < 1\}.$$

Now we show that the set $N(A)$ is bounded. Let $T=\{y\in P:\|y\|\leq R\}$ and $R=sup\left\{\|y\|:y=\lambda Ay,0<\lambda<1\right\}$. Then for all $y\in N(A)$, we have

$$\begin{aligned}|y(t)| &= \lambda|Ay(t)| \\ &\leq \lambda \sup_{t\in[a,b],\,y\in T} f(t,y(t))\left\{\left(\frac{\beta+n-2}{\alpha}+2b-2a\right)\int_a^b h(r)\nabla r\right\} \\ &\quad +\lambda\sum_{k=1}^{m} c_k.\end{aligned}$$

From (H1), (H2) and (H4), we obtain $N(A)$ is bounded. By Theorem 3.1, the BVP (1.1) has at least one positive solution. □

We will need also the following (Avery–Henderson) fixed-point theorem [16] to prove the existence of at least one positive solution for the BVP (1.1).

Theorem 3.3 [16]. *Let P be a cone in a real Banach space E. Set*

$$P(\phi,r)=\{u\in P:\phi(u)<r\}.$$

If η and ϕ are increasing, nonnegative continuous functionals on P, let θ be a nonnegative continuous functional on P with $\theta(0)=0$ such that, for some positive constants r and M,

$$\phi(u)\leq\theta(u)\leq\eta(u) \text{ and } \|u\|\leq M\phi(u)$$

for all $u\in\overline{P(\phi,r)}$. Suppose that there exist positive numbers $p<q<r$ such that

$$\theta(\lambda u)\leq\lambda\theta(u), \text{ for all } 0\leq\lambda\leq 1 \text{ and } u\in\partial P(\theta,q).$$

If $A:\overline{P(\phi,r)}\to P$ is a completely continuous operator satisfying

(i) *$\phi(Au)>r$ for all $u\in\partial P(\phi,r)$,*
(ii) *$\theta(Au)<q$ for all $u\in\partial P(\theta,q)$,*
(iii) *$P(\eta,p)\neq\emptyset$ and $\eta(Au)>p$ for all $u\in\partial P(\eta,p)$,*

then A has at least two fixed points u_1 and u_2 such that

$$p<\eta(u_1) \text{ with } \theta(u_1)<q \text{ and } q<\theta(u_2) \text{ with } \phi(u_2)<r.$$

Define the constants

$$M:=\left(\int_{\mu_{n-2}}^{b}\left(\frac{\beta+n-2}{\alpha}+\mu_{n-2}-a\right)h(s)\nabla s\right)^{-1} \tag{3.1}$$

and

$$N:=\left(\int_{a}^{b}\left(\frac{\beta+n-2}{\alpha}+s-a\right)h(s)\nabla s\right)^{-1}. \tag{3.2}$$

Theorem 3.4. *Assume (H1)–(H4) hold and $\alpha>0$, $\beta\geq 0$. Suppose there exist numbers $0<p<q<r$ such that the function f satisfies the following conditions:*

(i) $f(s,y) > rM$ *for* $(s,y) \in [\mu_{n-2}, b] \times \left[r, \frac{r(b-a)}{\mu_{n-2}-a}\right]$,

(ii) $f(s,y) < \frac{qN}{2}$ *for* $(s,y) \in [a,b] \times \left[0, \frac{q(b-a)}{\mu_{n-2}-a}\right]$ *and* $\sum\limits_{a<t_k<\mu_{n-2}} I_k(y(t_k)) \leq \frac{q}{2}$,

(iii) $f(s,y) > pM$ *for* $(s,y) \in [\mu_{n-2}, b] \times \left[\frac{p(\mu_{n-2}-a)}{b-a}, p\right]$,

where N and M are defined in (3.1) and (3.2), respectively. Then the BVP (1.1) has at least two positive solutions y_1 and y_2 such that

$$y_1(b) > p \text{ with } y_1(\mu_{n-2}) < q \text{ and } y_2(\mu_{n-2}) > q \text{ with } y_2(\mu_{n-2}) < r.$$

Proof. Define the cone P as in (2.6). We know that $AP \subset P$ is completely continuous. Let the nonnegative increasing continuous functionals ϕ, θ and η be defined on the cone P by

$$\phi(y) := y(\mu_{n-2}), \quad \theta(y) := y(\mu_{n-2}), \quad \eta(y) := y(b).$$

For each $y \in P$, we have

$$\phi(y) = \theta(y) \leq \eta(y)$$

and from Lemma (2.2) we have

$$\|y\| \leq \frac{b-a}{\mu_{n-2}-a}\phi(y). \tag{3.3}$$

In addition, $\theta(0) = 0$ and for all $y \in P$, $\lambda \in [0,1]$ we get $\theta(\lambda y) = \lambda\theta(y)$. We now verify that all of the conditions of Theorem 3.3 are satisfied.

If $y \in \partial P(\phi, r)$, from (3.3) we have $r = y(\mu_{n-2}) \leq y(s) \leq \|y\| \leq \frac{(b-a)r}{\mu_{n-2}-a}$ for $s \in [\mu_{n-2}, b]$. Then, from the hypothesis (i) and (3.1), we find

$$\begin{aligned}
\phi(Ay) &= \int_a^b \left(\frac{\beta}{\alpha} + s - a\right) h(s) f(s,y(s)) \nabla s + \frac{1}{\alpha}\sum_{i=1}^{n-2}\int_{\mu_i}^b h(s) f(s,y(s))\nabla s \\
&\quad + \int_{\mu_{n-2}}^b (\mu_{n-2}-s) h(s) f(s,y(s))\nabla s + \sum_{a<t_k<\mu_{n-2}} I_k(y(t_k)) \\
&\geq \int_{\mu_{n-2}}^b \left(\frac{\beta+n-2}{\alpha} + \mu_{n-2} - a\right) h(s) f(s,y(s))\nabla s \\
&> \int_{\mu_{n-2}}^b \left(\frac{\beta+n-2}{\alpha} + \mu_{n-2} - a\right) h(s) rM \nabla s \\
&= r.
\end{aligned}$$

Thus the condition (i) of Theorem 3.3 holds.

If $y \in \partial P(\theta, q)$, from (3.3) we have $0 \leq y(s) \leq \|u\| \leq \frac{(b-a)q}{\mu_{n-2}-a}$ for $s \in [a,b]$. Then, we obtain

$$\begin{aligned}\theta(Ay) &= \int_a^b \left(\frac{\beta}{\alpha}+s-a\right)h(s)f(s,y(s))\nabla s + \frac{1}{\alpha}\sum_{i=1}^{n-2}\int_{\mu_i}^b h(s)f(s,y(s))\nabla s \\ &\quad + \int_{\mu_{n-2}}^b (\mu_{n-2}-s)h(s)f(s,y(s))\nabla s + \sum_{a<t_k<\mu_{n-2}} I_k(y(t_k)) \\ &\le \int_a^b \left(\frac{\beta+n-2}{\alpha}+s-a\right)h(s)f(s,y(s))\nabla s + \sum_{a<t_k<\mu_{n-2}} I_k(y(t_k)) \\ &< \int_a^b \left(\frac{\beta+n-2}{\alpha}+s-a\right)h(s)\frac{qN}{2}\nabla s + \frac{q}{2} \\ &= q\end{aligned}$$

by hypothesis (ii) and (3.2). Hence the condition (ii) of Theorem 3.3 is satisfied.

Since $0 \in P$ and $p > 0$, $P(\eta, p) \neq \emptyset$. If $y \in \partial P(\eta, p)$, from Lemma (2.2) we have $\frac{p(\mu_{n-2}-a)}{b-a} \le y(s) \le \|y\| = p$ for $s \in [\mu_{n-2}, b]$. Then, we get

$$\begin{aligned}\eta(Ay) &= \int_a^b \left(\frac{\beta}{\alpha}+s-a\right)h(s)f(s,y(s))\nabla s + \frac{1}{\alpha}\sum_{i=1}^{n-2}\int_{\mu_i}^b h(s)f(s,y(s))\nabla s \\ &\quad + \sum_{a<t_k<b} I_k(y(t_k)) \\ &> \int_{\mu_{n-2}}^b \left(\frac{\beta+n-2}{\alpha}+\mu_{n-2}-a\right)h(s)pM\nabla s \\ &= p\end{aligned}$$

using hypothesis (iii) and (3.1). Since all the conditions of Theorem 3.3 are fulfilled, the BVP (1.1) has at least two positive solutions y_1 and y_2 such that

$$y_1(b) > p \text{ with } y_1(\mu_{n-2}) < q \text{ and } y_2(\mu_{n-2}) > q \text{ with } y_2(\mu_{n-2}) < r.$$

□

Now, we will use the following (Leggett–Williams) fixed-point theorem [17] to prove the existence of at least three positive solutions to the nonlinear BVP (1.1).

Theorem 3.5 [17]. *Let P be a cone in the real Banach space E. Set*

$$P_r := \{x \in P : \|x\| < r\}$$

$$P(\psi, a, b) := \{x \in P : a \le \psi(x),\ \|x\| \le b\}.$$

Suppose $A : \overline{P_r} \to \overline{P_r}$ be a completely continuous operator and ψ be a nonnegative continuous concave functional on P with $\psi(u) \le \|u\|$ for all $u \in \overline{P_r}$. If there exists $0 < p < q < l \le r$ such that the following condition hold,

(i) $\{u \in P(\psi, q, l) : \psi(u) > q\} \neq \emptyset$ *and* $\psi(Au) > q$ *for all* $u \in P(\psi, q, l)$*;*
(ii) $\|Au\| < p$ *for* $\|u\| \leq p$*;*
(iii) $\psi(Au) > q$ *for* $u \in P(\psi, q, r)$ *with* $\|Au\| > l$*,*

then A *has at least three fixed points* u_1, u_2 *and* u_3 *in* $\overline{P_r}$ *satisfying*

$$\|u_1\| < p,\ \psi(u_2) > q,\ p < \|u_3\| \text{ with } \psi(u_3) < q.$$

Theorem 3.6. *Assume (H1)–(H4) hold and* $\alpha > 0$, $\beta \geq 0$. *Suppose that there exist constants* $0 < p < q < \frac{q(b-a)}{\mu_{n-2}-a} \leq r$ *such that the function* f *satisfies the following conditions:*

(i) $f(s, y) \leq \frac{rN}{2}$ *for* $(s, y) \in [a, b] \times [0, r]$ *and* $I_k(y(t_k)) \leq \frac{r}{2m}$ *for* $k = 1, 2, \ldots, m$*,*
(ii) $f(s, y) > qM$ *for* $(s, y) \in [\mu_{n-2}, b] \times \left[q, \frac{q(b-a)}{\mu_{n-2}-a}\right]$*,*
(iii) $f(s, y) < \frac{pN}{2}$ *for* $(s, y) \in [a, b] \times [0, p]$*.*

Then the BVP (1.1) has at least three positive solutions y_1, y_2 *and* y_3 *satisfying*

$$y_1(b) < p,\ y_2(\mu_{n-2}) > q,\ y_3(b) > p \text{ with } y_3(\mu_{n-2}) < q.$$

Proof. The conditions of Theorem 3.5 will be shown to be satisfied. For this purpose we first define the nonnegative continuous concave functional $\psi : P \to [0, \infty)$ to be $\psi(y) := y(\mu_{n-2})$, the cone P as in (2.6), M as in (3.1) and N as in (3.2). For all $y \in P$, we have $\psi(y) \leq \|y\|$. If $y \in \overline{P_r}$, then $0 \leq y \leq r$ and from the hypothesis (i) and (3.2), we get

$$\begin{aligned}\|Ay\| &= \int_a^b \left(\frac{\beta}{\alpha} + s - a\right) h(s) f(s, y(s)) \nabla s + \frac{1}{\alpha} \sum_{i=1}^{n-2} \int_{\mu_i}^b h(s) f(s, y(s)) \nabla s \\ &\quad + \sum_{a<t_k<b} I_k(y(t_k)) \\ &\leq \int_a^b \left(\frac{\beta + n - 2}{\alpha} + s - a\right) h(s) \frac{rN}{2} \nabla s + \frac{r}{2} \\ &= r.\end{aligned}$$

This proves that $A : \overline{P_r} \to \overline{P_r}$.

Since $\frac{q(b-a)}{\mu_{n-2}-a} \in P\left(\psi, q, \frac{q(b-a)}{\mu_{n-2}-a}\right)$ and $\psi\left(\frac{q(b-a)}{\mu_{n-2}-a}\right) > q$, $\left\{y \in P(\psi, q, \frac{q(b-a)}{\mu_{n-2}-a}) : \psi(y) > q\right\} \neq \emptyset$. For $y \in P\left(\psi, q, \frac{q(b-a)}{\mu_{n-2}-a}\right)$, we have $q \leq y(\mu_{n-2}) \leq y(s) \leq \|y\| \leq \frac{q(b-a)}{\mu_{n-2}-a}$ for $s \in [\mu_{n-2}, b]$. Using the hypothesis (ii) and (3.1), we obtain

$$\begin{aligned}\psi(Ay) &= \int_a^b \left(\frac{\beta}{\alpha}+s-a\right) h(s)f(s,y(s))\nabla s + \frac{1}{\alpha}\sum_{i=1}^{n-2}\int_{\mu_i}^b h(s)f(s,y(s))\nabla s \\ &\quad + \int_{\mu_{n-2}}^b (\mu_{n-2}-s)h(s)f(s,y(s))\nabla s + \sum_{a<t_k<\mu_{n-2}} I_k(y(t_k)) \\ &\geq \int_{\mu_{n-2}}^b \left(\frac{\beta+n-2}{\alpha}+\mu_{n-2}-a\right) h(s)f(s,y(s))\nabla s \\ &> q.\end{aligned}$$

Hence, the condition (i) of Theorem 3.5 holds.

If $\|y\| \leq p$, we find

$$\begin{aligned}\|Ay\| &= \int_a^b \left(\frac{\beta}{\alpha}+s-a\right) h(s)f(s,y(s))\nabla s + \\ &\quad + \frac{1}{\alpha}\sum_{i=1}^{n-2}\int_{\mu_i}^b h(s)f(s,y(s))\nabla s + \sum_{a<t_k<b} I_k(y(t_k)) \\ &\leq \int_a^b \left(\frac{\beta+n-2}{\alpha}+s-a\right) h(s)f(s,y(s))\nabla s + \sum_{a<t_k<b} I_k(y(t_k)) \\ &< p.\end{aligned}$$

by hypothesis (iii) and (3.2). Thus, the condition (ii) of Theorem 3.5 is satisfied.

For the condition (iii) of Theorem 3.5, we suppose that $y \in P(\psi,q,r)$ with $\|Ay\| > \frac{q(b-a)}{\mu_{n-2}-a}$. Then, from Lemma (2.2) we get

$$\psi(Ay) = Ay(\mu_{n-2}) \geq \frac{\mu_{n-2}-a}{b-a}\|Ay\| > q.$$

Because all of the hypotheses of the Leggett–Williams fixed-point theorem are satisfied, the BVP (1.1) has at least three positive solutions y_1, y_2 and y_3 such that

$$y_1(b) < p,\ y_2(\mu_{n-2}) > q,\ y_3(b) > p \text{ with } y_3(\mu_{n-2}) < q.$$

$\square$

Example. Let $\mathbb{T} = \left\{(\frac{2}{3})^n : n \in \mathbb{N}\right\} \cup \{0\} \cup [3,10]$. Consider the following boundary value problem:

$$\begin{cases} y^{\Delta\nabla}(t) + t\frac{2013y^5}{y^5+1} = 0, \quad t \neq \frac{2}{3}, \quad t \in \left[\frac{2}{9},10\right] \subset \mathbb{T}^* \\ y^{\Delta}(10) = 0, \quad 2y\left(\frac{2}{9}\right) - y^{\Delta}\left(\frac{2}{9}\right) = y^{\Delta}(3) + y^{\Delta}(5), \\ y\left(\frac{2}{3}^+\right) - y\left(\frac{2}{3}^-\right) = 5. \end{cases}$$

If we take $p = 5$, $q = 11$ and $r = 473.893$, then all the conditions in Theorem 3.4 are satisfied. Thus, by Theorem 3.4, the BVP has at least two positive solutions y_1 and y_2 such that

$$y_1(10) > 5 \text{ with } y_1(5) < 11 \text{ and } y_2(5) > 11 \text{ with } y_2(5) < 473893.$$

If we take $p = 0.68$, $q = 1$ and $r = 11$, then all the conditions in Theorem 3.6 are satisfied. Thus, the BVP has at least three positive solutions y_1, y_2 and y_3 satisfying

$$y_1(10) < 0.68,\ y_2(5) > 1,\ y_3(5) < 1 \text{ with } y_3(10) > 0.68.$$

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Received: August 21, 2014.
Revised: March 9, 2015.
Accepted: June 4, 2015.