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Multivariate Normal α-Stable Exponential Families

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Abstract. The normal inverse Gaussian distributions are used to introduce the class of multivariate normal α-stable distributions. Some fundamental properties of these new distributions are established. We give the expression of the variance function of the generated natural exponential family and we use the Lévy–Khintchine representation to determine the associated Lévy measure. We also study the relationship between these distributions and the multivariate inverse Gaussian ones.

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1. Introduction

The natural exponential families (NEFs) represent a very important class of distributions in probability and statistical theory. In the last decades, it has drawn considerable attention of researchers and numerous works have been realized on their different aspects. For example, several characteristic properties of this class of distributions have been established (see Brown [4] and Louati [17]) and many classifications of the NEFs according to the form of their variance functions have been realized (see, for instance, Letac and Mora [15]).

The importance of the variance function can be explained as follows: on the one hand, it is a function of the mean, and on the other hand, it characterizes the family F within the class of all NEFs (see Tweedie [23]). Furthermore, for many common NEFs, the variance function takes a very simple form (see Kokonendji and Masmoudi [13, 14]).

An interesting class of NEFs is so called the stable one introduced by Paul Lévy in 1924. It is a rich class of probability distributions which has several mathematical properties. The stable distributions play a constantly increasing role. They are particularly important in probability because of their role as a natural generalization of the normal distribution and also in the practical work of the statisticians in the analysis of data in several areas of applications. That is why, several monographs have focused on stable models and have described the basic properties of the stable distributions, with an emphasis on practical applications. This class of probability distributions can be especially used as a model of different kinds of financial data. In fact, it is considered as a class of Lévy processes with stable distributions and used in financial applications: stock exchange, variation of stock market, financial returns, etc. (see Feller [6], McCulloch [18], Nolan [20,21] and Zolotarev [25]). Recently, Seshadri [22] gave a complete description of the class of univariate stable families, and Hassairi and Louati [10] characterized the multivariate stable exponential families by a homogeneous property of the variance function.

In 1997, Barndorff-Nielsen [2] defined the normal inverse Gaussian distribution by taking a random vector $(Y_1, \overline{Y})$ on $]0, +\infty[\times\mathbb{R}^{d-1}$ such that Y_1 is an inverse Gaussian distributed, and the conditional distribution of $\overline{Y}$, given $Y_1 = y_1$, is a centred normal random vector on $\mathbb{R}^{d-1}$ with a variance–covariance matrix $y_1 I_{d-1}$, where I_{d-1} denotes the identity matrix of $\mathbb{R}^{d-1}$. It follows that the random vector $\overline{Y}$ is a normal inverse Gaussian distributed. Many papers have used this distribution to determine a Lévy process, which is representable through subordination of Brownian motion by the inverse Gaussian process. Some of them applied this Lévy process in a stochastic volatility modelling (see Barndorff-Nielsen [2]) and used it to construct a market model for financial assets (see Jönsson et al. [12]).

Motivated by the definition of normal inverse Gaussian distribution given by Barndorff-Nielsen [2], in the present paper, we introduce a class of distributions called the class of normal α-stable distributions with $\alpha \in]0, 1[$. This class represents an extension of the inverse Gaussian one ($\alpha = \frac{1}{2}$) from taking the random vector $(Y_1, \overline{Y})$ on $]0, +\infty[\times\mathbb{R}^{d-1}$ with distribution $\mu_{\alpha,\beta}$ such that Y_1 is drifted α-stable distributed with Laplace transform $L_{\pi_{\alpha,\beta}}(s) = e^{1-(1-\beta s)^{\alpha}}$, and the conditional distribution of $\overline{Y}$, given $Y_1 = y_1$, is a centred normal random vector on $\mathbb{R}^{d-1}$ with a variance–covariance matrix $y_1 I_{d-1}$. Hence, the distribution $\overline{\mu}_{\alpha,\beta}$ of $\overline{Y}$ is called normal α-stable distribution. In this work, we explicit the expression of the variance functions of the NEFs generated respectively by $\mu_{\alpha,\beta}$ and $\overline{\mu}_{\alpha,\beta}$. Since the distribution $\mu_{\alpha,\beta}$ of $(Y_1, \overline{Y})$ and the normal α-stable distribution of $\overline{Y}$ are infinitely divisible, we determine their associated Lévy measures.

This paper is structured as follows. After recalling some properties of the NEFs and defining the normal α-stable distribution in Sect. 2, we determine the variance function of the NEF generated by the distribution $\mu_{\alpha,\beta}$ and the variance function of the NEF generated by the normal α-stable distribution in Sect. 3. In Sect. 4, we use the Lévy–Khintchine representation of the distribution $\mu_{\alpha,\beta}$ and the normal α-stable distribution $\overline{\mu}_{\alpha,\beta}$ to determine the Lévy measures of these distributions. Our work will be illustrated by some examples immediately followed by a conclusion.

2. Natural Exponential Families and Normal α-Stable Random Vector

2.1. Natural Exponential Families on $\mathbb{R}^d$

To clarify the results of this paper, we need to introduce some basic notations and definitions. Our notations are the ones used by Letac and Mora [15].

Let μ be a positive random measure on $\mathbb{R}^d$. We denote its Laplace transform by

$$L_\mu(\theta) = \int_{\mathbb{R}^d} e^{\langle\theta,x\rangle}\mu(\mathrm{d}x),$$

where $\langle\theta, x\rangle$ is the ordinary scalar product on $\mathbb{R}^d$. Moreover, the set $\Theta(\mu)$ is defined by

$$\Theta(\mu) = \text{interior}\{\theta \in \mathbb{R}^d;\ \ L_\mu(\theta) < \infty\}.$$

The set $\mathcal{M}(\mathbb{R}^d)$ is now defined as the set of positive measures μ such that μ is not concentrated on an affine hyperplane and $\Theta(\mu)$ is not empty. For each $\mu \in \mathcal{M}(\mathbb{R}^d)$ and $\theta \in \Theta(\mu)$, we define the cumulant function of μ by

$$k_\mu(\theta) = \ln\left(L_\mu(\theta)\right).$$

The probability set

$$F = F(\mu) = \{P(\theta,\mu)(\mathrm{d}x) = e^{\langle\theta,x\rangle - k_\mu(\theta)}\mu(\mathrm{d}x);\ \ \theta \in \Theta(\mu)\}$$

is called the NEF generated by μ.

It is well known that the mean of $P(\theta, \mu)$ is

$$k'_\mu(\theta) = \int_{\mathbb{R}^d} xP(\theta,\mu)(\mathrm{d}x). \tag{2.1}$$

The first derivative k'_μ defines a diffeomorphism between $\Theta(\mu)$ and its image $M_{F(\mu)}$, called the domain of the means of $F(\mu)$, and we denote its inverse function by $\psi_\mu\colon M_{F(\mu)} \longrightarrow \Theta(\mu)$.

The covariance operator $k''_\mu(\theta)$ of $P(\theta, \mu)$ is symmetric positive definite. It is given by

$$k''_\mu(\theta) = \int_{\mathbb{R}^d} x \otimes xP(\theta,\mu)(\mathrm{d}x) - k'_\mu(\theta) \otimes k'_\mu(\theta),$$

where, for all vectors u and v in $\mathbb{R}^d$, $(x \otimes x)(u, v) = \langle x, u\rangle\langle x, v\rangle$.

The map defined on $M_{F(\mu)}$ by $m \longmapsto V_{F(\mu)}(m) = k''_\mu(\psi_\mu(m))$ is called the variance function of the NEF $F(\mu)$. The importance of this function stems from the fact that it characterizes the family $F(\mu)$.

Finally, the set

$$\Lambda(\mu) = \{\lambda > 0;\ \exists\ \mu_\lambda \text{ such that } k_{\mu_\lambda}(\theta) = \lambda k_\mu(\theta), \forall\theta \in \Theta(\mu_\lambda) = \Theta(\mu)\}$$

is called the Jørgensen set of μ and the probability measure μ_λ is its λth power of convolution. The measure μ is infinitely divisible, if the set $\Lambda(\mu)$ is equal to $]0, +\infty[$ (see Seshadri [22, p. 155]). Recall that if a measure μ is infinitely divisible on $\mathbb{R}^d$, then it satisfies the Lévy–Khintchine representation

$$k''_\mu(\theta) = \Sigma + \int_{\mathbb{R}^d\setminus\{0\}} x \otimes xe^{\langle\theta,x\rangle}\nu(\mathrm{d}x),$$

where Σ is a symmetric positive matrix and the measure ν satisfies the condition $\int_{\mathbb{R}^d\setminus\{0\}} \min(1, \|x\|^2)\nu(\mathrm{d}x) < +\infty$. The measure ν is called Lévy measure. It is important to notice that Seshadri has given, in one dimension, the Lévy measure associated to the class of stable distributions (see Seshadri [22, p. 163]).

In what follows, we denote by $(e_1, \ldots, e_d)$ an orthonormal basis of $\mathbb{R}^d$ and each element x of $\mathbb{R}^d$ is represented as $x = (x_1, \overline{x}) \in \mathbb{R} \times \mathbb{R}^{d-1}$.

2.2. Normal α-Stable Random Vector

Let α be an element of $]0, 2]$. A random variable X on $\mathbb{R}$ is α-stable if, for each $n \geq 2$, there exist $f_n \in \mathbb{R}$ and n independent copies $X_1, X_2, \ldots, X_n$ of X, such that

$$X_1 + X_2 + \cdots + X_n =^d n^{1/\alpha} X + f_n, \tag{2.2}$$

where $=^d$ denotes the equality in distribution. Moreover, X is strictly stable if (2.2) holds with $f_n = 0$. Note that the case $\alpha = 2$ corresponds to a Gaussian random variable.

Let us consider $\alpha \in]0, 1[$. A positive random variable X has a standard positive stable distribution with index α, if there exists $c > 0$ such that, for all $\theta_1 \leq 0$, we have

$$L_X(\theta_1) = \mathbb{E}(e^{\theta_1 X}) = e^{-c(-\theta_1)^\alpha}. \tag{2.3}$$

We say that a random variable Z has a drifted positive stable distribution (see Hougaard [11]), if there exists $\theta_0 < 0$ such that, for all $\theta_1 \leq -\theta_0$, we have

$$L_Z(\theta_1) = \mathbb{E}(e^{\theta_1 Z}) = e^{-c(-\theta_1-\theta_0)^\alpha + c(-\theta_0)^\alpha}. \tag{2.4}$$

Remarks 2.1. 1. Note that the drifted distribution belongs to the exponential family generated by the standard one. In fact, according to (2.3) and (2.4), we can write, for all $\theta_1 \leq -\theta_0$,

$$L_Z(\theta_1) = e^{c(-\theta_0)^\alpha} e^{-c(-\theta_1-\theta_0)^\alpha} = e^{c(-\theta_0)^\alpha} L_X(\theta_1 + \theta_0).$$

This implies that the distribution P_Z of Z is an element of the NEF generated by P_X:

$$P_Z(dz) =^d e^{\theta_0 z + c(-\theta_0)^\alpha} P_X(\mathrm{d}z).$$

2. The Laplace transform of a drifted positive stable random variable Z has the following form

$$\begin{aligned} L_Z(\theta_1) &= e^{-c(-\theta_1-\theta_0)^\alpha + c(-\theta_0)^\alpha} \\ &= e^{c(-\theta_0)^\alpha\left(1-\left(1+\frac{\theta_1}{\theta_0}\right)^\alpha\right)} \\ &= e^{t(1-(1-\beta\theta_1)^\alpha)}, \quad \text{for all } \theta_1 < \frac{1}{\beta}, \end{aligned}$$

where $\beta = -\frac{1}{\theta_0} > 0$ and $t = c(-\theta_0)^\alpha > 0$. Note that, without loss of generality, we can take $t = 1$.

Let $Y = (Y_1, \overline{Y})$ be a random vector on $]0, +\infty[\times\mathbb{R}^{d-1}$ such that Y_1 is a drifted α-stable random variable on $]0, +\infty[$ with a distribution $\pi_{\alpha,\beta}$ $(0 < \alpha < 1)$ and the conditional distribution of $\overline{Y}$, given $Y_1 = y_1$, is a centred normal random vector on $\mathbb{R}^{d-1}$ with a variance–covariance matrix $y_1 I_{d-1}$, where I_{d-1} represents the identity matrix of order $d-1$. We denote the distribution of $Y = (Y_1, \overline{Y})$ by

$$\mu_{\alpha,\beta}(\mathrm{d}y_1, \mathrm{d}\overline{y}) = N(0, y_1 I_{d-1})(\mathrm{d}\overline{y}) \otimes \pi_{\alpha,\beta}(\mathrm{d}y_1). \tag{2.5}$$

Let $F_{\alpha,\beta} = F(\mu_{\alpha,\beta})$ be the NEF generated by $\mu_{\alpha,\beta}$. Suppose that there exists $\beta > 0$ such that the Laplace transform of the drifted α-stable random variable Y_1 is defined by

$$L_{\pi_{\alpha,\beta}}(\theta_1) = e^{1-(1-\beta\theta_1)^\alpha}, \quad \forall \theta_1 < 1/\beta. \tag{2.6}$$

Next, we characterize the distribution $\overline{\mu}_{\alpha,\beta}$ of the random vector $\overline{Y}$. More precisely, we have the following result.

Proposition 2.2. *The distribution $\overline{\mu}_{\alpha,\beta}$ of $\overline{Y}$ is called the normal α-stable distribution on $\mathbb{R}^{d-1}$, and for all $\overline{\theta} \in B_{d-1}(0, \sqrt{2/\beta})$, its Laplace transform is given by*

$$L_{\overline{\mu}_{\alpha,\beta}}(\overline{\theta}) = e^{1-\left(1-\beta\frac{\|\overline{\theta}\|^2}{2}\right)^\alpha},$$

where $B_{d-1}(0, \sqrt{2/\beta})$ is the ball in $\mathbb{R}^{d-1}$ centred at 0, with a radius $\sqrt{2/\beta}$.

Proof. Using (2.5) and (2.6), we get, for all $\theta \in \{(\theta_1, \overline{\theta}) \in \mathbb{R} \times \mathbb{R}^{d-1}\ ;\ \theta_1 + \frac{\|\overline{\theta}\|^2}{2} < 1/\beta\}$,

$$\begin{aligned} L_{\mu_{\alpha,\beta}}(\theta) &= \mathbb{E}(e^{\langle\theta,Y\rangle}) = \mathbb{E}(e^{\theta_1 Y_1 + \langle\overline{\theta},\overline{Y}\rangle}) \\ &= \int_0^{+\infty} \int_{\mathbb{R}^{d-1}} e^{\theta_1 y_1} e^{\langle\overline{\theta},\overline{y}\rangle} N(0, y_1 I_{d-1})(\mathrm{d}\overline{y}) \otimes \pi_{\alpha,\beta}(\mathrm{d}y_1) \\ &= \int_0^{+\infty} e^{\theta_1 y_1} \left(\int_{\mathbb{R}^{d-1}} e^{\langle\overline{\theta},\overline{y}\rangle} N(0, y_1 I_{d-1})(\mathrm{d}\overline{y}) \right) \pi_{\alpha,\beta}(\mathrm{d}y_1) \\ &= \int_0^{+\infty} e^{\theta_1 y_1} e^{\frac{y_1\|\overline{\theta}\|^2}{2}} \pi_{\alpha,\beta}(\mathrm{d}y_1) \\ &= \int_0^{+\infty} e^{y_1\left(\theta_1 + \frac{\|\overline{\theta}\|^2}{2}\right)} \pi_{\alpha,\beta}(\mathrm{d}y_1) \\ &= L_{\pi_{\alpha,\beta}}\left(\theta_1 + \frac{\|\overline{\theta}\|^2}{2}\right) \\ &= e^{1-\left(1-\beta\left(\theta_1 + \frac{\|\overline{\theta}\|^2}{2}\right)\right)^\alpha}. \end{aligned} \tag{2.7}$$

Taking $\theta_1 = 0$ in (2.7), we deduce that for all $\theta = (0, \overline{\theta})$ such that $\overline{\theta} \in B_{d-1}(0, \sqrt{2/\beta})$, we get

$$L_{\overline{\mu}_{\alpha,\beta}}(\overline{\theta}) = \mathbb{E}\left(e^{\langle\overline{\theta},\overline{Y}\rangle}\right) = L_{\mu_{\alpha,\beta}}(0, \overline{\theta}) = e^{1-\left(1-\beta\frac{\|\overline{\theta}\|^2}{2}\right)^\alpha}. \tag{2.8}$$

□

It is important to notice that when $\alpha = 1/2$, $\overline{\mu}_{1/2,\beta}$ corresponds to the normal inverse Gaussian distribution (see Barndorff-Nielsen [2]), and (2.8) becomes

$$L_{\overline{\mu}_{1/2,\beta}}(\overline{\theta}) = e^{1-\sqrt{1-\beta\frac{\|\overline{\theta}\|^2}{2}}}.$$

The class of normal inverse Gaussian distributions is absolutely continuous probability distribution that is defined as the normal variance–mean mixture where the mixing density is the inverse Gaussian distribution. It is well known that this class is infinitely divisible. In terms of processes, this distribution defines a Lévy process, which is determined through subordination of Brownian motion by the inverse Gaussian process (see Barndorff-Nielsen [1]).

3. Variance Function

In this section, we determine explicitly the expression of the variance function of the NEF $F(\overline{\mu}_{\alpha,\beta})$ generated by the normal α-stable distribution. The importance of the variance function comes from the fact that it characterizes the NEF and can be written as a function of the mean parameter m. Many characteristic properties of classes of distributions have been established using variance function (see Hassairi and Zarai [9], Letac and Mora [15] and Morris [19]). For this purpose, we give the expression of the Hessian of the cumulant function of $\mu_{\alpha,\beta}$.

Proposition 3.1. 1. *The covariance operator $k''_{\mu_{\alpha,\beta}}(\theta_1,\overline{\theta})$ of $\mu_{\alpha,\beta}$ is given by*

$$k''_{\mu_{\alpha,\beta}}(\theta_1,\overline{\theta}) = \frac{\alpha(1-\alpha)\beta^2}{\left(1-\beta\left(\theta_1+\frac{\|\overline{\theta}\|^2}{2}\right)\right)^{2-\alpha}}(1,\overline{\theta})\otimes(1,\overline{\theta}) + \frac{\alpha\beta}{\left(1-\beta\left(\theta_1+\frac{\|\overline{\theta}\|^2}{2}\right)\right)^{1-\alpha}}\left(I_d - e_1\otimes e_1\right),$$

where θ is an element of $\Theta(\mu_{\alpha,\beta}) = \{(\theta_1,\overline{\theta})\in\mathbb{R}^d \ ; \ \theta_1+\frac{\|\overline{\theta}\|^2}{2} < 1/\beta\}$.

2. *The covariance operator $k''_{\overline{\mu}_{\alpha,\beta}}(\overline{\theta})$ of $\overline{\mu}_{\alpha,\beta}$ is given by*

$$k''_{\overline{\mu}_{\alpha,\beta}}(\overline{\theta}) = \frac{\alpha\beta}{\left(1-\beta\frac{\|\overline{\theta}\|^2}{2}\right)^{1-\alpha}}I_{d-1} + \frac{\alpha(1-\alpha)\beta^2}{\left(1-\beta\frac{\|\overline{\theta}\|^2}{2}\right)^{2-\alpha}}\overline{\theta}\otimes\overline{\theta},$$

where $\overline{\theta}$ is an element of $\Theta(\overline{\mu}_{\alpha,\beta}) = B_{d-1}(0,\sqrt{2/\beta})$.

Proof.

1. According to (2.7), we have, for all $\theta\in\{(\theta_1,\overline{\theta})\in\mathbb{R}^d;\ \theta_1+\frac{\|\overline{\theta}\|^2}{2} < 1/\beta\}$,

$$k_{\mu_{\alpha,\beta}}(\theta) = k_{\mu_{\alpha,\beta}}(\theta_1,\overline{\theta}) = 1-\left(1-\beta\left(\theta_1+\frac{\|\overline{\theta}\|^2}{2}\right)\right)^{\alpha}. \tag{3.9}$$

The derivatives of $k_{\mu_{\alpha,\beta}}(\theta)$ with respect to θ_1 and $\overline{\theta}$ are given by

$$\frac{\partial k_{\mu_{\alpha,\beta}}(\theta_1,\overline{\theta})}{\partial\theta_1} = \alpha\beta\left(1-\beta\left(\theta_1+\frac{\|\overline{\theta}\|^2}{2}\right)\right)^{\alpha-1}, \tag{3.10}$$

and

$$\frac{\partial k_{\mu_{\alpha,\beta}}(\theta_1,\overline{\theta})}{\partial \overline{\theta}} = \alpha\beta\left(1-\beta\left(\theta_1+\frac{\|\overline{\theta}\|^2}{2}\right)\right)^{\alpha-1}\overline{\theta}. \tag{3.11}$$

Differentiating these with respect to θ_1 and $\overline{\theta}$, we get

$$\frac{\partial^2 k_{\mu_{\alpha,\beta}}(\theta_1,\overline{\theta})}{\partial \theta_1^2} = \frac{\alpha(1-\alpha)\beta^2}{\left(1-\beta\left(\theta_1+\frac{\|\overline{\theta}\|^2}{2}\right)\right)^{2-\alpha}}, \tag{3.12}$$

$$\frac{\partial^2 k_{\mu_{\alpha,\beta}}(\theta_1,\overline{\theta})}{\partial \theta_1 \partial \overline{\theta}} = \frac{\alpha(1-\alpha)\beta^2}{\left(1-\beta\left(\theta_1+\frac{\|\overline{\theta}\|^2}{2}\right)\right)^{2-\alpha}}\overline{\theta}, \tag{3.13}$$

and

$$\begin{aligned}\frac{\partial^2 k_{\mu_{\alpha,\beta}}(\theta_1,\overline{\theta})}{\partial \overline{\theta}^2} &= \frac{\alpha\beta}{\left(1-\beta\left(\theta_1+\frac{\|\overline{\theta}\|^2}{2}\right)\right)^{1-\alpha}} I_{d-1} \\ &\quad + \frac{\alpha(1-\alpha)\beta^2}{\left(1-\beta\left(\theta_1+\frac{\|\overline{\theta}\|^2}{2}\right)\right)^{2-\alpha}}\overline{\theta}\otimes\overline{\theta}.\end{aligned} \tag{3.14}$$

Therefore,

$$\begin{aligned}k''_{\mu_{\alpha,\beta}}(\theta_1,\overline{\theta}) &= \frac{\alpha(1-\alpha)\beta^2}{\left(1-\beta\left(\theta_1+\frac{\|\overline{\theta}\|^2}{2}\right)\right)^{2-\alpha}}\begin{pmatrix} 1 & \overline{\theta} \\ \overline{\theta}' & \overline{\theta}\otimes\overline{\theta}\end{pmatrix} \\ &\quad + \frac{\alpha\beta}{\left(1-\beta\left(\theta_1+\frac{\|\overline{\theta}\|^2}{2}\right)\right)^{1-\alpha}}\begin{pmatrix} 0 & 0 \\ 0 & I_{d-1}\end{pmatrix},\end{aligned} \tag{3.15}$$

where $\overline{\theta}'$ denotes the vector transpose of $\overline{\theta}$. This implies that the covariance operator $k''_{\mu_{\alpha,\beta}}(\theta_1,\overline{\theta})$ is equal to

$$\begin{aligned}&\frac{\alpha(1-\alpha)\beta^2}{\left(1-\beta\left(\theta_1+\frac{\|\overline{\theta}\|^2}{2}\right)\right)^{2-\alpha}}(1,\overline{\theta})\otimes(1,\overline{\theta}) \\ &\quad + \frac{\alpha\beta}{\left(1-\beta\left(\theta_1+\frac{\|\overline{\theta}\|^2}{2}\right)\right)^{1-\alpha}}\left(I_d - e_1\otimes e_1\right).\end{aligned}$$

2. Using (3.9) and the fact that

$$k_{\overline{\mu}_{\alpha,\beta}}(\overline{\theta}) = k_{\mu_{\alpha,\beta}}(0,\overline{\theta}), \tag{3.16}$$

we deduce that, for all $\overline{\theta}\in B_{d-1}(0,\sqrt{2/\beta})$, we have

$$k_{\overline{\mu}_{\alpha,\beta}}(\overline{\theta}) = 1-\left(1-\beta\frac{\|\overline{\theta}\|^2}{2}\right)^{\alpha}.$$

Differentiating this twice with respect to $\overline{\theta}$, we get the desired result □

Remark 3.2. The random vector $\dfrac{\overline{Y}}{\sqrt{\alpha\beta}}$ is standard normal α-stable distributed. In fact, using (3.11) and (3.16), we get $\mathbb{E}(\frac{\overline{Y}}{\sqrt{\alpha\beta}}) = \dfrac{k'_{\overline{\mu}_{\alpha,\beta}}(0)}{\sqrt{\alpha\beta}} = 0$. Furthermore, using (3.15), we deduce that the variance of the random variable $\frac{\overline{Y}}{\sqrt{\alpha\beta}}$ is

$$\mathrm{Var}\left(\frac{\overline{Y}}{\sqrt{\alpha\beta}}\right) = \frac{k''_{\mu_{\alpha,\beta}}(0)}{\alpha\beta} = I_{d-1}.$$

Next, we give the variance function of the NEF $F(\mu_{\alpha,\beta})$ generated by $\mu_{\alpha,\beta}$.

Theorem 3.3. *For all* $m = (m_1, \overline{m}) \in M_{F(\mu_{\alpha,\beta})} =]0, \infty[\times\mathbb{R}^{d-1}$, *the variance function of* $F(\mu_{\alpha,\beta})$ *is given by*

$$V_{F(\mu_{\alpha,\beta})}(m_1, \overline{m}) = \alpha^{\frac{1}{\alpha-1}}(1-\alpha)\beta^{\frac{\alpha}{\alpha-1}} m_1^{\frac{\alpha}{1-\alpha}} m \otimes m + m_1 \left(I_d - e_1 \otimes e_1\right). \quad (3.17)$$

Proof. Since

$$k'_{\mu_{\alpha,\beta}}(\theta_1, \overline{\theta}) = \left(\frac{\partial k_{\mu_{\alpha,\beta}}(\theta_1, \overline{\theta})}{\partial \theta_1}, \frac{\partial k_{\mu_{\alpha,\beta}}(\theta_1, \overline{\theta})}{\partial \overline{\theta}}\right) = m = (m_1, \overline{m}) \in]0, \infty[\times\mathbb{R}^{d-1},$$

using (3.10), we deduce that

$$1 - \beta\left(\theta_1 + \frac{\|\overline{\theta}\|^2}{2}\right) = \left(\frac{m_1}{\alpha\beta}\right)^{\frac{1}{\alpha-1}},$$

and according to (3.11), we have

$$\overline{\theta} = \frac{\overline{m}}{m_1}.$$

Inserting this in (3.15), we get

$$V_{F(\mu_{\alpha,\beta})}(m_1, \overline{m}) = \alpha^{\frac{1}{\alpha-1}}(1-\alpha)\beta^{\frac{\alpha}{\alpha-1}} m_1^{\frac{\alpha}{1-\alpha}} \begin{pmatrix} m_1^2 & m_1\overline{m} \\ m_1\overline{m}' & \overline{m} \otimes \overline{m} \end{pmatrix} + m_1 \begin{pmatrix} 0 & 0 \\ 0 & I_{d-1} \end{pmatrix}. \quad (3.18)$$

Then, the variance function of $F(\mu_{\alpha,\beta})$, for all $m = (m_1, \overline{m}) \in]0, \infty[\times\mathbb{R}^{d-1}$, is given by

$$V_{F(\mu_{\alpha,\beta})}(m) = \alpha^{\frac{1}{\alpha-1}}(1-\alpha)\beta^{\frac{\alpha}{\alpha-1}} m_1^{\frac{\alpha}{1-\alpha}} m \otimes m + m_1 \left(I_d - e_1 \otimes e_1\right).$$

□

Remark 3.4. Note that the case where $\alpha = 1/2$ corresponds to the inverse Gaussian family given by Hassairi [8]. In fact, let $Y = (Y_1, \overline{Y})$ be a random vector such that Y_1 is an inverse Gaussian distributed and $\overline{Y}$ is a normal inverse Gaussian distributed. Taking $\alpha = 1/2$ in (3.17), we obtain, for all $m = (m_1, \overline{m}) \in]0, \infty[\times\mathbb{R}^{d-1}$,

$$V_{F(\mu_{1/2,\beta})}(m_1, \overline{m}) = \frac{2}{\beta} m_1 m \otimes m + m_1(I_d - e_1 \otimes e_1).$$

This corresponds to a multivariate cubic variance function (Hassairi and Zarai [9]).

Next, we characterize the natural exponential family $F(\overline{\mu}_{\alpha,\beta})$ generated by the normal α-stable distribution by means its variance function. More precisely, we have the following theorem.

Theorem 3.5. *For all $\overline{m} \in M_{F(\overline{\mu}_{\alpha,\beta})} = \mathbb{R}^{d-1}$, the variance function of $F(\overline{\mu}_{\alpha,\beta})$ is given by*

$$V_{F(\overline{\mu}_{\alpha,\beta})}(\overline{m}) = \xi(\overline{m})I_{d-1} + \alpha^{\frac{1}{\alpha-1}}(1-\alpha)\beta^{\frac{\alpha}{\alpha-1}}\xi(\overline{m})^{\frac{\alpha}{1-\alpha}}\overline{m}\otimes\overline{m},$$

where $\xi(\overline{m})$ is the unique solution of the equation

$$(\alpha\beta)^{\frac{1}{1-\alpha}}s^{\frac{2\alpha-1}{\alpha-1}} - s^2 + \frac{\beta\|\overline{m}\|^2}{2} = 0, \quad \forall s > 0.$$

Before starting the proof of Theorem 3.5, we need to study and prove the two following technical lemmas.

Lemma 3.6. *Let α be an element of $]0;1[$ and β be a strictly positive real. Then, the function ϕ defined on $]0,+\infty[$ by*

$$\phi(x) = (\alpha\beta)^{\frac{1}{1-\alpha}}x^{\frac{2\alpha-1}{\alpha-1}} - x^2 + \frac{\beta\|\overline{m}\|^2}{2} \tag{3.19}$$

admits a unique positive solution $\xi(\overline{m})$.

Proof of Lemma 3.6. Differentiating ϕ with respect to x, we obtain, for all $x > 0$,

$$\phi'(x) = (\alpha\beta)^{\frac{1}{1-\alpha}}\frac{2\alpha-1}{\alpha-1}x^{\frac{\alpha}{\alpha-1}} - 2x. \tag{3.20}$$

We distinguish three cases.

<u>Case 1: $\alpha = 1/2$</u>

In this case, ϕ is strictly decreasing from $]0,+\infty[$ to $\phi(]0,+\infty[) =]-\infty, \frac{\beta^2}{4} + \frac{\beta\|\overline{m}\|^2}{2}[$. Since $\frac{\beta^2}{4} + \frac{\beta\|\overline{m}\|^2}{2} > 0$, we deduce that ϕ admits a unique positive zero.

<u>Case 2: $0 < \alpha < 1/2$</u>

In this case,

$$\phi'(x) = 0 \Leftrightarrow x = x_0 = \alpha\beta\left(\frac{2\alpha-1}{2\alpha-2}\right)^{1-\alpha} > 0.$$

Using the fact that ϕ' admits a unique zero x_0 in $]0,+\infty[$, $\lim_{x\mapsto 0}\phi'(x) = +\infty$ and $\lim_{x\mapsto+\infty}\phi'(x) = -\infty$, we deduce that

$$\phi'(x) > 0, \quad \forall x \in]0, x_0[$$

and

$$\phi'(x) < 0, \quad \forall x \in]x_0, +\infty[.$$

This implies that the function ϕ is strictly increasing on $]0, x_0[$ and strictly decreasing on $]x_0, +\infty[$. Furthermore, since $\phi(0) = \frac{\beta\|\overline{m}\|^2}{2} > 0$, $\phi(x_0) > 0$, and since $\lim_{x\mapsto+\infty}\phi(x)=-\infty$, we conclude that ϕ has a unique positive zero $\xi(\overline{m})>x_0$.

Case 3: $1/2<\alpha<1$

Using (3.20), we deduce that ϕ is strictly decreasing from $]0,+\infty[$ to $\phi(]0,+\infty[)=\mathbb{R}$. It follows that the equation $\phi(x)=0$ admits a unique solution. □

Lemma 3.7. *For all* $\overline{\theta}\in B_{d-1}(0,\sqrt{\beta/2})$, *one has*

$$k'_{\mu_{\alpha,\beta}}(0,\overline{\theta})=\left(\frac{\partial k_{\mu_{\alpha,\beta}}(0,\overline{\theta})}{\partial\theta_1},\quad k'_{\overline{\mu}_{\alpha,\beta}}(\overline{\theta})\right),$$

and

$$k''_{\mu_{\alpha,\beta}}(0,\overline{\theta})=\begin{pmatrix}\frac{\partial^2 k_{\mu_{\alpha,\beta}}(0,\overline{\theta})}{\partial\theta_1^2} & \frac{\partial^2 k_{\mu_{\alpha,\beta}}(0,\overline{\theta})}{\partial\overline{\theta}\partial\theta_1}\\ \left(\frac{\partial^2 k_{\mu_{\alpha,\beta}}(0,\overline{\theta})}{\partial\theta_1\partial\overline{\theta}}\right)' & k''_{\overline{\mu}_{\alpha,\beta}}(\overline{\theta})\end{pmatrix}.$$

Proof of Lemma 3.7. According to (2.1), we have for all $\theta\in\{(\theta_1,\overline{\theta})\in\mathbb{R}\times\mathbb{R}^{d-1}\ ;\ \theta_1+\frac{\|\overline{\theta}\|^2}{2}<1/\beta\}$

$$k'_{\mu_{\alpha,\beta}}(\theta_1,\overline{\theta})=\int_0^{+\infty}\int_{\mathbb{R}^{d-1}}(y_1,\overline{y})P((\theta_1,\overline{\theta}),\mu_{\alpha,\beta})(\mathrm{d}y_1,\mathrm{d}\overline{y}) \tag{3.21}$$

It follows that for all $\theta=(\theta_1,\overline{\theta})\in\Theta(\mu_{\alpha,\beta})=\{(\theta_1,\overline{\theta})\in\mathbb{R}\times\mathbb{R}^{d-1}\ ;\ \theta_1+\frac{\|\overline{\theta}\|^2}{2}<1/\beta\}$,

$$\begin{aligned}k'_{\mu_{\alpha,\beta}}(\theta_1,\overline{\theta})=&\left(\int_0^{+\infty}\int_{\mathbb{R}^{d-1}}y_1P((\theta_1,\overline{\theta}),\mu_{\alpha,\beta})(\mathrm{d}y_1,\mathrm{d}\overline{y}),\right.\\ &\left.\times\int_0^{+\infty}\int_{\mathbb{R}^{d-1}}\overline{y}P((\theta_1,\overline{\theta}),\mu_{\alpha,\beta})(\mathrm{d}y_1,\mathrm{d}\overline{y})\right).\end{aligned}$$

By setting $\theta_1=0$ and using the fact that

$$P((0,\overline{\theta}),\mu_{\alpha,\beta})(\mathrm{d}y_1,\mathrm{d}\overline{y})=e^{\langle\overline{\theta},\overline{y}\rangle-k_{\overline{\mu}_{\alpha,\beta}}(\overline{\theta})}\mu_{\alpha,\beta}(\mathrm{d}y_1,\mathrm{d}\overline{y}),$$

we deduce that for all $\overline{\theta}\in B_{d-1}(0,\sqrt{\beta/2})$, we have

$$\begin{aligned}\frac{\partial k_{\mu_{\alpha,\beta}}(0,\overline{\theta})}{\partial\overline{\theta}}&=\int_0^{+\infty}\int_{\mathbb{R}^{d-1}}\overline{y}P((0,\overline{\theta}),\mu_{\alpha,\beta})(\mathrm{d}y_1,\mathrm{d}\overline{y})\\&=\int_0^{+\infty}\int_{\mathbb{R}^{d-1}}\overline{y}e^{\langle\overline{\theta},\overline{y}\rangle-k_{\overline{\mu}_{\alpha,\beta}}(\overline{\theta})}\mu_{\alpha,\beta}(\mathrm{d}y_1,\mathrm{d}\overline{y})\\&=\int_{\mathbb{R}^{d-1}}\overline{y}e^{\langle\overline{\theta},\overline{y}\rangle-k_{\overline{\mu}_{\alpha,\beta}}(\overline{\theta})}\overline{\mu}_{\alpha,\beta}(\mathrm{d}\overline{y})\\&=k'_{\overline{\mu}_{\alpha,\beta}}(\overline{\theta}).\end{aligned} \tag{3.22}$$

Inserting this in (3.21), we conclude that for all $\overline{\theta} \in B_{d-1}(0, \sqrt{\beta/2})$, we have

$$\begin{aligned} k'_{\mu_{\alpha,\beta}}(0,\overline{\theta}) &= \Big(\int_0^{+\infty}\int_{\mathbb{R}^{d-1}} y_1 P((0,\overline{\theta}),\mu_{\alpha,\beta})(\mathrm{d}y_1,\mathrm{d}\overline{y}), \\ &\qquad \times \int_0^{+\infty}\int_{\mathbb{R}^{d-1}} \overline{y} P((0,\overline{\theta}),\mu_{\alpha,\beta})(\mathrm{d}y_1,\mathrm{d}\overline{y})\Big) \\ &= \Big(\int_0^{+\infty}\int_{\mathbb{R}^{d-1}} y_1 P((0,\overline{\theta}),\mu_{\alpha,\beta})(\mathrm{d}y_1,\mathrm{d}\overline{y}), k'_{\overline{\mu}_{\alpha,\beta}}(\overline{\theta})\Big) \\ &= \left(\frac{\partial k_{\mu_{\alpha,\beta}}(0,\overline{\theta})}{\partial \theta_1}, k'_{\overline{\mu}_{\alpha,\beta}}(\overline{\theta})\right). \end{aligned}$$

Furthermore, by differentiating (3.22) with respect to $\overline{\theta}$, we can write

$$\begin{aligned} k''_{\overline{\mu}_{\alpha,\beta}}(\overline{\theta}) &= \int_0^{+\infty}\int_{\mathbb{R}^{d-1}} \overline{y}\otimes\overline{y} P((0,\overline{\theta}),\mu_{\alpha,\beta})(\mathrm{d}y_1,\mathrm{d}\overline{y}) - k'_{\overline{\mu}_{\alpha,\beta}}(\overline{\theta})\otimes k'_{\overline{\mu}_{\alpha,\beta}}(\overline{\theta}) \\ &= \frac{\partial^2 k_{\mu_{\alpha,\beta}}(0,\overline{\theta})}{\partial \overline{\theta}^2}. \end{aligned}$$

This represents the desired result. □

Now, we are ready to prove Theorem 3.5.

Proof of Theorem 3.5. Using (3.10) and (3.11), we deduce that for a fixed $\theta_1 = 0$ and for $\overline{\theta} \in B_{d-1}(0, \sqrt{2/\beta})$, there exist $m_1(0,\overline{\theta}) \in]0, +\infty[$ and $\overline{m} \in \mathbb{R}^{d-1}$, such that

$$\frac{\partial k_{\mu_{\alpha,\beta}}(0,\overline{\theta})}{\partial \theta_1} = \alpha\beta\left(1-\beta\frac{\|\overline{\theta}\|^2}{2}\right)^{\alpha-1} = m_1(0,\overline{\theta}) = m_1 > 0,$$

and

$$\frac{\partial k_{\mu_{\alpha,\beta}}(0,\overline{\theta})}{\partial \overline{\theta}} = \alpha\beta\left(1-\beta\frac{\|\overline{\theta}\|^2}{2}\right)^{\alpha-1}\overline{\theta} = \overline{m}(0,\overline{\theta}) = \overline{m} \in \mathbb{R}^{d-1}.$$

Therefore,

$$m_1 = \alpha\beta\left(1-\beta\frac{\|\overline{m}\|^2}{2m_1^2}\right)^{\alpha-1}.$$

It follows that

$$(\alpha\beta)^{\frac{1}{1-\alpha}} m_1^{\frac{2\alpha-1}{\alpha-1}} - m_1^2 + \frac{\beta\|\overline{m}\|^2}{2} = 0. \tag{3.23}$$

Hence, m_1 is a positive zero of the function ϕ defined in (3.19). This, together with Lemma 3.6, implies that

$$m_1 = \xi(\overline{m}) \tag{3.24}$$

Furthermore, by combining Lemma 3.7 with the equality (3.18), we have

$$k''_{\overline{\mu}_{\alpha,\beta}}(\overline{\theta}) = V_{F(\overline{\mu}_{\alpha,\beta})}(\overline{m}) = m_1 I_{d-1} + \alpha^{\frac{1}{\alpha-1}}(1-\alpha)\beta^{\frac{\alpha}{\alpha-1}} m_1^{\frac{\alpha}{1-\alpha}}\overline{m}\otimes\overline{m}. \tag{3.25}$$

Inserting (3.24) in (3.25), we get for all $\overline{m} \in \mathbb{R}^{d-1}$,

$$V_{F(\overline{\mu}_{\alpha,\beta})}(\overline{m}) = \xi(\overline{m}) I_{d-1} + \alpha^{\frac{1}{\alpha-1}}(1-\alpha)\beta^{\frac{\alpha}{\alpha-1}}\xi(\overline{m})^{\frac{\alpha}{1-\alpha}}\overline{m}\otimes\overline{m}. \tag{3.26}$$

□

Remark 3.8. Note that, if $\alpha = 1/2$, then the solution of (3.23) is $\xi(\overline{m}) = \sqrt{\frac{\beta\|\overline{m}\|^2}{2} + \frac{\beta^2}{4}}$. In this case (3.26) becomes

$$V_{F(\overline{\mu}_{1/2,\beta})}(\overline{m}) = \sqrt{\frac{\beta\|\overline{m}\|^2}{2} + \frac{\beta^2}{4}}\left(I_{d-1} + \frac{2}{\beta}\overline{m}\otimes\overline{m}\right), \quad \text{for all } \overline{m} \in \mathbb{R}^{d-1}.$$

Then, $F(\overline{\mu}_{1/2,\beta})$ is the normal inverse Gaussian family on $\mathbb{R}^{d-1}$.

4. Lévy Measures

In this section, we determine the Lévy measures of the distribution $\mu_{\alpha,\beta}$ and the normal α-stable distribution $\overline{\mu}_{\alpha,\beta}$ which are infinitely divisible (see Letac and Seshadri [16]). For this purpose, we use the Lévy–Khintchine representation (see Barndorff-Nielsen and Hubalek [3] and Burnaev [5]).

Lemma 4.1. *There exists a Lévy measure $\nu_{\alpha,\beta}$ such that*

$$k''_{\mu_{\alpha,\beta}}(\theta_1,\overline{\theta}) = \int_{\mathbb{R}^d\setminus\{0\}} (y_1,\overline{y})\otimes(y_1,\overline{y})e^{\theta_1 y_1 + \langle\overline{\theta},\overline{y}\rangle}\nu_{\alpha,\beta}(\mathrm{d}y_1,\mathrm{d}\overline{y}).$$

Proof. Using Proposition 3.1, we have for all $\theta \in \{(\theta_1,\overline{\theta}) \in \mathbb{R}\times\mathbb{R}^{d-1}\ ;\ \theta_1 + \frac{\|\overline{\theta}\|^2}{2} < 1/\beta\}$,

$$k''_{\mu_{\alpha,\beta}}(\theta_1,\overline{\theta}) = \frac{\alpha(1-\alpha)\beta^2}{\left(1-\beta\left(\theta_1+\frac{\|\overline{\theta}\|^2}{2}\right)\right)^{2-\alpha}}(1,\overline{\theta})\otimes(1,\overline{\theta}) + \frac{\alpha\beta}{\left(1-\beta\left(\theta_1+\frac{\|\overline{\theta}\|^2}{2}\right)\right)^{1-\alpha}}(I_d - e_1\otimes e_1).$$

It follows that

$$\lim_{\theta_1\mapsto-\infty} k''_{\mu_{\alpha,\beta}}(\theta_1,\overline{\theta}) = 0. \tag{4.27}$$

Furthermore, since $\mu_{\alpha,\beta}$ is infinitely divisible, and according to Lévy–Khintchine representation, there exist a symmetric positive matrix Σ and a positive measure $\nu_{\alpha,\beta}$ such that

$$k''_{\mu_{\alpha,\beta}}(\theta_1,\overline{\theta}) = \Sigma + \int_{\mathbb{R}^d\setminus\{0\}} (y_1,\overline{y})\otimes(y_1,\overline{y})e^{\theta_1 y_1+\langle\overline{\theta},\overline{y}\rangle}\nu_{\alpha,\beta}(\mathrm{d}y_1,\mathrm{d}\overline{y}). \tag{4.28}$$

This, combined with (4.27), implies that

$$0 = \lim_{\theta_1\mapsto-\infty} k''_{\mu_{\alpha,\beta}}(\theta_1,\overline{\theta}) = \Sigma + \lim_{\theta_1\mapsto-\infty}\int_{\mathbb{R}^d\setminus\{0\}} (y_1,\overline{y})\otimes(y_1,\overline{y})e^{\theta_1 y_1+\langle\overline{\theta},\overline{y}\rangle} \nu_{\alpha,\beta}(\mathrm{d}y_1,\mathrm{d}\overline{y}).$$

Since Σ and $\int_{\mathbb{R}^d\setminus\{0\}}(y_1,\overline{y})\otimes(y_1,\overline{y})e^{\theta_1 y_1+\langle\overline{\theta},\overline{y}\rangle}\nu_{\alpha,\beta}(\mathrm{d}y_1,\mathrm{d}\overline{y})$ are positive matrices,

$$\Sigma = 0.$$

By inserting this in (4.28), we get the desired result. □

Next, we give the Lévy measures of $\mu_{\alpha,\beta}$ and $\overline{\mu}_{\alpha,\beta}$. More precisely, we have the following theorem.

Theorem 4.2. 1. *The Lévy measure of* $\mu_{\alpha,\beta}$ *is*

$$\nu_{\alpha,\beta}(\mathrm{d}y_1,\mathrm{d}\overline{y}) = \frac{\alpha\beta}{y_1}N(0,y_1I_{d-1})(\mathrm{d}\overline{y})\otimes\gamma(1-\alpha,\beta)(\mathrm{d}y_1), \tag{4.29}$$

where $\gamma(1-\alpha,\beta)$ *is the real Gamma distribution with a shape parameter* $1-\alpha>0$ *and a scale parameter* $\beta>0$.

2. *The Lévy measure of* $\overline{\mu}_{\alpha,\beta}$ *is*

$$\overline{\nu}_{\alpha,\beta}(\mathrm{d}\overline{y}) = \frac{\alpha\beta}{\Gamma(\beta)2\beta^{\alpha}(2\pi)^{\frac{d-1}{2}}}\left(\frac{2}{\beta\|\overline{y}\|^2}\right)^{\frac{2-d}{4}-\frac{\alpha}{2}}K_{\frac{2-d}{2}-\alpha}\left(\sqrt{2\|\overline{y}\|^2/\beta}\right)\mathrm{d}\overline{y}, \tag{4.30}$$

where $K_{\frac{2-d}{2}-\alpha}(\cdot)$ *denotes the modified Bessel function of the third kind with order* $\frac{2-d}{2}-\alpha$ *(see Seshadri [22, p. 27]).*

Proof. 1. Using (3.12), we have

$$\begin{aligned}\frac{\partial^2 k_{\mu_{\alpha,\beta}}(\theta_1,\overline{\theta})}{\partial\theta_1^2} &= \frac{\alpha(1-\alpha)\beta^2}{\left(1-\beta\left(\theta_1+\frac{\|\overline{\theta}\|^2}{2}\right)\right)^{2-\alpha}}\\ &= \alpha\beta\int_0^{+\infty}y_1e^{y_1\left(\theta_1+\frac{\|\overline{\theta}\|^2}{2}\right)}\gamma(1-\alpha,\beta)(\mathrm{d}y_1)\\ &= \alpha\beta\int_0^{+\infty}y_1e^{\theta_1y_1}\left(\int_{\mathbb{R}^{d-1}}e^{\langle\overline{\theta},\overline{y}\rangle}N(0,y_1I_{d-1})(\mathrm{d}\overline{y})\right)\\ &\quad\times\gamma(1-\alpha,\beta)(\mathrm{d}y_1)\\ &= \int_0^{+\infty}\int_{\mathbb{R}^{d-1}}y_1^2e^{\theta_1y_1+\langle\overline{\theta},\overline{y}\rangle}\frac{\alpha\beta}{y_1}N(0,y_1I_{d-1})(\mathrm{d}\overline{y})\\ &\quad\otimes\gamma(1-\alpha,\beta)(\mathrm{d}y_1).\end{aligned}$$

It follows that

$$\frac{\partial^2 k_{\mu_{\alpha,\beta}}(\theta_1,\overline{\theta})}{\partial\theta_1^2} = \int_{\mathbb{R}^d\setminus\{0\}}y_1^2e^{\theta_1y_1+\langle\overline{\theta},\overline{y}\rangle}\nu_{\alpha,\beta}(\mathrm{d}y_1,\mathrm{d}\overline{y}), \tag{4.31}$$

where the measure $\nu_{\alpha,\beta}$ is given by (4.29).
Moreover, according to (3.13), we have

$$\begin{aligned}\frac{\partial^2 k_{\mu_{\alpha,\beta}}(\theta_1,\overline{\theta})}{\partial\theta_1\partial\overline{\theta}} &= \frac{\alpha(1-\alpha)\beta^2}{\left(1-\beta\left(\theta_1+\frac{\|\overline{\theta}\|^2}{2}\right)\right)^{2-\alpha}}\overline{\theta} \\ &= \alpha\beta\overline{\theta}\int_0^{+\infty} y_1 e^{y_1\left(\theta_1+\frac{\|\overline{\theta}\|^2}{2}\right)}\gamma(1-\alpha,\beta)(\mathrm{d}y_1) \\ &= \alpha\beta\int_0^{+\infty} e^{\theta_1 y_1}\left(\int_{\mathbb{R}^{d-1}}\overline{y}e^{\langle\overline{\theta},\overline{y}\rangle}N(0,y_1I_{d-1})(\mathrm{d}\overline{y})\right) \\ &\quad\times\gamma(1-\alpha,\beta)(\mathrm{d}y_1) \\ &\quad\times\int_0^{+\infty}\int_{\mathbb{R}^{d-1}} y_1\overline{y}e^{\theta_1y_1+\langle\overline{\theta},\overline{y}\rangle}\frac{\alpha\beta}{y_1}N(0,y_1I_{d-1})(\mathrm{d}\overline{y}) \\ &\quad\otimes\gamma(1-\alpha,\beta)(\mathrm{d}y_1).\end{aligned}$$

This implies that

$$\frac{\partial^2 k_{\mu_{\alpha,\beta}}(\theta_1,\overline{\theta})}{\partial\theta_1\partial\overline{\theta}} = \int_{\mathbb{R}^d\setminus\{0\}} y_1\overline{y}e^{\theta_1y_1+\langle\overline{\theta},\overline{y}\rangle}\nu_{\alpha,\beta}(\mathrm{d}y_1,\mathrm{d}\overline{y}). \tag{4.32}$$

Finally, using (3.14), we get

$$\begin{aligned}\frac{\partial^2 k_{\mu_{\alpha,\beta}}(\theta_1,\overline{\theta})}{\partial\overline{\theta}^2} &= \frac{\alpha\beta}{\left(1-\beta\left(\theta_1+\frac{\|\overline{\theta}\|^2}{2}\right)\right)^{1-\alpha}}I_{d-1} + \frac{\alpha(1-\alpha)\beta^2}{\left(1-\beta\left(\theta_1+\frac{\|\overline{\theta}\|^2}{2}\right)\right)^{2-\alpha}}\overline{\theta}\otimes\overline{\theta} \\ &= \alpha\beta\int_0^{+\infty} e^{y_1\left(\theta_1+\frac{\|\overline{\theta}\|^2}{2}\right)}\gamma(1-\alpha,\beta)(\mathrm{d}y_1)I_{d-1} \\ &\quad+\alpha\beta\int_0^{+\infty} y_1e^{y_1\left(\theta_1+\frac{\|\overline{\theta}\|^2}{2}\right)}\gamma(1-\alpha,\beta)(\mathrm{d}y_1)\overline{\theta}\otimes\overline{\theta} \\ &= \alpha\beta\int_0^{+\infty} e^{y_1\left(\theta_1+\frac{\|\overline{\theta}\|^2}{2}\right)}(I_{d-1}+y_1\overline{\theta}\otimes\overline{\theta})\gamma(1-\alpha,\beta)(\mathrm{d}y_1) \\ &= \alpha\beta\int_0^{+\infty}\frac{1}{y_1}e^{\theta_1y_1}\left(\int_{\mathbb{R}^{d-1}}\overline{y}\otimes\overline{y}e^{\langle\overline{\theta},\overline{y}\rangle}N(0,y_1I_{d-1})(\mathrm{d}\overline{y})\right) \\ &\quad\times\gamma(1-\alpha,\beta)(\mathrm{d}y_1) \\ &= \int_0^{+\infty}\int_{\mathbb{R}^{d-1}}\overline{y}\otimes\overline{y}e^{\theta_1y_1+\langle\overline{\theta},\overline{y}\rangle}\frac{\alpha\beta}{y_1}N(0,y_1I_{d-1})(\mathrm{d}\overline{y}) \\ &\quad\otimes\gamma(1-\alpha,\beta)(\mathrm{d}y_1).\end{aligned}$$

Hence,

$$\frac{\partial^2 k_{\mu_{\alpha,\beta}}(\theta_1,\overline{\theta})}{\partial\overline{\theta}^2} = \int_{\mathbb{R}^d\setminus\{0\}}\overline{y}\otimes\overline{y}e^{\theta_1y_1+\langle\overline{\theta},\overline{y}\rangle}\nu_{\alpha,\beta}(\mathrm{d}y_1,\mathrm{d}\overline{y}). \tag{4.33}$$

Consequently, (4.31), (4.32) and (4.33) give

$$\begin{aligned}k''_{\mu_{\alpha,\beta}}(\theta_1,\overline{\theta}) = \int_{\mathbb{R}^d\setminus\{0\}}&(y_1,\overline{y})\otimes(y_1,\overline{y})e^{\theta_1y_1+\langle\overline{y},\overline{\theta}\rangle}\frac{\alpha\beta}{y_1}N(0,y_1I_{d-1})(\mathrm{d}\overline{y}) \\ &\otimes\gamma(1-\alpha,\beta)(\mathrm{d}y_1).\end{aligned}$$

2. According to Lemmas 3.7 and 4.1, we have for $\theta_1 = 0$,

$$k''_{\overline{\mu}_{\alpha,\beta}}(\overline{\theta}) = \int_{\mathbb{R}^d\setminus\{0\}} \overline{y} \otimes \overline{y} e^{\langle \overline{\theta}, \overline{y}\rangle} \nu_{\alpha,\beta}(\mathrm{d}y_1, \overline{y}).$$

Combined with (4.29), this gives

$$\begin{aligned} k''_{\overline{\mu}_{\alpha,\beta}}(\overline{\theta}) &= \int_{\mathbb{R}^d\setminus\{0\}} \overline{y} \otimes \overline{y} e^{\langle \overline{\theta}, \overline{y}\rangle} \frac{\alpha\beta}{y_1} N(0, y_1 I_{d-1})(\mathrm{d}\overline{y}) \otimes \gamma(1-\alpha, \beta)(\mathrm{d}y_1) \\ &= \int_{\mathbb{R}^{d-1}} \overline{y} \otimes \overline{y} e^{\langle \overline{\theta}, \overline{y}\rangle} \left(\frac{\alpha\beta}{\Gamma(\beta)\beta^{\alpha}(2\pi)^{\frac{d-1}{2}}} \int_0^{+\infty} \frac{e^{-\frac{\|\overline{y}\|^2}{2y_1}} e^{-\frac{y_1}{\beta}}}{y_1^{\frac{d+1}{2}+\alpha}} \mathrm{d}y_1 \right) \mathrm{d}\overline{y} \\ &= \int_{\mathbb{R}^{d-1}} \overline{y} \otimes \overline{y} e^{\langle \overline{\theta}, \overline{y}\rangle} \frac{\alpha\beta}{\Gamma(\beta) 2\beta^{\alpha}(2\pi)^{\frac{d-1}{2}}} \left(\frac{2}{\beta\|\overline{y}\|^2} \right)^{-\frac{d-2}{4}-\frac{\alpha}{2}} \\ &\quad \times K_{\frac{2-d}{2}-\alpha}\left(\sqrt{2\|\overline{y}\|^2/\beta}\right) \mathrm{d}\overline{y}. \end{aligned}$$

Therefore,

$$k''_{\overline{\mu}_{\alpha,\beta}}(\overline{\theta}) = \int_{\mathbb{R}^{d-1}} \overline{y} \otimes \overline{y} e^{\langle \overline{\theta}, \overline{y}\rangle} \overline{\nu}_{\alpha,\beta}(\mathrm{d}\overline{y}),$$

where the measure $\overline{\nu}_{\alpha,\beta}$ is defined in (4.30). This represents the desired result. □

Finally, we illustrate the result of the previous theorem by an example.

Example. For $\alpha = 1/2$, the Lévy measure of the distribution $\mu_{1/2,\beta}$ is given by

$$\nu_{1/2,\beta}(\mathrm{d}y_1, \mathrm{d}\overline{y}) = \frac{\beta}{2y_1} N(0, y_1 I_{d-1})(\mathrm{d}\overline{y}) \otimes \gamma(1/2, \beta)(\mathrm{d}y_1)$$

and the Lévy measure of the normal inverse Gaussian distribution $\overline{\mu}_{1/2,\beta}$ is given by

$$\overline{\nu}_{1/2,\beta}(\mathrm{d}\overline{y}) = \frac{\sqrt{\beta}}{4\Gamma(\beta)(2\pi)^{\frac{d-1}{2}}} \left(\frac{2}{\beta\|\overline{y}\|^2} \right)^{\frac{3-d}{4}} K_{\frac{3-d}{2}}\left(\sqrt{2\|\overline{y}\|^2/\beta}\right) \mathrm{d}\overline{y}.$$

5. Conclusion

In this paper, we have introduced a family of distributions called the multivariate normal α-stable family. It can be defined as the normal variance–mean mixtures where the mixing densities are the drifted α-stable distributions. It is important to notice that this family extends the normal inverse Gaussian one ($\alpha = 1/2$). So, as results, we have studied some important characteristic properties of this extended class of distributions. In fact, we have characterized the NEF generated by the normal α-stable distribution by its variance function which is explicitly written as a function of the mean. Furthermore, since the normal α-stable distribution is infinitely divisible and this property is strongly related to the Lévy measure which is important in mathematical finance and it is useful in the Ito–Lévy decomposition, we have given the Lévy measure associated to the normal α-stable distribution using the Lévy–Khintchine representation.

Since the normal inverse Gaussian distribution is used as a model in several works such as the GARECH-NIG model of Forsberg and Bollerslev [7] and the NIG-ACD model given by Wilhelmsson [24], so we will investigate these models (GARECH-NIG and NIG-ACD) to have the natural extension based on the normal α-stable distribution. Furthermore, we will estimate the parameters of this distribution using the maximum likelihood approach of the EM type.

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