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# Approximately $n$-Multiplicative Functionals on Banach Algebras

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**Abstract.** Let $A$ be a normed algebra, $\varphi : A \to \mathbb{C}$ be a linear functional. Then, the functional $(\varphi, n)^{\vee}$ is defined as $(\varphi, n)^{\vee}(a_1, \ldots, a_n) = \varphi(a_1 \ldots a_n) - \varphi(a_1) \ldots \varphi(a_n)$ for all elements $a_1, \ldots, a_n \in A$. If the norm of $(\varphi, n)^{\vee}$ is small, then $\varphi$ is approximately $n$-multiplicative linear functional and it is of interest whether or not $\|(\varphi, n)^{\vee}\|$ being small implies that $\varphi$ is near to an $n$-multiplicative linear functional. If this property holds for a Banach algebra $A$, then $A$ is an $n$-$AMNM$ algebra (approximately $n$-multiplicative linear functionals are near $n$-multiplicative linear functionals). We show that some properties of $AMNM$ (2-$AMNM$) algebras are also valid for $n$-$AMNM$ algebras. For example, we give some alternative definitions of $n$-$AMNM$. We also prove some theorems on the hereditary properties of $n$-$AMNM$ condition and we use an equivalent condition for the $n$-$AMNM$ property on certain Banach algebras when the Gelfand and norm topologies coincide on the character space of the algebra. We also give some examples which are $n$-$AMNM$ and finally, exhibit an example which is not $n$-$AMNM$.

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**Keywords.** Approximately $n$-multiplicative linear functional, Banach algebra, $(\varepsilon, n)$-multiplicative linear functional, $n$-$AMNM$ property.

## 1. Introduction

Let $A$ and $B$ be two complex algebras and $n \geq 2$ be an integer. A map $\varphi : A \to B$ is called an $n$-multiplicative if $\varphi(a_1 a_2 \ldots a_n) = \varphi(a_1)\varphi(a_2) \ldots \varphi(a_n)$ for all elements $a_1, a_2, \ldots, a_n \in A$. Moreover, if $\varphi$ is a linear mapping, then it is called an $n$-homomorphism. If $\varphi : A \to \mathbb{C}$ is a nonzero $n$-homomorphism, then $\varphi$ is called a complex $n$-character, or in brief, an $n$-character of $A$. If $A$ is a complex topological algebra, then the set of all continuous $n$-characters

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of $A$ is denoted by $M_{(A,n)}$. As usual, the set of all continuous characters of $A$ is denoted by $M_A$. The notion of $n$-homomorphism between Banach algebras was first introduced by Hejazian et al. and some of their significant properties were discussed in [7]. For further details on the above concepts and properties one can refer, for example, to [5,6,9–11,13] and [14].

Let $A$ and $B$ be normed algebras and $\varepsilon > 0$. A linear map $\varphi : A \to B$ is called an $\varepsilon$-multiplicative if for all $x, y \in A$,

$$\|\varphi(xy) - \varphi(x)\varphi(y)\| \leq \varepsilon\|x\|\|y\|.$$

The dual of a Banach algebra $A$ is denoted by $A^*$ and the set of all $n$-multiplicative (multiplicative) linear functionals on $A$ is thus $M_{(A,n)} \cup \{0\}$ $(M_A \cup \{0\})$.
For each $\varphi \in A^*$, define

$$d(\varphi) = \inf\{\|\varphi - \psi\| : \psi \in M_A \cup \{0\}\}.$$

An algebra $A$ is called an algebra in which approximately multiplicative linear functionals are near multiplicative linear functionals, or $A$ is $AMNM$ for short, if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $d(\varphi) < \varepsilon$, where $\varphi$ is a $\delta$-multiplicative linear functional. The notion of $AMNM$ algebras was first introduced by Johnson and some significant properties of this algebras were discussed in [1].

Let $A$ and $B$ be normed algebras and let $\varphi : A \to B$ be a linear map. We define the map $(\varphi, n)^{\vee}$ as follows:

$$(\varphi, n)^{\vee}(a_1, \cdots, a_n) = \varphi(a_1 \cdots a_n) - \varphi(a_1) \ldots \varphi(a_n)$$

for all elements $a_1, \ldots, a_n \in A$ with

$$\|(\varphi, n)^{\vee}\| = \sup\{\|(\varphi, n)^{\vee}(a_1, \ldots, a_n)\| : \ \|a_i\| \leq 1, \ 1 \leq i \leq n\}.$$

If $\|(\varphi, n)^{\vee}\| < \varepsilon$, then we say that $\varphi$ is an $(\varepsilon, n)$-multiplicative linear map. Clearly, every $(\varepsilon, 2)$-multiplicative linear map is just an $\varepsilon$-multiplicative linear map, in the usual sense. We also say that $\varphi$ is approximately $n$-multiplicative linear map, if there exists an $\varepsilon > 0$ such that $\varphi$ is $(\varepsilon, n)$-multiplicative linear map. For some properties of approximately $n$-multiplicative linear map, one may refer to [2,3]. The following example shows that the class of approximately $n$-multiplicative linear mappings is essentially wider than the class of approximately multiplicative linear mappings.

*Example* 1.1. Let $X$ be an infinite-dimensional Banach algebra and $f$ be a linear discontinuous functional on $X$. Now, consider the following two sets $A$ and $B$,

$$A = \left\{ \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} : x \in X \right\}, \qquad B = \left\{ \begin{pmatrix} 0 & 0 & x \\ y & 0 & z \\ 0 & 0 & 0 \end{pmatrix} : x, y, z \in \mathbb{C} \right\}.$$

It is easy to see that $A$ and $B$ are two Banach algebras with the usual matrix operations for addition, scalar multiplication and product if they are equipped with the maximum norm. Define $\varphi : A \to B$ with

$$\varphi\left(\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 & f(x) \\ f(x) & 0 & f(x) \\ 0 & 0 & 0 \end{pmatrix} \qquad (x \in X).$$

Since for all $P \in A$ and $Q \in B$, $P^2 = Q^3 = 0$, then $\varphi$ is a 3-homomorphism and so it is an $(\varepsilon, 3)$-multiplicative linear map for all $\varepsilon > 0$. On the other hand, $\varphi$ is not an approximately multiplicative linear map because $f$ is an unbounded linear functional on $X$.

This paper is concerned with approximately $n$-multiplicative linear functionals on Banach algebras, where $n \geq 2$ is an integer. We say that $A$ is an $n$-$AMNM$ (algebras in which approximately $n$-multiplicative linear functionals are near $n$-multiplicative linear functionals) if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $d_n(\varphi) < \varepsilon$ whenever $\varphi$ is $(\delta, n)$-multiplicative linear functional, and $d_n(\varphi) = \inf\{\|\varphi - \psi\| : \psi \in M_{(A,n)} \cup \{0\}\}$. Clearly, every 2-$AMNM$ algebra is just an $AMNM$ algebra, in the usual sense.

In this paper, we show that some properties of $AMNM$ algebras are also valid for $n$-$AMNM$ algebras. We prove that every $(\varepsilon, n)$-multiplicative linear functional $\varphi$ on a Banach algebra is bounded by $1 + \varepsilon$. We also give some alternative definitions of $n$-$AMNM$ and prove some theorems on the hereditary properties of $n$-$AMNM$ condition. Moreover, we show that if $A$ is a commutative separable unital Banach algebra where the Gelfand and norm topologies on $M_{(A,n)}$ are the same, then $A$ is $n$-$AMNM$ if and only if for all sequences $(\varphi_m)$ in $A^*$ with $\|(\varphi_m, n)^{\vee}\| \to 0$ and $\varphi_m \to \varphi$ in the *weak*$^*$ topology, $\varphi \neq 0$, then $\|\varphi_m - \varphi\| \to 0$.

Furthermore, we prove that if $A$ is a commutative separable unital Banach algebra where the Gelfand and norm topologies on $M_A$ are the same, then $A$ is $AMNM$ if and only if $A$ is $n$-$AMNM$. Finally, we give some examples of $n$-$AMNM$ Banach algebras and an example of a Banach algebra which is not $n$-$AMNM$.

## 2. Approximately $n$-Multiplicative Linear Functionals

Throughout this section, $n$ is an integer with $n \geq 2$. We first state and prove the following result, that will be needed later [3, 2.4].

**Theorem 2.1.** *Let $A$ be a normed algebra, and $p \geq 0$. If $\varphi : A \to \mathbb{C}$ satisfies $|\varphi(x_1 \cdots x_n) - \varphi(x_1) \cdots \varphi(x_n)| \leq \varepsilon \|x_1\|^p \cdots \|x_n\|^p$ for all $x_1 \cdots x_n \in A$, then $\varphi$ is $n$-multiplicative or there exists a constant $k$ such that $|\varphi(x)| \leq k\|x\|^p$ for all $x \in A$.*

*Proof.* Suppose that $\varphi$ is not $n$-multiplicative, that is, there exist $a_1, \ldots, a_n \in A$ such that

$$\varphi(a_1 \cdots a_n) \neq \varphi(a_1) \cdots \varphi(a_n).$$

Then, for every nonzero element $x \in A$, we have

$$
\begin{aligned}
&|\varphi(x)|^{(n-1)}|\varphi(a_1\cdots a_n)-\varphi(a_1)\cdots\varphi(a_n)| \\
&\quad = |\varphi(x)^{(n-1)}\varphi(a_1\cdots a_n)-\varphi(x)^{(n-1)}\varphi(a_1)\cdots\varphi(a_n) \\
&\qquad \pm\varphi(x^{(n-1)}a_1\cdots a_n)\pm\varphi(x^{(n-1)}a_1)\varphi(a_2)\cdots\varphi(a_n)| \\
&\quad \le |\varphi(x)^{(n-1)}\varphi(a_1\cdots a_n)-\varphi(x^{(n-1)}a_1\cdots a_n)| \\
&\qquad +|\varphi(x^{(n-1)}a_1\cdots a_n)-\varphi(x^{(n-1)}a_1)\varphi(a_2)\cdots\varphi(a_n)| \\
&\qquad +|\varphi(x^{(n-1)}a_1)\varphi(a_2)\cdots\varphi(a_n)-\varphi(x)^{(n-1)}\varphi(a_1)\cdots\varphi(a_n)| \\
&\quad \le 2\varepsilon\|x\|^{p(n-1)}\|a_1\|^p\cdots\|a_n\|^p+|\varphi(a_2)\cdots\varphi(a_n)|\varepsilon\|x\|^{p(n-1)}\|a_1\|^p \\
&\quad = \varepsilon\|x\|^{p(n-1)}\|a_1\|^p[2\|a_2\|^p\cdots\|a_n\|^p+|\varphi(a_2)\cdots\varphi(a_n)|].
\end{aligned}
$$

Therefore, if

$$
k=\left(\frac{\varepsilon\|a_1\|^p[2\|a_2\|^p\cdots\|a_n\|^p+|\varphi(a_2)\cdots\varphi(a_n)|]}{|\varphi(a_1\cdots a_n)-\varphi(a_1)\cdots\varphi(a_n)|}\right)^{\frac{1}{(n-1)}},
$$

then we have $|\varphi(x)|\le k\|x\|^p$. □

**Corollary 2.2.** *With the same hypotheses as in the theorem, if $A$ is Banach algebra and $\varphi$ is linear, then it is continuous.*

*Proof.* If $\varphi$ is $n$-multiplicative linear functional, then as in the proof of [14, Lemma 2.1] we can see that $\|\varphi\|\le 1$. Otherwise, by Theorem 2.1, the result follows. □

The following theorem has been proved by Jarosz in [12, 5.5] which shows that approximately multiplicative linear functionals are continuous.

**Theorem 2.3.** *Let $A$ be a Banach algebra and $\varphi$ be an $\varepsilon$-multiplicative linear functional. Then, $\|\varphi\|\le 1+\varepsilon$.*

Here, we extend the above theorem for $(\varepsilon,n)$-multiplicative linear functionals.

**Theorem 2.4.** *Let $A$ be a Banach algebra and $\varphi$ be an $(\varepsilon,n)$-multiplicative linear functional. Then, $\|\varphi\|\le 1+\varepsilon$.*

*Proof.* If $\varphi$ is $n$-multiplicative linear functional, then as in the proof of [14, Lemma 2.1], we can see that $\|\varphi\|\le 1$, so the result follows. Otherwise, by Corollary 2.2 $\varphi$ is bounded. Assume towards a contradiction that $\|\varphi\|>1+\varepsilon$, then there exists $a\in A$ with $\|a\|=1$ and $|\varphi(a)|>1+\varepsilon$, so $|\varphi(a)|=1+\varepsilon+p$ for some $p>0$. By induction on $m$, we shall prove that

$$
|\varphi(a^{n^m})|\ge 1+\varepsilon+mp. \tag{1}
$$

By the hypothesis, we have

$$
|\varphi(a^n)|\ge|\varphi^n(a)|-|\varphi^n(a)-\varphi(a^n)|\ge(1+\varepsilon+p)^n-\varepsilon\ge 1+\varepsilon+2p,
$$

so (1) is true, if $m=1$. Assume that (1) is true for $m$. Then, we have

$$\begin{aligned}\left|\varphi(a^{n^{m+1}})\right| &\geq \left|\varphi^n\left(a^{n^m}\right)\right| - \left|\varphi\left(a^{n^{m+1}}\right) - \varphi^n\left(a^{n^m}\right)\right| \\ &\geq (1+\varepsilon+mp)^n - \varepsilon \\ &\geq 1+\varepsilon+(m+1)p.\end{aligned}$$

This completes the proof of (1). Also, we have

$$|\varphi(x_1\cdots x_n) - \varphi(x_1)\cdots\varphi(x_n)||\varphi(x_{n+1})| \leq \varepsilon\|\varphi\|\|x_1\|\cdots\|x_{n+1}\| \qquad (2)$$

for all $x_1,\ldots,x_n,x_{n+1} \in A$. In particular, if $x_{n+1} = a^{n^m}$, then by (1) and (2) we have

$$|\varphi(x_1\cdots x_n) - \varphi(x_1)\cdots\varphi(x_n)| \leq \frac{\|\varphi\|\varepsilon\|x_1\|\cdots\|x_n\|}{1+\varepsilon+mp}$$

for all $x_1,\ldots,x_n \in A$ and all $m \in \mathbb{N}$. Letting $m \to \infty$ shows that $\varphi$ is $n$-multiplicative linear functional so this is a contradiction. □

In the following theorem, we prove a result similar to that of [1, Proposition 3.2] for $n$-$AMNM$ algebras, which gives us some alternative definitions of $n$-$AMNM$.

**Proposition 2.5.** *Let $A$ be a Banach algebra. Then, the following are equivalent:*

(i) *$A$ is $n$-$AMNM$;*
(ii) *for any sequence $(\varphi_m)$ in $A^*$ with $\|(\varphi_m,n)^\vee\| \to 0$, there is a sequence $(\psi_m)$ in $M_{(A,n)} \cup \{0\}$ with $\|\varphi_m - \psi_m\| \to 0$;*
(iii) *for any sequence $(\varphi_m)$ in $A^*$ with $\|(\varphi_m,n)^\vee\| \to 0$, there is a subsequence $(\varphi_{m_i})$ and a sequence $(\psi_i)$ in $M_{(A,n)} \cup \{0\}$ with $\|\varphi_{m_i} - \psi_i\| \to 0$;*
(iv) *for any sequence $(\varphi_m)$ in $A^*$ with $\|(\varphi_m,n)^\vee\| \to 0$ and $\inf_m \|\varphi_m\| > 0$, there is a sequence $(\psi_m)$ in $M_{(A,n)}$ with $\|\varphi_m - \psi_m\| \to 0$.*

*Proof.* (i) ⇒ (ii) Let $(\varphi_m)$ be a sequence in $A^*$ with $\|(\varphi_m,n)^\vee\| \to 0$. By the hypothesis, we have $d_n(\varphi_m) \to 0$. So, it is easy to see that, there exists a sequence $(\psi_m)$ in $M_{(A,n)} \cup \{0\}$ such that $\|\varphi_m - \psi_m\| \to 0$.

(ii) ⇒ (iii) This is trivial.

(iii) ⇒ (i) Suppose (i) is not true. Then, there is $\varepsilon > 0$ such that for any $m \in \mathbb{N}$ , there exists $(\frac{1}{m},n)$-multiplicative linear functional $\varphi_m$ such that $d_n(\varphi_m) \geq \varepsilon$. By the hypothesis, there exists a subsequence $(\varphi_{m_i})$ of sequence $(\varphi_m)$ such that $d_n(\varphi_{m_i}) \to 0$, so we get a contradiction.

(i) ⇒ (iv) This is similar to (i) ⇒ (ii).

(iv) ⇒ (iii) If there exists a subsequence $(\varphi_{m_i})$ such that $\inf_i \|\varphi_{m_i}\| > 0$, then (iv) implies (iii). Otherwise, there exists a subsequence $(\varphi_{m_k})$ such that $\|\varphi_{m_k}\| \to 0$, so the result follows. □

**Corollary 2.6.** *Every finite-dimensional Banach algebra $A$ is $n$-$AMNM$.*

*Proof.* Let $(\varphi_m)$ be a sequence in $A^*$ with $\|(\varphi_m,n)^\vee\| \to 0$. Then, by Theorem 2.4, the sequence $(\varphi_m)$ is bounded and so has a convergent subsequence $(\varphi_{m_i})$ with limit $\psi$. Since for any $x_1,\ldots,x_n \in A$,

$$|\varphi_{m_i}(x_1\cdots x_n) - \varphi_{m_i}(x_1)\cdots\varphi_{m_i}(x_n)| \leq \|(\varphi_{m_i},n)^\vee\|\|x_1\|\cdots\|x_n\|,$$

by passing to the limit as $i \to \infty$, we see that $\psi \in M_{(A,n)} \cup \{0\}$. So, by the condition (iii) of Proposition 2.5 for $\psi_i = \psi$, the result follows. $\square$

**Theorem 2.7.** [1, 2.3] *Let $A$ be a Banach algebra with a bounded approximate identity $(e_\alpha)$ of bound $k > 0$. If $0 < \varepsilon < \frac{1}{4k^2}$ and $\varphi$ is an $\varepsilon$-multiplicative linear functional, then either $\|\varphi\| \leq 2\varepsilon k$ or $\|\varphi\| \geq \frac{1}{k} - 2\varepsilon k$.*

**Corollary 2.8.** *Let $A$ be a Banach algebra with a bounded approximate identity of bound $k$. If $\varphi$ is an $n$-character on $A$, then $\|\varphi\| > \frac{1}{k}$.*

*Proof.* By the proof of [14, Lemma 2.1] and Theorem 2.7, the result follows. $\square$

**Theorem 2.9.** *Let $A$ be a commutative Banach algebra and let $J$ be a closed ideal in $A$. If $J$ and $A/J$ are $n$-AMNM, then $A$ is $n$-AMNM.*

*Proof.* If $(\varphi_m)$ is a sequence in $A^*$ with $\|(\varphi_m, n)^\vee\| \to 0$, then $\|(\varphi_m, n)^\vee|_J\| \to 0$. Now, we can assume that there exists a subsequence of $(\varphi_m)$, again denoted by $(\varphi_m)$, such that $\inf_m \|\varphi_m|_J\| > \eta$ for some $\eta > 0$ or $\|\varphi_m|_J\| \to 0$. If there is an $\eta > 0$ such that $\inf_m \|\varphi_m|_J\| > \eta$, then by the hypothesis and Proposition 2.5, there is a sequence $({\psi'}_m) \subseteq M_{(J,n)}$ such that

$$\|\varphi_m|_J - {\psi'}_m\| \to 0.$$

Since each ${\psi'}_m$ is nonzero for all $m \in \mathbb{N}$, there exists $j_m \in J$ which ${\psi'}_m(j_m) = 1$. We now define $\psi_m(a) = {\psi'}_m(a j_m^{(n-1)})$ for all $a \in A$, so that $\psi_m \in A^*$ and we also have

$$\begin{aligned}\psi_m(a_1 \cdots a_n) &= {\psi'}_m\left(a_1 \cdots a_n j_m^{(n-1)}\right) \frac{{\psi'}_m(j_m^{(n-1)})^{(n-1)}}{{\psi'}_m(j_m^{(n-1)})^{(n-1)}} \\ &= \frac{\psi_m(a_1) \cdots \psi_m(a_n)}{{\psi'}_m(j_m^{(n-1)})^{(n-1)}}\end{aligned}$$

for all $a_1, \ldots, a_n \in A$. On the other hand,

$$\begin{aligned}{\psi'}_m\left(j_m^{(n-1)}\right)^{(n-1)} &= {\psi'}_m\left(j_m^{(n-1)}\right)^{(n-1)} {\psi'}_m(j_m) \\ &= {\psi'}_m\left(j_m^{(n-1)(n-1)} j_m\right) = {\psi'}_m(j_m)^{(n-1)} {\psi'}_m\left(j_m^{(n-1)(n-2)} j_m\right) \\ &= {\psi'}_m\left(j_m^{(n-1)(n-2)} j_m\right) = {\psi'}_m\left(j_m^{(n-1)(n-3)} j_m\right)^{(n-1)} = \cdots \\ &= {\psi'}_m\left(j_m^{(n-1)} j_m\right) = {\psi'}_m(j_m)^n = 1,\end{aligned}$$

therefore, $\psi_m$ is $n$-character and so $\psi_m \in M_{(A,n)}$. By the Hahn–Banach theorem, $(\varphi_m - \psi_m)|_J$ can be extended to an element $\theta_m$ of $A^*$ with $\|\theta_m\| = \|(\varphi_m - \psi_m)|_J\| = \|\varphi_m|_J - {\psi'}_m\|$, so $\|\theta_m\| \to 0$. Since

$$\begin{aligned}\|(\varphi_m - \theta_m, n)^\vee(a_1, \ldots, a_n)\| &\leq (\|(\varphi_m, n)^\vee\| + \|\theta_m\| \\ &\quad + \sum_{k=1}^{n} \binom{n}{k} \|\theta_m\|^k \|\varphi_m\|^{n-k} \|a_1\| \cdots \|a_n\|,\end{aligned}$$

then $\|(\varphi_m - \theta_m, n)^\vee\| \to 0$. For all $a \in A$,

$$(\varphi_m - \theta_m)(a) = (\varphi_m - \theta_m)(aj_m^{(n-1)}) - (\varphi_m - \theta_m, n)^\vee(a, j_m, \dots, j_m)$$
$$= \psi_m(a) - (\varphi_m - \theta_m, n)^\vee(a, j_m, \dots, j_m),$$

so $\|\varphi_m - \theta_m - \psi_m\| \to 0$ and hence

$$\|\varphi_m - \psi_m\| \le \|\varphi_m - \psi_m - \theta_m\| + \|\theta_m\| \to 0,$$

thus the result follows by Proposition 2.5.

Now, consider the case where $\|\varphi_m|_J\| \to 0$. Let $\theta_m$ be an extension of $\varphi_m|_J$ to $A$ with $\|\theta_m\| = \|\varphi_m|_J\|$. Set $\phi_m = \varphi_m - \theta_m$, then by a similar argument, $\|(\phi_m, n)^\vee\| \to 0$. Since $\phi_m = 0$ on $J$, we can consider $(\phi_m)$ as a sequence in $(A/J)^*$, and so, there is a sequence $(\psi_m) \subseteq M_{(A/J,n)} \cup \{0\} \subseteq M_{(A,n)} \cup \{0\}$ such that $\|\phi_m - \psi_m\| \to 0$. Then

$$\|\varphi_m - \psi_m\| \le \|\varphi_m - \theta_m - \psi_m\| + \|\theta_m\| \to 0,$$

and by Proposition 2.5, the theorem is proved. □

**Theorem 2.10.** *Let $A$ be a commutative Banach algebra and let $J$ be a closed ideal in $A$. If $A$ is $n$-AMNM, then $J$ is $n$-AMNM. Moreover, if $A$ is $n$-AMNM and $J$ has a bounded approximate identity of bound $k > 0$, then $A/J$ is $n$-AMNM.*

*Proof.* Suppose that $(\varphi_m)$ is a sequence in $J^*$ with $\|(\varphi_m, n)^\vee\| \to 0$ and $k = \inf \|\varphi_m\| > 0$. Then, there exists a sequence $(j_m)$ in $J$ with $\|j_m\| < \frac{2}{k}$ and $\varphi_m(j_m) = 1$. Set $\phi_m(a) = \varphi_m(aj_m^{(n-1)})$ for all $a \in A$ and $m \in \mathbb{N}$. Then, $\phi_m \in A^*$ and for $a_1, \dots, a_n \in A$ we have

$$|(\phi_m, n)^\vee(a_1, \cdots, a_n)| \le \left|\varphi_m\left(a_1 \cdots a_n j_m^{(n-1)}\right) - \varphi_m\left(a_1 \cdots a_n j_m^{2(n-1)}\right)\right|$$
$$+ \left|\varphi_m\left(a_1 \cdots a_n j_m^{2(n-1)}\right) - \varphi_m\left(a_1 \cdots a_n j_m^{3(n-1)}\right)\right|$$
$$+ \cdots + \left|\varphi_m\left(a_1 \cdots a_n j_m^{n(n-1)}\right) - \varphi_m\left(a_1 j_m^{(n-1)}\right) \cdots \varphi_m\left(a_n j_m^{(n-1)}\right)\right|,$$

so $\|(\phi_m, n)^\vee\| \to 0$. By the hypothesis and Proposition 2.5, there is a sequence $(\psi_m)$ in $M_{(A,n)} \cup \{0\}$ with $\|\phi_m - \psi_m\| \to 0$. We now have

$$|\phi_m(a) - \varphi_m(a)| = |(\varphi_m, n)^\vee(a, j_m, \dots, j_m)| \to 0$$

for any $a \in J$. If $\theta_m = \psi_m|_J$, then it is easy to see that, $\theta_m \in M_{(J,n)} \cup \{0\}$ and

$$\|\varphi_m - \theta_m\| \le \|\varphi_m - \phi_m\| + \|\phi_m - \theta_m\| \to 0.$$

Now, the result follows by Proposition 2.5.

For the proof of the second part of the theorem, let $(\varphi_m)$ be a sequence in $(A/J)^* \subset A^*$ with $\|(\varphi_m, n)^\vee\| \to 0$. Then, by Proposition 2.5, there is a sequence $(\psi_m) \subset M_{(A,n)} \cup \{0\}$ such that $\|\varphi_m - \psi_m\| \to 0$. By Corollary 2.8, either $\|\psi_m|_J\| = 0$ or $\|\psi_m|_J\| \ge \frac{1}{k}$. Since $\|\varphi_m - \psi_m\| \to 0$ and $\varphi_m|_J = 0$, then $\psi_m|_J = 0$ and hence each $\psi_m \in M_{(A/J,n)} \cup \{0\}$. So, by Proposition 2.5, the result follows. □

**Corollary 2.11.** *Let $A$ be a commutative Banach algebra and $J$ be a closed ideal in $A$ such that $A/J$ is finite dimensional. Then, $A$ is $n$-AMNM if and only if $J$ is. In particular $A$ is $n$-AMNM if and only if the unitization of $A$, $A^+$ is.*

*Proof.* It is an immediate consequence of Corollary 2.6, Theorem 2.9 and Theorem 2.10. ☐

In the next result, $A\hat{\otimes}B$ is the completion of $A\otimes B$ in the projective tensor norm [4].

**Theorem 2.12.** *Let $A$ and $B$ be commutative $n$-AMNM Banach algebras. Then, $A\hat{\otimes}B$ is $n$-AMNM.*

*Proof.* Suppose that $(\varphi_m)$ is a sequence in $(A\hat{\otimes}B)^*$ with $k=\inf_m\|\varphi_m\|>0$ and $\|(\varphi_m,n)^{\vee}\|\to 0$. Using the canonical isometric identification between $(A\hat{\otimes}B)^*$ and the set of bounded bilinear forms on $A\times B$, there exist sequences $(a_m)$ and $(b_m)$ in $A$ and $B$, respectively such that $\|a_m\|\|b_m\|<\frac{2}{k}$ and $\varphi_m(a_m\otimes b_m)=1$. Now, define the functions $\theta_m:A\to\mathbb{C}$ and $\psi_m:B\to\mathbb{C}$ by

$$\theta_m(x)=\varphi_m(xa_m\otimes b_m),\ \psi_m(y)=\varphi_m(a_m^{(n-1)}\otimes yb_m^{(n-1)}),\ x\in A,\ y\in B.$$

So for any $x_1,\dots,x_n\in A$, we have

$$\begin{aligned}|\theta_m(x_1\cdots x_n)-\theta_m(x_1)\cdots\theta_m(x_n)| &= |\varphi_m(x_1\cdots x_na_m\otimes b_m)\\ &\quad-\varphi_m(x_1a_m\otimes b_m)\cdots\varphi_m(x_na_m\otimes b_m)|\\ &\le|\varphi_m(x_1\cdots x_na_m\otimes b_m)\varphi_m(a_m\otimes b_m)^{(n-1)}\\ &\quad-\varphi_m(x_1\cdots x_na_m^n\otimes b_m^n)|\\ &\quad+|\varphi_m(x_1\cdots x_na_m^n\otimes b_m^n)\\ &\quad-\varphi_m(x_1a_m\otimes b_m)\cdots\varphi_m(x_na_m\otimes b_m)|.\end{aligned}$$

Since $\|(\varphi_m,n)^{\vee}\|\to 0$, then $\|(\theta_m,n)^{\vee}\|\longrightarrow 0$. By the hypothesis and Proposition 2.5 there is a sequence $(\acute{\theta}_m)$ in $M_{(A,n)}\cup\{0\}$ with $\|\theta_m-\acute{\theta}_m\|\longrightarrow 0$. Also for any $y_1,\dots,y_n\in B$, we have

$$\begin{aligned}|\psi_m(y_1\cdots y_n)-\psi_m(y_1)\cdots\psi_m(y_n)| &= \Big|\varphi_m\left(a_m^{(n-1)}\otimes y_1\cdots y_nb_m^{(n-1)}\right)\\ &\quad-\varphi_m\left(a_m^{(n-1)}\otimes y_1b_m^{(n-1)}\right)\cdots\varphi_m\left(a_m^{(n-1)}\otimes y_nb_m^{(n-1)}\right)\Big|\\ &\le\Big|\varphi_m\left(a_m^{(n-1)}\otimes y_1\cdots y_nb_m^{(n-1)}\right)\varphi_m(a_m\otimes b_m)^{(n-1)}\\ &\quad-\varphi_m\left(a_m^{2(n-1)}\otimes y_1\cdots y_nb_m^{2(n-1)}\right)\Big|\\ &\quad+\Big|\varphi_m\left(a_m^{2(n-1)}\otimes y_1\cdots y_nb_m^{2(n-1)}\right)\varphi_m(a_m\otimes b_m)^{(n-1)}\\ &\quad-\varphi_m\left(a_m^{3(n-1)}\otimes y_1\cdots y_nb_m^{3(n-1)}\right)\Big|\\ &\quad\vdots\\ &\quad+\Big|\varphi_m\left(a_m^{n(n-1)}\otimes y_1\cdots y_nb_m^{n(n-1)}\right)\end{aligned}$$

$$-\varphi_m\left(a_m^{(n-1)}\otimes y_1 b_m^{(n-1)}\right)\cdots$$
$$\varphi_m\left(a_m^{(n-1)}\otimes y_n b_m^{(n-1)}\right)\Big|,$$

therefore, $\|(\psi_m,n)^{\vee}\|\longrightarrow 0$. By the hypothesis and Proposition 2.5, there is a sequence $(\acute{\psi_m})$ in $M_{(B,n)}\cup\{0\}$ with $\|\psi_m-\acute{\psi_m}\|\longrightarrow 0$. Now, consider the function $\phi_m : A\hat{\otimes}B\to\mathbb{C}$ defined by $\phi_m(x\otimes y)=\acute{\theta}_m(x)\acute{\psi}_m(y)$ for every $x\otimes y\in A\otimes B$. It is easy to see that $\phi_m\in M_{(A\hat{\otimes}B,n)}\cup\{0\}$. On the other hand, we have

$$\begin{aligned}|\varphi_m(x\otimes y)-\phi_m(x\otimes y)|\leq&\Big|\varphi_m(x\otimes y)\varphi_m(a_m\otimes b_m)^{(n-1)}\\
&-\varphi_m\left(xa_m^{(n-1)}\otimes yb_m^{(n-1)}\right)\Big|\\
&+\Big|\varphi_m\left(xa_m^{(n-1)}\otimes yb_m^{(n-1)}\right)\varphi_m(a_m\otimes b_m)^{(n-1)}\\
&-\varphi_m\left(xa_m^{2(n-1)}\otimes yb_m^{2(n-1)}\right)\Big|\\
&+\Big|\varphi_m\left(xa_m^{2(n-1)}\otimes yb_m^{2(n-1)}\right)\\
&-\varphi_m(xa_m\otimes b_m)\varphi_m\left(a_m^{(n-1)}\otimes yb_m^{(n-1)}\right)\varphi_m(a_m\otimes b_m)^{n-2}\Big|\\
&+|\theta_m(x)\psi_m(y)-\phi_m(x\otimes y)|.\end{aligned}$$

Since for any $x\in A$, $y\in B$, $|\theta_m(x)\psi_m(y)-\phi_m(x\otimes y)|\longrightarrow 0$, then $\|\varphi_m-\phi_m\|\longrightarrow 0$ and so by Proposition 2.5, the result follows. □

**Theorem 2.13.** *Let $B$ be a unital commutative Banach algebra and let $A$ be a commutative Banach algebra. If $A\hat{\otimes}B$ is $n$-AMNM, then $A$ is $n$-AMNM.*

*Proof.* Let $(\varphi_m)$ be a sequence in $A^*$ with $k=\inf_m\|\varphi_m\|>0$ and $\|(\varphi_m,n)^{\vee}\|\to 0$. Now, by the universal property of projective tensor product, we define $\phi_m\in(A\hat{\otimes}B)^*$ by $\phi_m(a\otimes b)=\varphi_m(a)\psi(b)$, where $\psi\in M_B$. It is easy to see that $\|(\phi_m,n)^{\vee}\|=\|(\varphi_m,n)^{\vee}\|$ and $\|\phi_m\|=\|\varphi_m\|$, so $\|\phi_m\|\geq k$. By the hypothesis and Proposition 2.5, there is a sequence $(\acute{\phi_m})$ in $M_{(A\hat{\otimes}B,n)}$ with $\|\phi_m-\acute{\phi_m}\|\longrightarrow 0$. Now, consider the function $\theta_m : A\to\mathbb{C}$ defined by $\theta_m(a)=\acute{\phi_m}(a\otimes 1)$ for every $a\in A$. It is easy to see that $\theta_m\in M_{(A,n)}\cup\{0\}$ and $\|\varphi_m-\theta_m\|\leq\|\phi_m-\acute{\phi_m}\|$. Therefore, $\|\varphi_m-\theta_m\|\longrightarrow 0$ and by Proposition 2.5 the result follows. □

**Corollary 2.14.** *Let $A$ and $B$ be unital commutative Banach algebras. If $A\hat{\otimes}B$ is $n$-AMNM, then $A$ and $B$ are $n$-AMNM.*

**Proposition 2.15.** *Let $A$ be a unital AMNM Banach algebra and $n>2$ be an integer. Then, for every $\varepsilon>0$, there exists $\delta>0$ such that for any $(\delta,n)$-multiplicative linear functional $\varphi$, either $d(\varphi)<\varepsilon$ or $d(\varphi(1)^{n-2}\varphi)<\varepsilon$.*

*Proof.* Suppose that $\varepsilon>0$. Then, by the hypothesis there exists a $\gamma>0$ such that for any $\gamma$-multiplicative linear functional $\theta$, $d(\theta)<\varepsilon$. Set $\delta=\min\{1,\frac{\gamma}{2^n},\varepsilon\}$, and let $\varphi$ be a $(\delta,n)$-multiplicative linear functional. If $\varphi(1)=0$, then it is easy to see that $\|\varphi\|\leq\delta\leq\varepsilon$ and so $d(\varphi)<\varepsilon$. Otherwise, we define $\phi : A \to \mathbb{C}$ by $\phi = \varphi(1)^{n-2}\varphi$. Since $|\varphi(1) - \varphi(1)^n| < 1$ so it is easy to see that $|\varphi(1)| < 2$ and for every $a, b \in A$, we have

$$\begin{aligned}|\phi(ab) - \phi(a)\phi(b)| &= |\varphi(1)|^{n-2}|\varphi(ab) - \varphi(1)^{n-2}\varphi(a)\varphi(b)|\\ &\leq 2^{n-2}\delta\|a\|\|b\|\\ &< 2^n\delta\|a\|\|b\|.\end{aligned}$$

Hence, $\phi$ is a $2^n\delta$-multiplicative linear functional and then $d(\varphi(1)^{n-2}\varphi) < \varepsilon$. □

**Lemma 2.16.** *Let $A$ be a Banach algebra and let $\varphi$ be an $(\varepsilon, n)$-multiplicative linear functional on $A$ such that $\varphi(a) = 1$. If $\psi : A \to \mathbb{C}$ is defined by $\psi(x) = \varphi(ax)$, then $\psi$ is approximately multiplicative linear functional.*

*Proof.* By the hypothesis for every $x, y \in A$, we have

$$\begin{aligned}|\psi(xy) - \psi(x)\psi(y)| &= |\varphi(axy) - \varphi(ax)\varphi(ay)|\\ &= |\varphi(axy) \pm \varphi(a^{n-1}xya) \pm \varphi(ax)\varphi(ya) \pm \varphi(axaya^{n-2})|\\ &\leq |\varphi(a)^{n-2}\varphi(axy)\varphi(a) - \varphi(a^{n-1}xya)|\\ &\quad + |\varphi(a^{n-1}xya) - \varphi(a)^{n-2}\varphi(ax)\varphi(ya)|\\ &\quad + |\varphi(ax)\varphi(a)\varphi(ya)\varphi(a)^{n-3} - \varphi(axaya^{n-2})|\\ &\quad +|\varphi(axaya^{n-2}) - \varphi(ax)\varphi(ay)\varphi(a)^{n-2}|\\ &\leq 4\varepsilon\|a\|^n\|x\|\|y\|.\end{aligned}$$

Therefore, $\psi$ is $4\delta$-multiplicative linear functional, such that $\delta = 4\varepsilon\|a\|^n$. □

**Theorem 2.17.** *Let $X$ be a locally compact Hausdorff space and $n > 2$ be an integer. Then, for every $\varepsilon > 0$, there exists $\delta > 0$ such that for any $(\delta, n)$-multiplicative linear functional $\varphi$ on $C_0(X)$, either $d(\varphi) < \varepsilon$ or there exists $f_0 \in C_0(X)$ with $\varphi(f_0) \neq 0$ and $d(\psi) < \varepsilon$, where $\psi(f) = \varphi(f_0 f)$ for all $f \in C_0(X)$.*

*Proof.* Let $\varepsilon > 0$. By [1, Theorem 4.1] there exists a $\gamma > 0$ such that for any $\gamma$-multiplicative linear functional $\theta$, $d(\theta) < \varepsilon$. Suppose that $(e_\alpha)_{\alpha\in\wedge}$ is an approximate identity for $C_0(X)$ with $\|e_\alpha\| \leq 1$, $\delta = \min\{\varepsilon, \frac{\gamma}{4}\}$ and $\varphi$ is a $(\delta, n)$-multiplicative linear functional on $C_0(X)$. Since $\varphi$ is a $(\delta, n)$-multiplicative linear functional, we have

$$|\varphi(fe_\alpha^{n-1}) - \varphi(f)\varphi(e_\alpha)^{n-1}| \leq \delta\|f\| \quad f \in C_0(X).$$

If for all $\alpha \in \wedge$, $\varphi(e_\alpha) = 0$, then it is easy to see that $\|\varphi\| \leq \delta$ and so $d(\varphi) < \varepsilon$. If there exists $\alpha_0 \in \wedge$ such that $\varphi(e_{\alpha_0}) \neq 0$, then we define $\psi : C_0(X) \to \mathbb{C}$ by $\psi(f) = \varphi(e_{\alpha_0} f)$. By Lemma 2.16, $\psi$ is $4\delta$-multiplicative linear functional and so $d(\psi) < \varepsilon$. □

The following theorem has been proved by Howey in [8, Theorem 3.1].

**Theorem 2.18.** *Let $A$ be a commutative separable unital Banach algebra where the Gelfand and norm topologies coincide on $M_A$ . Then, $A$ is AMNM if and only if for all sequences $(\varphi_m)$ in $A^*$ with $\|(\varphi_m, 2)^\vee\| \to 0$ and $\varphi_m \to \varphi$ in the weak$^*$ topology, $\varphi \neq 0$, then $\|\varphi_m - \varphi\| \to 0$.*

**Lemma 2.19.** *Let $A$ be a commutative unital Banach algebra. Then, the Gelfand and norm topologies coincide on $M_A$ if and only if they are the same on $M_{(A,n)}$.*

*Proof.* Let $(\varphi_m)$ be a sequence in $M_{(A,n)}$ *weak*$^*$ converges to $\varphi$ in $M_{(A,n)}$. Then, $(\varphi_m)$ is bounded and $\varphi_m(1) \to \varphi(1)$. Define $\phi_m, \phi : A \to \mathbb{C}$ by $\phi_m = \varphi_m(1)^{n-2}\varphi_m$ and $\phi = \varphi(1)^{n-2}\varphi$. By [7, Theorem 2.2] $\phi_m$, $\phi \in M_A$ for each $m \in \mathbb{N}$. Now by the hypothesis, we have $\phi_m \to \phi$ in the *weak*$^*$ topology, and hence $\|\phi_m - \phi\| \to 0$. Since $\varphi_m = \varphi_m(1)\phi_m$ and $\varphi = \varphi(1)\phi$, then $\|\varphi_m - \varphi\| \to 0$, so the result follows. □

The following theorem is an extension of Theorem 2.18 for $n > 2$.

**Theorem 2.20.** *Let $A$ be a commutative separable unital Banach algebra where the Gelfand and norm topologies coincide on $M_A$. Then, $A$ is $n$-AMNM if and only if for all sequences $(\varphi_m)$ in $A^*$ with $\|(\varphi_m, n)^\vee\| \to 0$ and $\varphi_m \to \varphi$ in the weak$^*$ topology, $\varphi \neq 0$, then $\|\varphi_m - \varphi\| \to 0$.*

*Proof.* By Lemma 2.19 and a modification of the proof of Theorem 2.18 the result follows. □

**Theorem 2.21.** *Let $A$ be a commutative separable unital Banach algebra where the Gelfand and norm topologies coincide on $M_A$. Then, $A$ is AMNM if and only if $A$ is $n$-AMNM.*

*Proof.* Suppose that $A$ is $AMNM$, $(\varphi_m)$ is a sequence in $A^*$ with $\|(\varphi_m, n)^\vee\| \to 0$ and $\varphi_m \to \varphi$ in the *weak*$^*$ topology, where $\varphi$ is a nonzero element of $A^*$. Then, $\varphi$ is $n$-character so by the proof of [14, Lemma 2.1] we can see that $|\varphi(1)| = \|\varphi\| = 1$. Since $\varphi_m(1) \to \varphi(1)$, there exists $M \in \mathbb{N}$ such that $|\varphi_m(1)| > \frac{1}{2}$ for each $m \geq M$. Now define $\phi_m, \phi : A \to \mathbb{C}$ for all $m \geq M$ by $\phi_m = \varphi_m/\varphi_m(1)$ and $\phi = \varphi/\varphi(1)$. By Lemma 2.19, for all $x, y \in A$, we have

$$\left|\frac{\varphi_m(xy)}{\varphi_m(1)} - \frac{\varphi_m(x)\varphi_m(y)}{\varphi_m(1)^2}\right| \leq \frac{4}{|\varphi_m(1)|^n}\|(\varphi_m, n)^\vee\|\|x\|\|y\|,$$

then $\|(\phi_m, 2)^\vee\| \to 0$, and for all $x \in A$,

$$\left|\frac{\varphi_m(x)}{\varphi_m(1)} - \frac{\varphi(x)}{\varphi(1)}\right| \leq \frac{|\varphi_m(x)||\varphi(1) - \varphi_m(1)| + |\varphi_m(1)||\varphi_m(x) - \varphi(x)|}{|\varphi_m(1)||\varphi(1)|},$$

so $\phi_m \to \phi$ in the *weak*$^*$ topology. Now by the hypothesis, we have $\|\phi_m - \phi\| \to 0$ and thus it is easy to see that $\|\varphi_m - \varphi\| \to 0$. Then by Theorem 2.20 $A$ is $n$-$AMNM$. Conversely, let $(\varphi_m)$ be a sequence in $A^*$ with $\|(\varphi_m, 2)^\vee\| \to 0$ and $\varphi_m \to \varphi$ in the *weak*$^*$ topology, where $\varphi$ is a nonzero element of $A^*$. It is easy to see that $\|(\varphi_m, n)^\vee\| \to 0$ and by the hypothesis, we have $\|\varphi_m - \varphi\| \to 0$. Then, by Theorem 2.18, $A$ is $AMNM$. □

*Remark* 2.22. Howey in [8] proved that the Gelfand and norm topologies are the same on $M(c^N[0,1]^M)$ where $c^N[0,1]^M$ is the algebra of complex-valued functions defined on $[0,1]^M$ with all $N$th order partial derivatives continuous. In fact, he proved that it is $AMNM$ and so by Theorem 2.21, it is $n$-$AMNM$.

*Example* 2.23. Let $L^1 = L^1(\mathbb{Z})$ be the space of all functions $f : \mathbb{Z} \to \mathbb{C}$ such that $\|f\| = \sum_{k\in\mathbb{Z}} |f(k)| < \infty$. Clearly, $L^1$ is a separable commutative unital Banach algebra with usual convolution. Johnson in [1, Theorem 5.2] proved that $L^1$ is $AMNM$. It is easy to see that the character space of $L^1$ is homeomorphic to $\mathbb{T} = \{z \in \mathbb{C} : \ |z| = 1\}$. Also, let $z_m$ converges to $z_0$ in the $weak^*$ topology of $\mathbb{T}$ (that is in the standard topology of $\mathbb{T}$) and let $\mathcal{P}$ be the set of all polynomials of $z$ and $z^{-1}$. For any $f$ with norm 1 take $p$ in $\mathcal{P}$ close to $f$, then $z_m$ at $p$ is close to $z_0$ at $p$ for large $m$, so it is convergent in norm. Therefore, the Gelfand and norm topologies are the same on $\mathbb{T} = M_{L^1}$. Now by Theorem 2.21, $L^1$ is $n$-$AMNM$.

Johnson [1] gave the following example to show that not all the classical commutative Banach algebras are $AMNM$. We show that the Banach algebra obtained in this example is not $n$-$AMNM$.

*Example* 2.24. For each positive integer $m$, let $A_m$ be the algebra $\mathbb{C}^m$ with multiplication $(ab)_j = a_j b_j$. The standard basis of $A_m$ will be denoted by $e_1, \ldots, e_m$ and the unit by $1_m$. We set $S_m = \{0, e_1, \ldots, e_m, 1_m\}$ and let $U_m$ be the absolutely convex cover of $S_m$, that is,

$$U_m = \left\{ \sum_{i=1}^{k-1} \lambda_i e_i + \lambda_k 1_m : \sum_{i=1}^{k} |\lambda_i| \le 1 \ , \ k \in N \right\}.$$

We take the norm on $A_m$ for which the unit ball is $U_m$. As $S_m$ is closed under multiplication so is $U_m$ and $A_m$ is a Banach algebra. We define $A$ to be the set of all sequences $(a_j)$ with $a_j \in A_j$ and $\|a\| = (\sum \|a_j\|^2)^{\frac{1}{2}} < \infty$. Then, $A$ is a Banach algebra. Let $f_m \in A_m^*$ such that $f_m(e_j) = \frac{1}{m}$ for all $j = 1, \ldots, m$, and $p_m$ be the projection of $A$ to $A_m$ and $g_m = p_m^* f_m$. We show that $\|(g_m, n)^\vee\| \le \frac{1}{m}$. For all $x_1, \ldots, x_n \in S_m$, we have

$$\begin{aligned} &f_m(x_1 \ldots x_n) - f_m(x_1) \ldots f_m(x_n) \\ &\quad = \begin{cases} 0 & \text{if } x_1 = \cdots = x_n = 1_m \text{ or } \exists\ x_j = 0, \\ -\frac{1}{m^{n-r}} & \text{if } x_{k_1} = \cdots = x_{k_r} = 1_m \text{ and } \exists\ i, j,\ x_{n_i} \neq x_{n_j}, \\ \frac{1}{m} - \frac{1}{m^{n-r}} & \text{if } x_{k_1} = \cdots = x_{k_r} = 1_m \text{ and } \exists\ i,\ x_{k_{r+1}} = \cdots = x_{k_n} = e_i. \end{cases} \end{aligned}$$

Thus, for all $x_1, \ldots, x_n \in S_m$, $|f_m(x_1, \ldots, x_n) - f_m(x_1) \ldots f_m(x_n)| \le \frac{1}{m}$ and so, as $U_m$ is the absolutely convex cover of $S_m$, we get the same inequality for all $x_1, \ldots, x_n \in U_m$, showing that $\|(f_m, n)^\vee\| \le \frac{1}{m}$. Since $p_m$ is a norm decreasing algebra homomorphism, we get $(g_m, n)^\vee(x_1, \ldots, x_n) = (f_m, n)^\vee(p(x_1), \ldots, p(x_n))$. Therefore, $\|(g_m, n)^\vee\| \le \frac{1}{m}$. Let $\phi \in M_{(A,n)} \cup \{0\}$. If $\phi(1_m) = 0$, then

$$\|\phi - g_m\| \ge |\phi(1_m) - g(1_m)| = 1,$$

because $\|1_m\| \le 1$ and $g_m(1_m) = 1$. If $\phi(1_m) \neq 0$, then $\psi_m = \phi_{|_{A_m}} \in M_{(A_m,n)}$. Hence, there exists $0 \le \theta \le 2\pi$ such that $\psi_m(1_m) = \cos\theta + i \sin\theta$. Define $\phi_m : A_m \to \mathbb{C}$ by $\phi_m = \psi_m(1)^{n-2} \psi_m$. Then, by [7, Theorem 2.2], $\phi_m \in M_A$ and $\psi_m = \psi_m(1_m)\phi_m$. Since $\phi_m$ is a character on $A_m$, we have it is of the form $x \mapsto x_k$ for some $k \in \{1, \ldots, m\}$. Therefore, $\psi_m(e_k) = \psi_m(1_m)\phi_m(e_k) = \psi_m(1_m) = \cos\theta + i \sin\theta$. We have

$$\|f_m - \psi_m\| \geq |f_m(e_k) - \psi_m(e_k)| = \sqrt{1 + \frac{1}{m^2} - \frac{2}{m}\cos\theta} \geq 1 - \frac{1}{m},$$

thus

$$\|g_m - \phi\| \geq \|(g_m - \phi)_{|A_m}| = \|f_m - \psi_m\| \geq 1 - \frac{1}{m}.$$

So by Proposition 2.5 $A$ is not $n$-$AMNM$.

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