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Generalization of Popoviciu-Type Inequalities Via Fink's Identity

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Abstract. We obtained useful identities via Fink's identity, by which the inequality of Popoviciu for convex functions is generalized for higher order convex functions. We investigate the bounds for the identities related to the generalization of the Popoviciu inequality using inequalities for the Čebyšev functional. Some results relating to the Grüss- and Ostrowski-type inequalities are constructed. Further, we also construct new families of exponentially convex functions and Cauchy-type means by looking at linear functional associated with the obtained inequalities.

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1. Introduction

The theory developed in the study of convex functions, arising from intuitive geometrical observations, may be readily applied to topics in real analysis and economics. Convexity is a simple and natural notion which can be traced back to Archimedes (circa 250 B.C.), in connection with his famous estimate of the value of π (using inscribed and circumscribed regular polygons). He noticed the important fact that the perimeter of a convex figure is smaller than the perimeter of any other convex figure, surrounding it. The theory of convex functions has experienced a rapid development. This can be attributed to several causes: first, so many areas in modern analysis directly or indirectly involve the application of convex functions; second, convex functions are closely related to the theory of inequalities and many important inequalities are consequences of the applications of convex functions (see [12]).

Definition 1. A function $f\colon I \to \mathbb{R}$ is convex on I if

$$(x_3 - x_2)\, f(x_1) + (x_1 - x_3)\, f(x_2) + (x_2 - x_1)\, f(x_3) \geq 0 \tag{1}$$

holds for all $x_1, x_2, x_3 \in I$ such that $x_1 < x_2 < x_3$.

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An important characterization of convex function is stated in [12, p. 2].

Theorem 1.1. *If f is a convex function defined on I and if $x_1 \leq y_1$, $x_2 \leq y_2$, $x_1 \neq x_2$, $y_1 \neq y_2$, then the following inequality is valid*

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(y_2) - f(y_1)}{y_2 - y_1}. \tag{2}$$

If the function f is concave, then the inequality reverses.

Divided differences are fairly ascribed to Newton, and the term divided difference was used by Augustus de Morgan in 1842. Divided differences are found to be very helpful when we are dealing with functions having different degrees of smoothness. One of the important references on divided difference is the book by de Boor [2].

Definition 2. The nth order divided difference of a function $f\colon [a,b] \to \mathbb{R}$ at mutually distinct points $x_0, \ldots, x_n \in [a,b]$ is defined recursively by

$$[x_i; f] = f(x_i), \quad i = 0, \ldots, n,$$
$$[x_0, \ldots, x_n; f] = \frac{[x_1, \ldots, x_n; f] - [x_0, \ldots, x_{n-1}; f]}{x_n - x_0}. \tag{3}$$

It is easy to see that (3) is equivalent to

$$[x_0, \ldots, x_n; f] = \sum_{i=0}^{n} \frac{f(x_i)}{q'(x_i)}, \quad \text{where } q(x) = \prod_{j=0}^{n} (x - x_j).$$

The following definition of a real valued convex function is characterized by nth order divided difference (see [12, p.15]).

Definition 3. A function $f\colon [a,b] \to \mathbb{R}$ is said to be n-convex $(n \geq 0)$ if and only if for all choices of $(n+1)$ distinct points $x_0, \ldots, x_n \in [a,b]$, $[x_0, \ldots, x_n; f] \geq 0$ holds.

If this inequality is reversed, then f is said to be n-concave. If the inequality is strict, then f is said to be a strictly n-convex (n-concave) function.

Remark 1.2. Note that $0-$convex functions are non-negative functions, $1-$convex functions are increasing functions and $2-$convex functions are simply the convex functions.

The following theorem gives an important criteria to examine the n-convexity of a function f (see [12, p. 16]).

Theorem 1.3. *If $f^{(n)}$ exists, then f is n-convex if and only if $f^{(n)} \geq 0$.*

In 1965, Popoviciu introduced a characterization of convex function [13]. In 1976, the inequality of Popoviciu as given by Vasić and Stanković in [14] can be written in the following form (see [12, p. 173]):

Theorem 1.4. *Let $m, k \in \mathbb{N}$, $m \geq 3$, $2 \leq k \leq m-1$, $[\alpha, \beta] \subset \mathbb{R}$, $\mathbf{x} = (x_1, \ldots, x_m) \in [\alpha, \beta]^m$, $\mathbf{p} = (p_1, \ldots, p_m)$ be a positive m-tuple such that $\sum_{i=1}^{m} p_i = 1$. Also let $f\colon [\alpha, \beta] \to \mathbb{R}$ be a convex function. Then*

$$p_{k,m}(\mathbf{x}, \mathbf{p}; f) \leq \frac{m-k}{m-1} p_{1,m}(\mathbf{x}, \mathbf{p}; f) + \frac{k-1}{m-1} p_{m,m}(\mathbf{x}, \mathbf{p}; f), \tag{4}$$

where

$$p_{k,m}(\mathbf{x},\mathbf{p};f) = p_{k,m}(\mathbf{x},\mathbf{p};f(x)) := \frac{1}{C_{k-1}^{m-1}} \sum_{1\le i_1<\cdots<i_k\le m} \left(\sum_{j=1}^{k} p_{i_j}\right) f\left(\frac{\sum_{j=1}^{k} p_{i_j}x_{i_j}}{\sum_{j=1}^{k} p_{i_j}}\right)$$

is the linear functional with respect to f.

By inequality (4), we write

$$\Upsilon(\mathbf{x},\mathbf{p};f) := \frac{m-k}{m-1}p_{1,m}(\mathbf{x},\mathbf{p};f) + \frac{k-1}{m-1}p_{m,m}(\mathbf{x},\mathbf{p};f) - p_{k,m}(\mathbf{x},\mathbf{p};f). \quad (5)$$

Remark 1.5. It is important to note that under the assumptions of Theorem 1.4, if the function f is convex then $\Upsilon(\mathbf{x},\mathbf{p};f) \ge 0$ and $\Upsilon(\mathbf{x},\mathbf{p};f) = 0$ for $f(x) = x$ or f is a constant function.

The mean value theorems and exponential convexity of the linear functional $\Upsilon(\mathbf{x},\mathbf{p};f)$ are given in [9] for a positive m-tuple $\mathbf{p}$. Some special classes of convex functions are considered to construct the exponential convexity of $\Upsilon(\mathbf{x},\mathbf{p};f)$ in [9]. In [10] (see also [6]), the results related to $\Upsilon(\mathbf{x},\mathbf{p};f)$ are generalized with help of Green function and n-exponential convexity is proved instead of exponential convexity.

In the present paper, we use A. M. Fink's identity and prove many interesting results. The following theorem is proved by Fink in [5].

Theorem 1.6. *Let $a, b \in \mathbb{R}$, $\phi : [a,b] \to \mathbb{R}$, $n \ge 1$ and $\phi^{(n-1)}$ is absolutely continuous on $[a,b]$. Then*

$$\begin{aligned}\phi(x) &= \frac{n}{b-a}\int_a^b \phi(t)\,\mathrm{d}t \\ &\quad - \sum_{w=1}^{n-1}\left(\frac{n-w}{w!}\right)\left(\frac{\phi^{(w-1)}(a)(x-a)^w - \phi^{(w-1)}(b)(x-b)^w}{b-a}\right) \\ &\quad + \frac{1}{(n-1)!(b-a)}\int_a^b (x-t)^{n-1} w^{[a,b]}(t,x)\,\phi^{(n)}(t)\,\mathrm{d}t, \end{aligned} \quad (6)$$

where

$$w^{[a,b]}(t,x) = \begin{cases} t-a, & a \le t \le x \le b, \\ t-b, & a \le x < t \le b. \end{cases} \quad (7)$$

The organization of the paper is the following: in Sect. 2, we present the generalization of the Popoviciu's inequality using A. M. Fink's identity combined together with the n-convexity of the function ϕ. In Sect. 3, we present some interesting results using Čebyšev functional and Grüss-type inequalities along with some results relating to the Ostrowski-type inequality. In Sect. 4, we study the functional defined as the difference between the R.H.S. and the L.H.S. of the generalized inequality and our objective is to investigate the properties of functional, such as n-exponential and logarithmic convexity. Furthermore, we prove monotonicity property of the generalized

Cauchy means obtained via this functional. Finally, in Sect. 5 we give several examples of the families of functions for which the obtained results can be applied.

2. Generalization of Popoviciu's Inequality for n-convex Functions Via A. M. Fink's Identity

Motivated by identity (5), we construct the following identity with help of Fink identity.

Theorem 2.1. *Let $\phi\colon [\alpha,\beta] \to \mathbb{R}$ be such that for $n \geq 1$, $\phi^{(n-1)}$ is absolutely continuous and let $m,k \in \mathbb{N}$, $m \geq 3$, $2 \leq k \leq m-1$, $[\alpha,\beta] \subset \mathbb{R}$, $\mathbf{x} = (x_1,\ldots,x_m) \in [\alpha,\beta]^m$, $\mathbf{p} = (p_1,\ldots,p_m)$ be a real m-tuple such that $\sum_{j=1}^{k} p_{i_j} \neq 0$ for any $1 \leq i_1 < \cdots < i_k \leq m$ and $\sum_{i=1}^{m} p_i = 1$. Also let $\frac{\sum_{j=1}^{k} p_{i_j} x_{i_j}}{\sum_{j=1}^{k} p_{i_j}} \in [\alpha,\beta]$ for any $1 \leq i_1 < \cdots < i_k \leq m$ with $w^{[\alpha,\beta]}(t,x)$ be the same as defined in (7). Then we have the following identity:*

$$\begin{aligned}\Upsilon(\mathbf{x},\mathbf{p};\phi(x)) &= \sum_{w=2}^{n-l}\left(\frac{n-w}{w!(\beta-\alpha)}\right)\\ &\quad\times\left(\phi^{(w-1)}(\beta)\,\Upsilon(\mathbf{x},\mathbf{p};(x-\beta)^{w}) - \phi^{(w-1)}(\alpha)\,\Upsilon(\mathbf{x},\mathbf{p};(x-\alpha)^{w})\right)\\ &\quad+\frac{1}{(n-1)!\,(\beta-\alpha)}\int_{\alpha}^{\beta}\Upsilon(\mathbf{x},\mathbf{p};(x-t)^{n-1}\,w^{[\alpha,\beta]}(t,x))\phi^{(n)}(t)\mathrm{d}t. \end{aligned} \tag{8}$$

Proof. Using (6) in (5) and using linearity of the functional $\Upsilon(\cdot)$, we have

$$\begin{aligned}\Upsilon(\mathbf{x},\mathbf{p};\phi(x)) &= \Upsilon\left(\mathbf{x},\mathbf{p};\frac{n}{\beta-\alpha}\int_{\alpha}^{\beta}\phi(t)\mathrm{d}t\right)\\ &\quad+\sum_{w=1}^{n-l}\left(\frac{n-w}{w!(\beta-\alpha)}\right)\phi^{(w-1)}(\beta)\,\Upsilon(\mathbf{x},\mathbf{p};(x-\beta)^{w})\\ &\quad-\sum_{w=1}^{n-l}\left(\frac{n-w}{w!(\beta-\alpha)}\right)\phi^{(w-1)}(\alpha)\,\Upsilon(\mathbf{x},\mathbf{p};(x-\alpha)^{w})\\ &\quad+\frac{1}{(n-1)!\,(\beta-\alpha)}\int_{\alpha}^{\beta}\Upsilon(\mathbf{x},\mathbf{p};(x-t)^{n-1}\,w^{[\alpha,\beta]}(t,x))\phi^{(n)}(t)\mathrm{d}t.\end{aligned} \tag{9}$$

After simplification and following Remark 1.5, we get (8). □

In the following theorem we obtain generalizations of Popoviciu's inequality for n-convex functions.

Theorem 2.2. *Let all the assumptions of Theorem 2.1 be satisfied and let for $n \geq 1$*

$$\Upsilon(\mathbf{x},\mathbf{p};(x-t)^{n-1}\,w^{[\alpha,\beta]}(t,x)) \geq 0, \quad t \in [\alpha,\beta]. \tag{10}$$

If ϕ is n-convex, then we have

$$\Upsilon(\mathbf{x},\mathbf{p};\phi(x)) \geq \sum_{w=2}^{n-l}\left(\frac{n-w}{w!(\beta-\alpha)}\right) \times \left(\phi^{(w-1)}(\beta)\,\Upsilon(\mathbf{x},\mathbf{p};(x-\beta)^{w})-\phi^{(w-1)}(\alpha)\,\Upsilon(\mathbf{x},\mathbf{p};(x-\alpha)^{w})\right). \tag{11}$$

Proof. Since $\phi^{(n-1)}$ is absolutely continuous on $[\alpha,\beta]$, $\phi^{(n)}$ exists almost everywhere. As ϕ is n-convex, applying Theorem 1.3, we have, $\phi^{(n)} \geq 0$ for all $x \in [\alpha,\beta]$. Hence, we can apply Theorem 2.1 to obtain (11). □

Now we will give generalization of Popoviciu's inequality for $m-$tuples.

Theorem 2.3. *Let all the assumptions of Theorem 2.1 be satisfied in addition with the condition that $\mathbf{p}=(p_1,\ldots,p_m)$ be a positive m-tuple such that $\sum_{i=1}^{m} p_i = 1$ and consider $\phi\colon [\alpha,\beta]\to\mathbb{R}$ is n-convex function.*

(i) If n be even and $n > 3$, then (11) holds.
(ii) Let the inequality (11) be satisfied. If for even w: $\phi^{(w-1)}(\beta) \geq 0$ and $\phi^{(w-1)}(\alpha) \leq 0$ and for odd w: $\phi^{(w-1)}(\beta) \leq 0$ and $\phi^{(w-1)}(\alpha) \leq 0$, the R.H.S. of (11) is non-negative and we have inequality

$$\Upsilon(\mathbf{x},\mathbf{p};\phi(x)) \geq 0 \tag{12}$$

Proof. (i) For

$$\vartheta(x) := (x-t)^{n-1} w^{[\alpha,\beta]}(t,x) = \begin{cases} (x-t)^{n-1}(t-\alpha), & \alpha \leq t \leq x \leq \beta,\\ (x-t)^{n-1}(t-\beta), & \alpha \leq x < t \leq \beta, \end{cases}$$

we have,

$$\vartheta''(x) := \begin{cases} (n-1)(n-2)(x-t)^{n-3}(t-\alpha), & \alpha \leq t \leq x \leq \beta,\\ (n-1)(n-2)(x-t)^{n-3}(t-\beta), & \alpha \leq x < t \leq \beta, \end{cases}$$

showing that ϑ is convex for even n, where $n > 3$. Hence, by virtue of Remark 1.5, (10) holds for even n, where $n > 3$. Therefore, following Theorem 2.2, we can obtain (11).

(ii) It is easy to see that the function $(x-\alpha)^{w}$ is convex for $w = 2,\ldots,n-1$ and $(x-\beta)^{w}$ is convex for even w and concave for odd w, where $x \in [\alpha,\beta]$. Therefore using the given conditions and by following Remark 1.5, the non negativity of the R.H.S. of (11) is immediate and we have (12). □

3. Bounds for Identities Related to Generalization of Popoviciu's Inequality

In this section, we present some interesting results using Čebyšev functional and Grüss-type inequalities. For two Lebesgue integrable functions $f,h\colon [\alpha,\beta] \to \mathbb{R}$, we consider the Čebyšev functional

$$\Delta(f,h)=\frac{1}{\beta-\alpha}\int_\alpha^\beta f(t)h(t)\mathrm{d}t-\frac{1}{\beta-\alpha}\int_\alpha^\beta f(t)\mathrm{d}t\cdot\frac{1}{\beta-\alpha}\int_\alpha^\beta h(t)\mathrm{d}t.$$

The following Grüss-type inequalities are given in [4].

Theorem 3.1. *Let $f\colon[\alpha,\beta]\to\mathbb{R}$ be a Lebesgue integrable function and $h\colon[\alpha,\beta]\to\mathbb{R}$ be an absolutely continuous function with $(\cdot-\alpha)(\beta-\cdot)[h']^2\in L[\alpha,\beta]$. Then we have the inequality*

$$|\Delta(f,h)|\le\frac{1}{\sqrt{2}}[\Delta(f,f)]^{\frac{1}{2}}\frac{1}{\sqrt{\beta-\alpha}}\left(\int_\alpha^\beta(x-\alpha)(\beta-x)[h'(x)]^2\mathrm{d}x\right)^{\frac{1}{2}}.\quad(13)$$

The constant $\frac{1}{\sqrt{2}}$ in (13) is the best possible.

Theorem 3.2. *Assume that $h\colon[\alpha,\beta]\to\mathbb{R}$ is monotonic nondecreasing on $[\alpha,\beta]$ and $f\colon[\alpha,\beta]\to\mathbb{R}$ be absolutely continuous with $f'\in L_\infty[\alpha,\beta]$. Then we have the inequality*

$$|\Delta(f,h)|\le\frac{1}{2(\beta-\alpha)}||f'||_\infty\int_\alpha^\beta(x-\alpha)(\beta-x)[h'(x)]^2\mathrm{d}h(x).\quad(14)$$

The constant $\frac{1}{2}$ in (14) is the best possible.

In the sequel, we consider above theorems to derive generalizations of the results proved in the previous section. To avoid many notions let us denote

$$\mathfrak{R}(t)=\Upsilon(\mathbf{x},\mathbf{p};(x-t)^{n-1}\,w^{[\alpha,\beta]}(t,x),\quad t\in[\alpha,\beta],\quad(15)$$

Consider the Čebyšev functional $\Delta(\mathfrak{R},\mathfrak{R})$ given as:

$$\Delta(\mathfrak{R},\mathfrak{R})=\frac{1}{\beta-\alpha}\int_\alpha^\beta\mathfrak{R}^2(t)\mathrm{d}t-\left(\frac{1}{\beta-\alpha}\int_\alpha^\beta\mathfrak{R}(t)\mathrm{d}t\right)^2,\quad(16)$$

Theorem 3.3. *Let $\phi\colon[\alpha,\beta]\to\mathbb{R}$ be such that for $n\ge1$, $\phi^{(n)}$ is absolutely continuous with $(\cdot-\alpha)(\beta-\cdot)[\phi^{(n+1)}]^2\in L[\alpha,\beta]$. Let $m,k\in\mathbb{N}$, $m\ge3$, $2\le k\le m-1$, $[\alpha,\beta]\subset\mathbb{R}$, $\mathbf{x}=(x_1,\ldots,x_m)\in[\alpha,\beta]^m$, $\mathbf{p}=(p_1,\ldots,p_m)$ be a real m-tuple such that $\sum_{j=1}^k p_{i_j}\neq0$ for any $1\le i_1<\cdots<i_k\le m$ and $\sum_{i=1}^m p_i=1$. Also let $\frac{\sum_{j=1}^k p_{i_j}x_{i_j}}{\sum_{j=1}^k p_{i_j}}\in[\alpha,\beta]$ for any $1\le i_1<\cdots<i_k\le m$ with $\mathfrak{R}$ defined in (15).*
Then

$$\begin{aligned}\Upsilon(\mathbf{x},\mathbf{p};\phi(x))=&\sum_{w=2}^{n-l}\left(\frac{n-w}{w!(\beta-\alpha)}\right)\\&\times\left(\phi^{(w-1)}(\beta)\,\Upsilon(\mathbf{x},\mathbf{p};(x-\beta)^w)-\phi^{(w-1)}(\alpha)\,\Upsilon(\mathbf{x},\mathbf{p};(x-\alpha)^w)\right)\\&+\frac{\phi^{(n-1)}(\beta)-\phi^{(n-1)}(\alpha)}{(\beta-\alpha)^2\,(n-1)!}\int_\alpha^\beta\mathfrak{R}(t)\mathrm{d}t+\mathfrak{K}_n(\alpha,\beta;\phi),\end{aligned}\quad(17)$$

where the remainder $\mathfrak{K}_n(\alpha,\beta;\phi)$ satisfies the bound

$$|\mathfrak{K}_n(\alpha,\beta;\phi)|\le\frac{1}{\sqrt{2}\,(n-1)!}[\Delta(\mathfrak{R},\mathfrak{R})]^{\frac{1}{2}}\frac{1}{\sqrt{\beta-\alpha}}\left|\int_\alpha^\beta(t-\alpha)(\beta-t)[\phi^{(n+1)}(t)]^2\mathrm{d}t\right|^{\frac{1}{2}}.\quad(18)$$

Proof. (i) If we apply Theorem 3.1 for $f \to \mathfrak{R}$ and $h \to \phi^{(n)}$, we get

$$\left|\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\mathfrak{R}(t)\phi^{(n)}(t)\mathrm{d}t-\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\mathfrak{R}(t)\mathrm{d}t\cdot\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\phi^{(n)}(t)\mathrm{d}t\right| \leq \frac{1}{\sqrt{2}}[\Delta(\mathfrak{R},\mathfrak{R})]^{\frac{1}{2}}\frac{1}{\sqrt{\beta-\alpha}}\left|\int_{\alpha}^{\beta}(t-\alpha)(\beta-t)[\phi^{(n+1)}(t)]^{2}\mathrm{d}t\right|^{\frac{1}{2}}. \tag{19}$$

Divide both sides of (19) by $(n-1)$, we have

$$\left|\frac{1}{(n-1)!\beta-\alpha}\int_{\alpha}^{\beta}\mathfrak{R}(t)\phi^{(n)}(t)\mathrm{d}t - \frac{1}{(n-1)!\beta-\alpha}\int_{\alpha}^{\beta}\mathfrak{R}(t)\mathrm{d}t\cdot\frac{\phi^{(n-1)}(\beta)-\phi^{(n-1)}(\alpha)}{(\beta-\alpha)}\right| \leq \frac{1}{\sqrt{2}\,(n-1)!}[\Delta(\mathfrak{R},\mathfrak{R})]^{\frac{1}{2}}\frac{1}{\sqrt{\beta-\alpha}}\left|\int_{\alpha}^{\beta}(t-\alpha)(\beta-t)[\phi^{(n+1)}(t)]^{2}\mathrm{d}t\right|^{\frac{1}{2}}. \tag{20}$$

By denoting

$$\mathfrak{K}_n(\alpha,\beta;\phi)=\frac{1}{(n-1)!\beta-\alpha}\int_{\alpha}^{\beta}\mathfrak{R}(t)\phi^{(n)}(t)\mathrm{d}t - \frac{1}{(n-1)!\beta-\alpha}\int_{\alpha}^{\beta}\mathfrak{R}(t)\mathrm{d}t\cdot\frac{\phi^{(n-1)}(\beta)-\phi^{(n-1)}(\alpha)}{(\beta-\alpha)}. \tag{21}$$

In (20), we have (18). Hence, we have

$$\frac{1}{(n-1)!\beta-\alpha}\int_{\alpha}^{\beta}\mathfrak{R}(t)\phi^{(n)}(t)\mathrm{d}t = \frac{\phi^{(n-1)}(\beta)-\phi^{(n-1)}(\alpha)}{(\beta-\alpha)^{2}\,(n-1)!}\int_{\alpha}^{\beta}\mathfrak{R}(t)\mathrm{d}t+\mathfrak{K}_n(\alpha,\beta;\phi),$$

where the remainder $\mathfrak{K}_n(\alpha,\beta;\phi)$ satisfies the estimation (18). Now from identity (8), we obtain (17). □

The following Grüss-type inequalities can be obtained using Theorem 3.2.

Theorem 3.4. *Let $\phi : [\alpha,\beta] \to \mathbb{R}$ be such that for $n \geq 1$, $\phi^{(n)}$ is absolutely continuous and let $\phi^{(n+1)} \geq 0$ on $[\alpha,\beta]$ with $\mathfrak{R}$ defined in (15), respectively. Then the representation (17) and the remainder $\mathfrak{K}_n(\alpha,\beta;\phi)$ satisfies the estimation*

$$|\mathfrak{K}_n(\alpha,\beta;\phi)| \leq \frac{||\mathfrak{R}'||_{\infty}}{(n-1)!}\left[\frac{\phi^{(n-1)}(\beta)+\phi^{(n-1)}(\alpha)}{2}-\frac{\phi^{(n-2)}(\beta)-\phi^{(n-2)}(\alpha)}{\beta-\alpha}\right]. \tag{22}$$

Proof. Applying Theorem 3.2 for $f \to \mathfrak{R}$ and $h \to \phi^{(n)}$, we get

$$\left|\frac{1}{\beta-\alpha}\int_\alpha^\beta \mathfrak{R}(t)\phi^{(n)}(t)\mathrm{d}t - \frac{1}{\beta-\alpha}\int_\alpha^\beta \mathfrak{R}(t)\mathrm{d}t\cdot\frac{1}{\beta-\alpha}\int_\alpha^\beta \phi^{(n)}(t)\mathrm{d}t\right|$$
$$\leq \frac{1}{2(\beta-\alpha)}||\mathfrak{R}'||_\infty\int_\alpha^\beta (t-\alpha)(\beta-t)\phi^{(n+1)}(t)\mathrm{d}t. \tag{23}$$

Since

$$\begin{aligned}\int_\alpha^\beta (t-\alpha)(\beta-t)\phi^{(n+1)}(t)\mathrm{d}t &= \int_\alpha^\beta [2t-(\alpha+\beta)]\phi^{(n)}(t)\mathrm{d}t\\ &= (\beta-\alpha)\big[\phi^{(n-1)}(\beta)+\phi^{(n-1)}(\alpha)\big]\\ &\quad -2\big(\phi^{(n-2)}(\beta)-\phi^{(n-2)}(\alpha)\big).\end{aligned}$$

Therefore, using identity (8) and the inequality (23), we deduce (22). □

Now we intend to give the Ostrowski-type inequalities related to generalizations of Popoviciu's inequality.

Theorem 3.5. *Suppose all the assumptions of Theorem 2.1 be satisfied. Moreover, assume (p,q) is a pair of conjugate exponents, that is $p,q \in [1,\infty]$ such that $1/p+1/q=1$. Let $|\phi^{(n)}|^p : [\alpha,\beta] \to \mathbb{R}$ be an R-integrable function for some $n \geq 2$. Then, we have*

$$\left|\begin{aligned}&\Upsilon(\mathbf{x},\mathbf{p};\phi(x)) - \textstyle\sum_{w=2}^{n-l}\left(\frac{n-w}{w!(\beta-\alpha)}\right)\\ &\quad\times\left(\phi^{(w-1)}(\beta)\,\Upsilon(\mathbf{x},\mathbf{p};(x-\beta)^w) - \phi^{(w-1)}(\alpha)\,\Upsilon(\mathbf{x},\mathbf{p};(x-\alpha)^w)\right)\end{aligned}\right|$$
$$\leq \frac{1}{(n-1)!\beta-\alpha}||\phi^{(n)}||_p\left(\int_\alpha^\beta\left|\Upsilon(\mathbf{x},\mathbf{p};(x-t)^{n-1}w^{[\alpha,\beta]}(t,x))\right|^q\mathrm{d}t\right)^{1/q} \tag{24}$$

The constant on the R.H.S. of (24) is sharp for $1 < p \leq \infty$ and the best possible for $p=1$.

Proof. Let us denote

$$\mathfrak{I} = \frac{1}{(n-1)!\beta-\alpha}\left(\Upsilon(\mathbf{x},\mathbf{p};(x-t)^{n-1}w^{[\alpha,\beta]}(t,x))\right), \quad t \in [\alpha,\beta].$$

Using identity (8), we obtain

$$\left|\begin{aligned}&\Upsilon(\mathbf{x},\mathbf{p};\phi(x)) - \textstyle\sum_{w=2}^{n-l}\left(\frac{n-w}{w!(\beta-\alpha)}\right)\\ &\quad\times\left(\phi^{(w-1)}(\beta)\,\Upsilon(\mathbf{x},\mathbf{p};(x-\beta)^w) - \phi^{(w-1)}(\alpha)\,\Upsilon(\mathbf{x},\mathbf{p};(x-\alpha)^w)\right)\end{aligned}\right|$$
$$= \left|\int_\alpha^\beta \mathfrak{I}(t)\phi^{(n)}(t)\mathrm{d}t\right|. \tag{25}$$

Apply Hölder's inequality for integrals on the right-hand side of (25), we have

$$\left|\int_\alpha^\beta \mathfrak{I}(t)\phi^{(n)}(t)\mathrm{d}t\right| \leq \left(\int_\alpha^\beta \left|\phi^{(n)}(t)\right|^p\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_\alpha^\beta |\mathfrak{I}(t)|^q\,\mathrm{d}t\right)^{\frac{1}{q}},$$

which combine together with (25) gives (24).

For the proof of the sharpness of the constant $\left(\int_\alpha^\beta \left|\Im(t)\right|^q \mathrm{d}t\right)^{1/q}$, let us define the function ϕ for which the equality in (24) is obtained.

For $1 < p \leq \infty$ take ϕ to be such that

$$\phi^{(n)}(t) = \mathrm{sgn}\Im(t)|\Im(t)|^{\frac{1}{p-1}}.$$

For $p = \infty$ take $\phi^{(n)}(t) = \mathrm{sgn}\Im(t)$.

For $p = 1$, we prove that

$$\left|\int_\alpha^\beta \Im(t)\phi^{(n)}(t)\mathrm{d}t\right| \leq \max_{t\in[\alpha,\beta]} |\Im(t)|\left(\int_\alpha^\beta \phi^{(n)}(t)\mathrm{d}t\right) \tag{26}$$

is the best possible inequality. Suppose that $|\Im(t)|$ attains its maximum at $t_0 \in [\alpha, \beta]$. To start with first we assume that $\Im(t_0) > 0$. For δ small enough we define $\phi_\delta(t)$ by

$$\phi_\delta(t) = \begin{cases} 0, & \alpha \leq t \leq t_0, \\ \frac{1}{\delta n!}(t-t_0)^n, & t_o \leq t \leq t_0+\delta, \\ \frac{1}{n!}(t-t_0)^{n-1}, & t_0+\delta \leq t \leq \beta. \end{cases}$$

Then for δ small enough

$$\left|\int_\alpha^\beta \Im(t)\phi^{(n)}(t)\mathrm{d}t\right| = \left|\int_{t_0}^{t_0+\delta} \Im(t)\frac{1}{\delta}\mathrm{d}t\right| = \frac{1}{\delta}\int_{t_0}^{t_0+\delta} \Im(t)\mathrm{d}t.$$

Now from inequality (26), we have

$$\frac{1}{\delta}\int_{t_0}^{t_0+\delta} \Im(t)\mathrm{d}t \leq \Im(t_0)\int_{t_0}^{t_0+\delta} \frac{1}{\delta}\mathrm{d}t = \Im(t_0).$$

Since

$$\lim_{\delta\to 0} \frac{1}{\delta}\int_{t_0}^{t_0+\delta} \Im(t)\mathrm{d}t = \Im(t_0),$$

the statement follows. The case when $\Im(t_0) < 0$, we define $\phi_\delta(t)$ by

$$\phi_\delta(t) = \begin{cases} \frac{1}{n!}(t-t_0-\delta)^{n-1}, & \alpha \leq t \leq t_0, \\ \frac{-1}{\delta n!}(t-t_0-\delta)^n, & t_o \leq t \leq t_0+\delta, \\ 0, & t_0+\delta \leq t \leq \beta, \end{cases}$$

and rest of the proof is the same as above. □

4. Mean Value Theorems and n-exponential Convexity

We recall some definitions and basic results from [1,7,11] which are required in sequel.

Definition 4. A function $\phi\colon I \to \mathbb{R}$ is n-exponentially convex in the Jensen sense on I if

$$\sum_{i,j=1}^{n} \xi_i\xi_j\,\phi\left(\frac{x_i+x_j}{2}\right) \geq 0,$$

hold for all choices $\xi_1, \ldots, \xi_n \in \mathbb{R}$ and all choices $x_1, \ldots, x_n \in I$. A function $\phi : I \to \mathbb{R}$ is n-exponentially convex if it is n-exponentially convex in the Jensen sense and continuous on I.

Definition 5. A function $\phi\colon I \to \mathbb{R}$ is exponentially convex in the Jensen sense on I if it is n-exponentially convex in the Jensen sense for all $n \in \mathbb{N}$.

A function $\phi\colon I \to \mathbb{R}$ is exponentially convex if it is exponentially convex in the Jensen sense and continuous.

Proposition 4.1. *If $\phi\colon I \to \mathbb{R}$ is an n-exponentially convex in the Jensen sense, then the matrix $\left[\phi\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{m}$ is a positive semi-definite matrix for all $m \in \mathbb{N}, m \le n$. Particularly,*

$$\det\left[\phi\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{m} \ge 0$$

for all $m \in \mathbb{N}$, $m = 1, 2, \ldots, n$.

Remark 4.2. It is known that $\phi\colon I \to \mathbb{R}$ is a log-convex in the Jensen sense if and only if

$$\alpha^2\phi(x) + 2\alpha\beta\phi\left(\frac{x+y}{2}\right) + \beta^2\phi(y) \ge 0,$$

holds for every $\alpha, \beta \in \mathbb{R}$ and $x, y \in I$. It follows that a positive function is log-convex in the Jensen sense if and only if it is $2-$exponentially convex in the Jensen sense.

A positive function is log-convex if and only if it is $2-$exponentially convex.

Remark 4.3. By the virtue of Theorem 2.2, we define the positive linear functional with respect to n-convex function ϕ as follows

$$\Lambda(\phi) := \Upsilon(\mathbf{x}, \mathbf{p}; \phi(x)) - \sum_{w=2}^{n-l}\left(\frac{n-w}{w!(\beta-\alpha)}\right) \times \left(\phi^{(w-1)}(\beta)\,\Upsilon(\mathbf{x}, \mathbf{p}; (x-\beta)^w) - \phi^{(w-1)}(\alpha)\,\Upsilon(\mathbf{x}, \mathbf{p}; (x-\alpha)^w)\right) \ge 0. \tag{27}$$

Lagrange- and Cauchy-type mean value theorems related to defined functional are given in the following theorems.

Theorem 4.4. *Let $\phi\colon [\alpha, \beta] \to \mathbb{R}$ be such that $\phi \in C^n[\alpha, \beta]$. If the inequality in (10) holds, then there exist $\xi \in [\alpha, \beta]$ such that*

$$\Lambda(\phi) = \phi^{(n)}(\xi)\Lambda(\varphi), \tag{28}$$

where $\varphi(x) = \frac{x^n}{n!}$ and $\Lambda(\cdot)$ is defined by (27).

Proof. Similar to the proof of Theorem 4.1 in [8] (see also [3]). □

Theorem 4.5. *Let $\phi, \psi : [\alpha, \beta] \to \mathbb{R}$ be such that $\phi, \psi \in C^n[\alpha, \beta]$. If the inequality in* (10) *holds, then there exist $\xi \in [\alpha, \beta]$ such that*

$$\frac{\Lambda(\phi)}{\Lambda(\psi)} = \frac{\phi^{(n)}(\xi)}{\psi^{(n)}(\xi)}, \tag{29}$$

provided that the denominators are non-zero and $\Lambda(\cdot)$ is defined by (27).

Proof. Similar to the proof of Corollary 4.2 in [8] (see also [3]). □

Theorem 4.5 enables us to define Cauchy means, because if

$$\xi = \left(\frac{\phi^{(n)}}{\psi^{(n)}}\right)^{-1} \left(\frac{\Lambda(\phi)}{\Lambda(\psi)}\right),$$

which means that ξ is mean of α, β for given functions ϕ and ψ.

Next we construct the non-trivial examples of n-exponentially and exponentially convex functions from positive linear functional $\Lambda(\cdot)$. We use the idea given in [11]. In the sequel I and J are intervals in $\mathbb{R}$.

Theorem 4.6. *Let $\Gamma = \{\phi_t : t \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval I in $\mathbb{R}$ such that the function $t \mapsto [x_0, \ldots, x_n; \phi_t]$ is n-exponentially convex in the Jensen sense on J for every $(n+1)$ mutually different points $x_0, \ldots, x_n \in I$. Then for the linear functional $\Lambda(\phi_t)$ as defined by* (27), *the following statements are valid:*

(i) *The function $t \to \Lambda(\phi_t)$ is n-exponentially convex in the Jensen sense on J and the matrix $[\Lambda(\phi_{\frac{t_j+t_l}{2}})]_{j,l=1}^m$ is a positive semi-definite for all $m \in \mathbb{N}, m \leq n$, $t_1, \ldots, t_m \in J$. Particularly,*

$$\det[\Lambda(\phi_{\frac{t_j+t_l}{2}})]_{j,l=1}^m \geq 0 \quad \text{for all } m \in \mathbb{N}, \quad m = 1, 2, \ldots, n.$$

(ii) *If the function $t \to \Lambda(\phi_t)$ is continuous on J, then it is n-exponentially convex on J.*

Proof. (i) For $\xi_j \in \mathbb{R}$ and $t_j \in J$, $j = 1, \ldots, n$, we define the function

$$h(x) = \sum_{j,l=1}^{n} \xi_j \xi_l \phi_{\frac{t_j+t_l}{2}}(x).$$

Using the assumption that the function $t \mapsto [x_0, \ldots, x_n; \phi_t]$ is n-exponentially convex in the Jensen sense, we have

$$[x_0, \ldots, x_n, h] = \sum_{j,l=1}^{n} \xi_j \xi_l \left[x_0, \ldots, x_n; \phi_{\frac{t_j+t_l}{2}}\right] \geq 0,$$

which in turn implies that h is a n-convex function on J, therefore, from Remark 4.3 we have $\Lambda(h) \geq 0$. The linearity of $\Lambda(\cdot)$ gives

$$\sum_{j,l=1}^{n} \xi_j \xi_l \Lambda\left(\phi_{\frac{t_j+t_l}{2}}\right) \geq 0.$$

We conclude that the function $t \mapsto \Lambda(\phi_t)$ is n-exponentially convex on J in the Jensen sense.

The remaining part follows from Proposition 4.1.

(ii) If the function $t \to \Lambda(\phi_t)$ is continuous on J, then it is n-exponentially convex on J by definition. □

The following corollary is an immediate consequence of the above theorem.

Corollary 4.7. *Let $\Gamma = \{\phi_t : t \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval I in $\mathbb{R}$, such that the function $t \mapsto [x_0, \ldots, x_n; \phi_t]$ is exponentially convex in the Jensen sense on J for every $(n+1)$ mutually different points $x_0, \ldots, x_n \in I$. Then for the linear functional $\Lambda(\phi_t)$ as defined by (27), the following statements hold:*

(i) *The function $t \to \Lambda(\phi_t)$ is exponentially convex in the Jensen sense on J and the matrix $\left[\Lambda\left(\phi_{\frac{t_j+t_l}{2}}\right)\right]_{j,l=1}^{m}$ is a positive semi-definite for all $m \in \mathbb{N}, m \leq n$, $t_1, \ldots, t_m \in J$. Particularly,*

$$\det\left[\Lambda\left(\phi_{\frac{t_j+t_l}{2}}\right)\right]_{j,l=1}^{m} \geq 0 \quad \textit{for all } m \in \mathbb{N}, \quad m = 1, 2, \ldots, n.$$

(ii) *If the function $t \to \Lambda(\phi_t)$ is continuous on J, then it is exponentially convex on J.*

Corollary 4.8. *Let $\Gamma = \{\phi_t : t \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval I in $\mathbb{R}$, such that the function $t \mapsto [x_0, \ldots, x_n; \phi_t]$ is $2-$exponentially convex in the Jensen sense on J for every $(n+1)$ mutually different points $x_0, \ldots, x_n \in I$. Let $\Lambda(\cdot)$ be linear functional defined by (27). Then the following statements hold:*

(i) *If the function $t \mapsto \Lambda(\phi_t)$ is continuous on J, then it is $2-$exponentially convex function on J. If $t \mapsto \Lambda(\phi_t)$ is additionally strictly positive, then it is also log-convex on J. Furthermore, the following inequality holds true:*

$$[\Lambda(\phi_s)]^{t-r} \leq [\Lambda(\phi_r)]^{t-s} [\Lambda(\phi_t)]^{s-r},$$

for every choice $r, s, t \in J$, such that $r < s < t$.

(ii) *If the function $t \mapsto \Lambda(\phi_t)$ is strictly positive and differentiable on J, then for every $p, q, u, v \in J$, such that $p \leq u$ and $q \leq v$, we have*

$$\mu_{p,q}(\Lambda, \Gamma) \leq \mu_{u,v}(\Lambda, \Gamma), \tag{30}$$

where

$$\mu_{p,q}(\Lambda, \Gamma) = \begin{cases} \left(\frac{\Lambda(\phi_p)}{\Lambda(\phi_q)}\right)^{\frac{1}{p-q}}, & p \neq q, \\ \exp\left(\frac{\frac{d}{dp}\Lambda(\phi_p)}{\Lambda(\phi_p)}\right), & p = q, \end{cases} \tag{31}$$

for $\phi_p, \phi_q \in \Gamma$.

Proof. (i) This is an immediate consequence of Theorem 4.6 and Remark 4.2.

(ii) Since $p \mapsto \Lambda(\phi_t)$ is positive and continuous, by (i) we have that $t \mapsto \Lambda(\phi_t)$ is log-convex on J, that is, the function $t \mapsto \log \Lambda(\phi_t)$ is convex on J. Hence, we get

$$\frac{\log \Lambda(\phi_p) - \log \Lambda(\phi_q)}{p-q} \leq \frac{\log \Lambda(\phi_u) - \log \Lambda(\phi_v)}{u-v}, \tag{32}$$

for $p \leq u, q \leq v, p \neq q, u \neq v$. So, we conclude that

$$\mu_{p,q}(\Lambda, \Gamma) \leq \mu_{u,v}(\Lambda, \Gamma).$$

Cases $p = q$ and $u = v$ follow from (32) as limit cases. □

5. Applications to Cauchy Means

In this section, we present some families of functions which fulfil the conditions of Theorem 4.6, Corollaries 4.7 and 4.8. This enables us to construct large families of functions which are exponentially convex. Explicit form of this functions is obtained after we calculate explicit action of functionals on a given family.

Example 5.1. Let us consider a family of functions

$$\Gamma_1 = \{\phi_t : \mathbb{R} \to \mathbb{R} : t \in \mathbb{R}\}$$

defined by

$$\phi_t(x) = \begin{cases} \frac{e^{tx}}{t^n}, & t \neq 0, \\ \frac{x^n}{n!}, & t = 0. \end{cases}$$

Since $\frac{d^n \phi_t}{\mathrm{d}x^n}(x) = e^{tx} > 0$, the function ϕ_t is n-convex on $\mathbb{R}$ for every $t \in \mathbb{R}$ and $t \mapsto \frac{d^n \phi_t}{\mathrm{d}x^n}(x)$ is exponentially convex by definition. Using analogous arguing as in the proof of Theorem 4.6 we also have that $t \mapsto [x_0, \ldots, x_n; \phi_t]$ is exponentially convex (and so exponentially convex in the Jensen sense). Now, using Corollary 4.7 we conclude that $t \mapsto \Lambda(\phi_t)$ is exponentially convex in the Jensen sense. It is easy to verify that this mapping is continuous (although the mapping $t \mapsto \phi_t$ is not continuous for $t = 0$), so it is exponentially convex. For this family of functions, $\mu_{t,q}(\Lambda, \Gamma_1)$, from (31), becomes

$$\mu_{t,q}(\Lambda, \Gamma_1) = \begin{cases} \left(\frac{\Lambda(\phi_t)}{\Lambda(\phi_q)}\right)^{\frac{1}{t-q}}, & t \neq q, \\ \exp\left(\frac{\Lambda(\mathrm{id} \cdot \phi_t)}{\Lambda(\phi_t)} - \frac{n}{t}\right), & t = q \neq 0, \\ \exp\left(\frac{1}{n+1} \frac{\Lambda(\mathrm{id} \cdot \phi_0)}{\Lambda(\phi_0)}\right), & t = q = 0, \end{cases}$$

where "id" is the identity function. By Corollary 4.8 $\mu_{t,q}(\Lambda, \Gamma_1)$ is a monotone function in parameters t and q.

Since

$$\left(\frac{\frac{d^n f_t}{\mathrm{d}x^n}}{\frac{d^n f_q}{\mathrm{d}x^n}}\right)^{\frac{1}{t-q}} (\log x) = x,$$

using Theorem 4.5 it follows that:

$$M_{t,q}(\Lambda, \Gamma_1) = \log \mu_{t,q}(\Lambda, \Gamma_1),$$

satisfies

$$\alpha \leq M_{t,q}(\Lambda, \Gamma_1) \leq \beta.$$

Hence, $M_{t,q}(\Lambda, \Gamma_1)$ is a monotonic mean.

Example 5.2. Let us consider a family of functions

$$\Gamma_2 = \{g_t : (0,\infty) \to \mathbb{R} : t \in \mathbb{R}\}$$

defined by

$$g_t(x) = \begin{cases} \frac{x^t}{t(t-1)\cdots(t-n+1)}, & t \notin \{0,1,\ldots,n-1\}, \\ \frac{x^j \log x}{(-1)^{n-1-j} j!(n-1-j)!}, & t = j \in \{0,1,\ldots,n-1\}. \end{cases}$$

Since $\frac{d^n g_t}{\mathrm{d}x^n}(x) = x^{t-n} > 0$, the function g_t is n-convex for $x > 0$ and $t \mapsto \frac{d^n g_t}{\mathrm{d}x^n}(x)$ is exponentially convex by definition. Arguing as in Example 5.1 we get that the mappings $t \mapsto \Lambda(g_t)$ is exponentially convex. Hence, for this family of functions $\mu_{p,q}(\Lambda, \Gamma_2)$, from (31), are equal to

$$\mu_{t,q}(\Lambda,\Gamma_2) = \begin{cases} \left(\frac{\Lambda(g_t)}{\Lambda(g_q)}\right)^{\frac{1}{t-q}}, & t \neq q, \\ \exp\left((-1)^{n-1}(n-1)!\frac{\Lambda(g_0 g_t)}{\Lambda(g_t)} + \sum_{k=0}^{n-1}\frac{1}{k-t}\right), & t = q \notin \{0,1,\ldots,n-1\}, \\ \exp\left((-1)^{n-1}(n-1)!\frac{\Lambda(g_0 g_t)}{2\Lambda(g_t)} + \sum_{\substack{k=0\\k\neq t}}^{n-1}\frac{1}{k-t}\right), & t = q \in \{0,1,\ldots,n-1\}. \end{cases}$$

Again, using Theorem 4.5 we conclude that

$$\alpha \le \left(\frac{\Lambda(g_t)}{\Lambda(g_q)}\right)^{\frac{1}{t-q}} \le \beta. \tag{33}$$

Hence, $\mu_{t,q}(\Lambda, \Gamma_2)$ is a mean and its monotonicity is followed by (30).

Example 5.3. Let

$$\Gamma_3 = \{\zeta_t : (0,\infty) \to \mathbb{R} : t \in (0,\infty)\}$$

be a family of functions defined by

$$\zeta_t(x) = \begin{cases} \frac{t^{-x}}{(-\log t)^n}, & t \neq 1; \\ \frac{x^n}{(n)!}, & t = 1. \end{cases}$$

Since $\frac{d^n \zeta_t}{\mathrm{d}x^n}(x) = t^{-x}$ is the Laplace transform of a non-negative function (see [15]) it is exponentially convex. Obviously ζ_t are n-convex functions for every $t > 0$.

For this family of functions, $\mu_{t,q}(\Lambda, \Gamma_3)$, in this case for $[\alpha, \beta] \subset \mathbb{R}^+$, from (31) becomes

$$\mu_{t,q}(\Lambda,\Gamma_3) = \begin{cases} \left(\frac{\Lambda(\zeta_t)}{\Lambda(\zeta_q)}\right)^{\frac{1}{t-q}}, & t \neq q; \\ exp\left(-\frac{\Lambda(\mathrm{id}\cdot\zeta_t)}{t\Lambda(\zeta_t)} - \frac{n}{t\log t}\right), & t = q \neq 1; \\ exp\left(-\frac{1}{n+1}\frac{\Lambda(\mathrm{id}\cdot\zeta_1)}{\Lambda(\zeta_1)}\right), & t = q = 1, \end{cases}$$

where id is the identity function. By Corollary 4.8 $\mu_{p,q}(\Lambda, \Gamma_3)$ is a monotone function in parameters t and q.

Using Theorem 4.5 it follows that

$$M_{t,q}\left(\Lambda, \Gamma_3\right) = -L(t,q) log \mu_{t,q}\left(\Lambda, \Gamma_3\right),$$

satisfies

$$\alpha \leq M_{t,q}\left(\Lambda, \Gamma_3\right) \leq \beta.$$

This shows that $M_{t,q}\left(\Lambda, \Gamma_3\right)$ is a mean. Because of the inequality (30), this mean is monotonic. Furthermore, $L(t,q)$ is logarithmic mean defined by

$$L(t,q) = \begin{cases} \frac{t-q}{\log t - \log q}, & \text{t} \neq \text{q;} \\ \\ t, & \text{t} = \text{q.} \end{cases}$$

Example 5.4. Let

$$\Gamma_4 = \{\Lambda_t : (0,\infty) \to \mathbb{R} : t \in (0,\infty)\}$$

be a family of functions defined by

$$\Lambda_t(x) = \frac{e^{-x\sqrt{t}}}{\left(-\sqrt{t}\right)^n}.$$

Since $\frac{d^n \Lambda_t}{\mathrm{d}x^n}(x) = e^{-x\sqrt{t}}$ is the Laplace transform of a non-negative function (see [15]) it is exponentially convex. Obviously Λ_t are n-convex function for every $t > 0$.

For this family of functions, $\mu_{t,q}\left(\Lambda, \Gamma_4\right)$, in this case for $[\alpha, \beta] \subset \mathbb{R}^+$, from (31) becomes

$$\mu_{t,q}\left(\Lambda, \Gamma_4\right) = \begin{cases} \left(\frac{\Lambda(\Lambda_t)}{\Lambda(\Lambda_q)}\right)^{\frac{1}{t-q}}, & t \neq q; \\ exp\left(-\frac{\Lambda(\mathrm{id}\cdot\Lambda_t)}{2\sqrt{t}\Lambda(\Lambda_t)} - \frac{n}{2t}\right), & t = q; \end{cases} \quad i = 1, 2.$$

By Corollary 4.8, it is a monotone function in parameters t and q.
Using Theorem 4.5 it follows that

$$M_{t,q}\left(\Lambda, \Gamma_4\right) = -\left(\sqrt{t} + \sqrt{q}\right) \ln \mu_{t,q}\left(\Lambda, \Gamma_4\right),$$

satisfies

$$\alpha \leq M_{t,q}\left(\Lambda, \Gamma_4\right) \leq \beta.$$

This shows that $M_{t,q}\left(\Lambda, \Gamma_4\right)$ is a mean. Because of the above inequality (30), this mean is monotonic.

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