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The Category of Maximal Cohen–Macaulay Modules as a Ring with Several Objects

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Abstract. Over a commutative local Cohen–Macaulay ring, we view and study the category of maximal Cohen–Macaulay modules as a ring with several objects. We compute the global dimension of this category and thereby extend some results of Iyama and Leuschke.

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1. Introduction

Let R be a commutative local Cohen–Macaulay ring with Krull dimension d. Suppose that R has *finite CM-type*; this means that, up to isomorphism, R admits only finitely many indecomposable maximal Cohen–Macaulay modules $X_1, \ldots, X_n$. In this case, the category MCM of maximal Cohen–Macaulay R-modules has a *representation generator*, i.e., a module $X \in \mathsf{MCM}$ that contains as direct summands all indecomposable maximal Cohen–Macaulay R-modules (for example, $X = X_1 \oplus \cdots \oplus X_n$ would be such a module). A result, proved independently by Iyama [12] and Leuschke [14], shows that the endomorphism ring $E = \mathrm{End}_R(X)$ has global dimension $\leqslant \max\{2, d\}$, and that equality holds if $d \geqslant 2$.

If R does not have finite CM-type, then MCM has no representation generator and there is a priori no endomorphism ring E to consider. However, regardless of CM-type, one can always view the entire category MCM as a "ring with several objects"[1] and then study its (finitely presented) left/right "modules", i.e., covariant/contravariant additive functors from MCM to abelian groups. The category $\mathsf{MCM}\text{-}\mathsf{mod}$ of finitely presented left modules over the "several object ring" MCM is the natural object to investigate in the general case. Indeed, if R has finite CM-type, then this category is equivalent to the

[1] In Sect. 3, we recapitulate a few points from the theory of rings with several objects. The classic references for this theory are Freyd [9,10] and Mitchell [15]. In [2, §2], Auslander uses the terminology "uncoordinized ring" for a ring with several objects.

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category E-mod of finitely generated left E-modules, where E is the endomorphism ring introduced above. It turns out that MCM-mod and mod-MCM, i.e., the categories of finitely presented left and right modules over MCM, are abelian with enough projectives. Thus, one can naturally speak of the global dimensions of these categories; they are called the left and right global dimensions of MCM, and they are denoted $\mathrm{l.gldim}\,(\mathsf{MCM})$ and $\mathrm{r.gldim}\,(\mathsf{MCM})$. We show that there is an equality $\mathrm{l.gldim}\,(\mathsf{MCM}) = \mathrm{r.gldim}\,(\mathsf{MCM})$; this number is simply called the *global dimension of* MCM, and it is denoted by $\mathrm{gldim}\,(\mathsf{MCM})$. Our first main result, Theorem 4.10, shows that there are inequalities,

$$d \leqslant \mathrm{gldim}\,(\mathsf{MCM}) \leqslant \max\{2, d\}, \qquad (*)$$

and thus it extends Iyama's and Leuschke's theorem to the case of arbitrary CM-type. We prove the left inequality in $(*)$ by showing that MCM always admits a finitely presented module with projective dimension d. Actually, we show that if M is any Cohen–Macaulay R-module of dimension t, then $\mathrm{Ext}_R^{d-t}(M,-)$ is a finitely presented left MCM-module and $\mathrm{Hom}_R(-,M)$ is a finitely presented right MCM-module both with projective dimension equal to $d-t$. Our second main result, Theorem 4.15, shows that if $d = 0, 1$, then the left inequality in $(*)$ is an equality if and only if R is regular, that is, there are equivalences:

$$\mathrm{gldim}\,(\mathsf{MCM}) = 0 \iff R \text{ is a field.}$$
$$\mathrm{gldim}\,(\mathsf{MCM}) = 1 \iff R \text{ is a discrete valuation ring.}$$

Note that, for an Artin algebra or a (possibly non-commutative) order R over a complete regular local ring, the results in this paper were established by Iyama [13, Thm. 3.6.2] (if, in addition, R is an isolated singularity, then the results go back to Auslander [2, Thm. A.1]). The present paper employs nothing, but elementary techniques from commutative algebra.

2. Preliminaries

2.1. Setup

Throughout, $(R, \mathfrak{m}, k)$ is a commutative noetherian local Cohen–Macaulay ring with Krull dimension d. It is assumed that R has a dualizing (or canonical) module Ω.

The category of finitely generated projective R-modules is denoted proj; the category of maximal Cohen–Macaulay R-modules (defined below) is denoted MCM; and the category of all finitely generated R-modules is denoted mod.

The *depth* of a finitely generated R-module $M \neq 0$, denoted $\mathrm{depth}_R\, M$, is the supremum of the lengths of all M-regular sequences $x_1, \ldots, x_n \in \mathfrak{m}$. This numerical invariant can be computed homologically as follows:

$$\mathrm{depth}_R\, M \;=\; \inf\{\, i \in \mathbb{Z} \mid \mathrm{Ext}_R^i(k, M) \neq 0 \,\}.$$

By definition, $\operatorname{depth}_R 0 = \inf \emptyset = +\infty$. For a finitely generated R-module $M \neq 0$, one always has $\operatorname{depth}_R M \leqslant d$, and M is called *maximal Cohen–Macaulay* if equality holds. The zero module is also considered to be maximal Cohen–Macaulay; thus an arbitrary finitely generated R-module M is maximal Cohen–Macaulay if and only if $\operatorname{depth}_R M \geqslant d$.

2.2 It is well-known that the dualizing module Ω gives rise to a duality on the category of maximal Cohen–Macaulay modules; more precisely, there is an equivalence of categories:

$$\mathsf{MCM} \underset{\operatorname{Hom}_R(-,\Omega)}{\overset{\operatorname{Hom}_R(-,\Omega)}{\rightleftarrows}} \mathsf{MCM}^{\mathrm{op}}.$$

We use the shorthand notation $(-)^\dagger$ for the functor $\operatorname{Hom}_R(-,\Omega)$. For any finitely generated R-module M, there is a canonical homomorphism $\delta_M \colon M \to M^{\dagger\dagger}$, which is natural in M, and because of the equivalence above, δ_M is an isomorphism if M belongs to MCM.

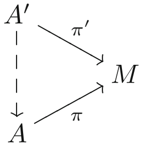
We will need the following results about depth; they are folklore and easily proved.[2]

Lemma 2.1. *Let $n \geqslant 0$ be an integer and let $0 \to X_n \to \cdots \to X_0 \to M \to 0$ be an exact sequence of finitely generated R-modules. If $X_0, \ldots, X_n$ are maximal Cohen–Macaulay, then one has* $\operatorname{depth}_R M \geqslant d - n$.

Lemma 2.2. *Let $m \geqslant 0$ be an integer and let $0 \to K_m \to X_{m-1} \to \cdots \to X_0 \to M \to 0$ be an exact sequence of finitely generated R-modules. If $X_0, \ldots, X_{m-1}$ are maximal Cohen–Macaulay, then one has* $\operatorname{depth}_R K_m \geqslant \min\{d, \operatorname{depth}_R M + m\}$. *In particular, if $m \geqslant d$ then the R-module K_m is maximal Cohen–Macaulay.*

We will also need a few notions from relative homological algebra.

Definition 2.3. Let $\mathcal{A}$ be a full subcategory of a category $\mathcal{M}$. Following Enochs and Jenda [8, def. 5.1.1], we say that $\mathcal{A}$ is *precovering* (or *contravariantly finite*) in $\mathcal{M}$ if every $M \in \mathcal{M}$ has an $\mathcal{A}$-*precover* (or a *right $\mathcal{A}$-approximation*); that is, a morphism $\pi \colon A \to M$ with $A \in \mathcal{A}$ such that every other morphism $\pi' \colon A' \to M$ with $A' \in \mathcal{A}$ factors through π, as illustrated by the following diagram:



The notion of $\mathcal{A}$-*preenvelopes* (or *left $\mathcal{A}$-approximations*) is categorically dual to the notion defined above. The subcategory $\mathcal{A}$ is said to be *preenveloping* (or *covariantly finite*) in $\mathcal{M}$ if every $M \in \mathcal{M}$ has an $\mathcal{A}$-preenvelope.

[2] One way to prove Lemmas 2.1 and 2.2 is by induction on n and m, using the behavior of depth on short exact sequences recorded in Bruns and Herzog [5, Prop. 1.2.9].

The following result is a consequence of Auslander and Buchweitz's maximal Cohen–Macaulay approximations.

Theorem 2.4. *Every finitely generated R-module has an* MCM*-precover.*

Proof. By [3, Thm. A], any finitely generated R-module M has a maximal Cohen–Macaulay approximation, that is, a short exact sequence,

$$0 \longrightarrow I \longrightarrow X \stackrel{\pi}{\longrightarrow} M \longrightarrow 0,$$

where X is maximal Cohen–Macaulay and I has finite injective dimension. A classic result of Ischebeck [11] (see also [5, Exerc. 3.1.24]) shows that $\mathrm{Ext}^1_R(X', I) = 0$ for every X' in MCM, and hence $\mathrm{Hom}_R(X', \pi) : \mathrm{Hom}_R(X', X) \to \mathrm{Hom}_R(X', M)$ is surjective. □

3. Rings with Several Objects

The classic references for the theory of rings with several objects are Freyd [9,10] and Mitchell [15]. Below, we recapitulate a few definitions and results that we need.

A ring A can be viewed as a preadditive category $\bar{A}$ with a single object $*$ whose endo hom-set $\mathrm{Hom}_{\bar{A}}(*, *)$ is A, and where composition is given by ring multiplication. The category $(\bar{A}, \mathsf{Ab})$ of additive covariant functors from $\bar{A}$ to the category Ab of abelian groups is naturally equivalent to the category A-Mod of left A-modules. Indeed, an additive functor $F\colon \bar{A} \to \mathsf{Ab}$ yields a left A-module whose underlying abelian group is $M = F(*)$ and where left A-multiplication is given by $am = F(a)(m)$ for $a \in A = \mathrm{Hom}_{\bar{A}}(*, *)$ and $m \in M = F(*)$. Note that, the preadditive category associated to the opposite ring A^{o} of A is the opposite (or dual) category of $\bar{A}$; in symbols: $\overline{A^{\mathrm{o}}} = \bar{A}^{\mathrm{op}}$. It follows that the category $(\bar{A}^{\mathrm{op}}, \mathsf{Ab})$ of additive covariant functors $\bar{A}^{\mathrm{op}} \to \mathsf{Ab}$ (which correspond to additive contravariant functors $\bar{A} \to \mathsf{Ab}$) is naturally equivalent to the category Mod-A of right A-modules.

These considerations justify the well-known viewpoint that any skeletally small preadditive category $\mathcal{A}$ may be thought of as a *ring with several objects*. A *left* $\mathcal{A}$*-module* is an additive covariant functor $\mathcal{A} \to \mathsf{Ab}$, and the category of all such is denoted by $\mathcal{A}$-Mod. Similarly, a *right* $\mathcal{A}$*-module* is an additive covariant functor $\mathcal{A}^{\mathrm{op}} \to \mathsf{Ab}$ (which corresponds to an additive contravariant functor $\mathcal{A} \to \mathsf{Ab}$), and the category of all such is denoted Mod-$\mathcal{A}$.

From this point, we assume for simplicity that $\mathcal{A}$ is a skeletally small *additive* category which is *closed under direct summands* (i.e., every idempotent splits). The category $\mathcal{A}$-Mod is a Grothendieck category, see [9, prop. 5.21], with enough projectives. In fact, it follows from Yoneda's lemma that the representable functors $\mathcal{A}(A, -)$, where A is in $\mathcal{A}$, constitute a generating set of projective objects in $\mathcal{A}$-Mod. A left $\mathcal{A}$-module F is called *finitely generated*, respectively, *finitely presented* (or *coherent*), if there exists an exact sequence $\mathcal{A}(A, -) \to F \to 0$, respectively, $\mathcal{A}(B, -) \to \mathcal{A}(A, -) \to F \to 0$, for some $A, B \in \mathcal{A}$.[3] The category of finitely presented left $\mathcal{A}$-modules is denoted by

[3] If the category $\mathcal{A}$ is only assumed to be preadditive, then one would have to modify the definitions of finitely generated/presented accordingly. For example, in this case, a

$\mathcal{A}$-mod. The Yoneda functor,

$$\mathcal{A}^{\mathrm{op}} \longrightarrow \mathcal{A}\text{-Mod} \quad \text{given by} \quad A \longmapsto \mathcal{A}(A,-),$$

is fully faithful, see [9, thm. 5.36]. Moreover, this functor identifies the objects in $\mathcal{A}$ with the finitely generated projective left $\mathcal{A}$-modules, that is, a finitely generated left $\mathcal{A}$-module is projective if and only if it is isomorphic to $\mathcal{A}(A,-)$ for some $A \in \mathcal{A}$; cf. [9, exerc. 5-G].

Here is a well-known, but important, example:

Example 3.1. Let A be any ring and let $\mathcal{A} = A\text{-proj}$ be the category of all finitely generated projective left A-modules. In this case, the category $\mathcal{A}\text{-mod} = (A\text{-proj})\text{-mod}$ is equivalent to the category mod-A of finitely presented right A-modules. Let us explain why:

Let F be a left (A-proj)-module, that is, an additive covariant functor $F\colon A\text{-proj} \to \mathsf{Ab}$. For $a \in A$, the homothety map $\chi_a\colon A \to A$ given by $b \mapsto ba$ is left A-linear and so it induces an endomorphism $F(\chi_a)$ of the abelian group $F(A)$. Thus, $F(A)$ has a natural structure of a right A-module given by $xa = F(\chi_a)(x)$ for $a \in A$ and $x \in F(A)$. This right A-module is denoted $\mathsf{e}(F)$, and we get a functor e, called *evaluation*, displayed in the diagram below. The other functor f in the diagram, called *functorfication*, is given by $\mathsf{f}(M) = M \otimes_A -$ (restricted to A-proj) for a right A-module M.

$$(A\text{-proj})\text{-Mod} \underset{\mathsf{f}}{\overset{\mathsf{e}}{\rightleftarrows}} \text{Mod-}A$$

The functors e and f yield an equivalence of categories: For every right A-module M, there is obviously an isomorphism $(\mathsf{e}\circ\mathsf{f})(M) = M \otimes_A A \cong M$. We must also show that every left (A-proj)-module F is isomorphic to $(\mathsf{f}\circ\mathsf{e})(F) = F(A)\otimes_A -$. For every $P \in A\text{-proj}$ and $y \in P$ the left A-linear map $\mu_P^y\colon A \to P$ given by $a \mapsto ay$ induces a group homomorphism $F(\mu_P^y)\colon F(A) \to F(P)$, and thus one has a group homomorphism $\tau_P\colon F(A) \otimes_A P \to F(P)$ given by $x \otimes y \mapsto F(\mu_P^y)(x)$. It is straightforward to verify that τ is a natural transformation. To prove that τ_P is an isomorphism for every $P \in A\text{-proj}$ it suffices, since the functors $F(A) \otimes_A -$ and F are both additive, to check that $\tau_A\colon F(A) \otimes_A A \to F(A)$ is an isomorphism. However, this is evident.

It is not hard to verify that the functors e and f restrict to an equivalence between finitely presented objects, as claimed.

Observation 3.2. Example 3.1 shows that for any ring A, the category (A-proj)-mod is equivalent to mod-A. Since there is an equivalence of categories,

$$A\text{-proj} \underset{\mathrm{Hom}_{A^{\circ}}(-,A)}{\overset{\mathrm{Hom}_A(-,A)}{\rightleftarrows}} (\text{proj-}A)^{\mathrm{op}},$$

Footnote 3 continued
left $\mathcal{A}$-module F is called finitely generated if there is an exact sequence of the form $\bigoplus_{i=1}^{n} \mathcal{A}(A_i,-) \to F \to 0$ for some $A_1,\ldots,A_n \in \mathcal{A}$.

it follows[4] that (A-proj)-mod is further equivalent to ((proj-A)$^{\mathrm{op}}$)-mod, which is the same as mod-(proj-A). In conclusion, there are equivalences of categories:

$$(A\text{-proj})\text{-mod} \;\simeq\; \text{mod-}A \;\simeq\; \text{mod-}(\text{proj-}A).$$

Of course, by applying this to the opposite ring A^{o}, one obtains equivalences:

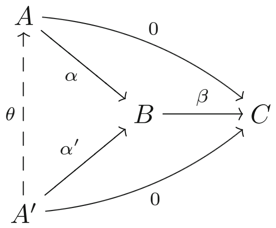
$$(\text{proj-}A)\text{-mod} \;\simeq\; A\text{-mod} \;\simeq\; \text{mod-}(A\text{-proj}).$$

In general, the category $\mathcal{A}$-mod of finitely presented left $\mathcal{A}$-modules is an additive category with cokernels, but it is not necessarily an abelian subcategory of $\mathcal{A}$-Mod. A classic result of Freyd describes the categories $\mathcal{A}$ for which $\mathcal{A}$-mod is abelian. This result is stated in Theorem 3.4 below, but first we explain some terminology.

A *pseudo-kernel* (also called a *weak kernel*) of a morphism $\beta\colon B \to C$ in $\mathcal{A}$ is a morphism $\alpha\colon A \to B$ such that the sequence

$$\mathcal{A}(-,A) \xrightarrow{\mathcal{A}(-,\alpha)} \mathcal{A}(-,B) \xrightarrow{\mathcal{A}(-,\beta)} \mathcal{A}(-,C)$$

is exact in Mod-$\mathcal{A}$. Equivalently, one has $\beta\alpha = 0$ and for every morphism $\alpha'\colon A' \to B$ with $\beta\alpha' = 0$ there is a (not necessarily unique!) morphism $\theta\colon A' \to A$ with $\alpha\theta = \alpha'$.



We say that the category $\mathcal{A}$ *has pseudo-kernels* is every morphism in $\mathcal{A}$ has a pseudo-kernel.

Pseudo-cokernels (also called *weak cokernels*) are defined dually.

Observation 3.3. Suppose that $\mathcal{A}$ is a full subcategory of an abelian category $\mathcal{M}$.

If $\mathcal{A}$ is precovering in $\mathcal{M}$, see Definition 2.3, then $\mathcal{A}$ has pseudo-kernels. Indeed, given a morphism $\beta\colon B \to C$ in $\mathcal{A}$ it has a kernel $\iota\colon M \to B$ in the abelian category $\mathcal{M}$; and it is easily verified that if $\pi\colon A \to M$ is any $\mathcal{A}$-precover of M, then $\alpha = \iota\pi\colon A \to B$ is a pseudo-kernel in $\mathcal{A}$ of β.

A similar argument shows that if $\mathcal{A}$ is preenveloping in $\mathcal{M}$, then $\mathcal{A}$ has pseudo-cokernels.

Theorem 3.4. *The category* mod-$\mathcal{A}$ (*respectively,* $\mathcal{A}$-mod) *of finitely presented right* (*respectively, left*) *$\mathcal{A}$-modules is an abelian subcategory of* Mod-$\mathcal{A}$ (*respectively,* $\mathcal{A}$-Mod) *if and only if $\mathcal{A}$ has pseudo-kernels* (*respectively, has pseudo-cokernels*).

Proof. See Freyd [10, thm. 1.4] or Auslander and Reiten [4, prop. 1.3]. □

[4] cf. the proof of Proposition 4.4.

Example 3.5. Let A be a left and right noetherian ring. As A is left noetherian, the category $\mathcal{M} = A\text{-}\mathsf{mod}$ of finitely presented left A-modules is abelian, and evidently $\mathcal{A} = A\text{-}\mathsf{proj}$ is precovering herein. As A is right noetherian, A-proj is also preenveloping in A-mod; cf. [8, Exa. 8.3.10]. It follows from Observation 3.3 that A-proj has both pseudo-kernels and pseudo-cokernels, and therefore the categories mod-(A-proj) and (A-proj)-mod are abelian by Theorem 3.4. Of course, this also follows directly from Observation 3.2 which shows that mod-(A-proj) and (A-proj)-mod are equivalent to A-mod and mod-A, respectively.

Note that if $\mathcal{A}$-mod is abelian, i.e., if $\mathcal{A}$ has pseudo-cokernels, then every finitely presented left $\mathcal{A}$-module F admits a projective resolution in $\mathcal{A}$-mod, that is, an exact sequence

$$\cdots \longrightarrow \mathcal{A}(A_1,-) \longrightarrow \mathcal{A}(A_0,-) \longrightarrow F \longrightarrow 0$$

where $A_0, A_1, \ldots$ belong to $\mathcal{A}$. Thus, one can naturally speak of the *projective dimension* of F (i.e., the length, possibly infinite, of the shortest projective resolution of F in $\mathcal{A}$-mod) and of the *global dimension* of the category $\mathcal{A}$-mod (i.e., the supremum of projective dimensions of all objects in $\mathcal{A}$-mod).

Definition 3.6. In the case where the category $\mathcal{A}$-mod (respectively, mod-$\mathcal{A}$) is abelian, then its global dimension is called the *left* (respectively, *right*) *global dimension of* $\mathcal{A}$, and it is denoted $\mathrm{l.\,gldim}\,\mathcal{A}$ (respectively, $\mathrm{r.\,gldim}\,\mathcal{A}$).

Note that, $\mathrm{l.\,gldim}\,(\mathcal{A}^{\mathrm{op}})$ is the same as $\mathrm{r.\,gldim}\,\mathcal{A}$ (when these numbers make sense).

Example 3.7. Let A be a left and right noetherian ring whose global dimension[5] we denote gldim A. Recall that gldim A can be computed as the supremum of projective dimensions of all *finitely generated* (left or right) A-modules. It follows from Observation 3.2 that

$$\mathrm{l.\,gldim}\,(A\text{-}\mathsf{proj}) = \mathrm{gldim}\ A = \mathrm{r.\,gldim}\,(A\text{-}\mathsf{proj}).$$

4. The Global Dimension of the Category MCM

We are now in a position to prove the results announced in the Introduction.

Example 4.1. Suppose that R has finite CM-type and let X be any representation generator of the category MCM, cf. Sect. 1. This means that $\mathsf{MCM} = \mathsf{add}_R X$ where $\mathsf{add}_R X$ denotes the category of direct summands of finite direct sums of copies of X. Write $E = \mathrm{End}_R(X)$ for the endomorphism ring of X; this R-algebra is often referred to as the *Auslander algebra*. Note that, X has a canonical structure as a left-R–left-E–bimodule ${}_{R,E}X$. It is easily verified that there is an equivalence, known as Auslander's *projectivization*, given by:

$$\mathsf{MCM} = \mathsf{add}_R X \underset{-\otimes_E X}{\overset{\mathrm{Hom}_R(X,-)}{\rightleftarrows}} \mathsf{proj}\text{-}E.$$

[5] Recall that for a ring which is both left and right noetherian, the left and right global dimensions are equal; indeed, they both coincide with the weak global dimension.

It now follows from Observation 3.2 that there are equivalences of categories:

$$\mathsf{MCM}\text{-}\mathsf{mod} \simeq (\mathsf{proj}\text{-}E)\text{-}\mathsf{mod} \simeq E\text{-}\mathsf{mod}.$$

Similarly, there is an equivalence of categories: $\mathsf{mod}\text{-}\mathsf{MCM} \simeq \mathsf{mod}\text{-}E$.

Proposition 4.2. *The category* MCM *has pseudo-kernels and pseudo-cokernels.*

Proof. As MCM is precovering in the abelian category mod, see Theorem 2.4, we get from Observation 3.3 that MCM has pseudo-kernels. To prove that MCM has pseudo-cokernels, let $\alpha\colon X \to Y$ be any homomorphism between maximal Cohen–Macaulay R-modules. With the notation from 2.2 we let $\iota\colon Z \to Y^\dagger$ be a pseudo-kernel in MCM of $\alpha^\dagger\colon Y^\dagger \to X^\dagger$. We claim that $\iota^\dagger\delta_Y\colon Y \to Z^\dagger$ is a pseudo-cokernel of α, i.e., that the sequence

$$\mathrm{Hom}_R(Z^\dagger,U)\xrightarrow{\mathrm{Hom}_R(\iota^\dagger\delta_Y,U)}\mathrm{Hom}_R(Y,U)\xrightarrow{\mathrm{Hom}_R(\alpha,U)}\mathrm{Hom}_R(X,U) \qquad (1)$$

is exact for every $U \in \mathsf{MCM}$. From the commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{\alpha} & Y & \xrightarrow{\iota^\dagger\delta_Y} & Z^\dagger \\ \cong\big\downarrow\delta_X & & \cong\big\downarrow\delta_Y & & \big\| \\ X^{\dagger\dagger} & \xrightarrow{\alpha^{\dagger\dagger}} & Y^{\dagger\dagger} & \xrightarrow{\iota^\dagger} & Z^\dagger \end{array}$$

it follows that the sequence (1) is isomorphic to

$$\mathrm{Hom}_R(Z^\dagger,U)\xrightarrow{\mathrm{Hom}_R(\iota^\dagger,U)}\mathrm{Hom}_R(Y^{\dagger\dagger},U)\xrightarrow{\mathrm{Hom}_R(\alpha^{\dagger\dagger},U)}\mathrm{Hom}_R(X^{\dagger\dagger},U)\,. \qquad (2)$$

Recall from 2.2 that there is an isomorphism $U \cong U^{\dagger\dagger}$. From this fact and from the "swap" isomorphism [7, (A.2.9)], it follows that the sequence (2) is isomorphic to

$$\mathrm{Hom}_R(U^\dagger,Z^{\dagger\dagger})\xrightarrow{\mathrm{Hom}_R(U^\dagger,\iota^{\dagger\dagger})}\mathrm{Hom}_R(U^\dagger,Y^{\dagger\dagger\dagger})\xrightarrow{\mathrm{Hom}_R(U^\dagger,\alpha^{\dagger\dagger\dagger})}\mathrm{Hom}_R(U^\dagger,X^{\dagger\dagger\dagger})\,. \qquad (3)$$

Finally, the commutative diagram

$$\begin{array}{ccccc} Z & \xrightarrow{\iota} & Y^\dagger & \xrightarrow{\alpha^\dagger} & X^\dagger \\ \cong\big\downarrow\delta_Z & & \cong\big\downarrow\delta_{Y^\dagger} & & \cong\big\downarrow\delta_{X^\dagger} \\ Z^{\dagger\dagger} & \xrightarrow{\iota^{\dagger\dagger}} & Y^{\dagger\dagger\dagger} & \xrightarrow{\alpha^{\dagger\dagger\dagger}} & X^{\dagger\dagger\dagger} \end{array}$$

shows that the sequence (3) is isomorphic to

$$\mathrm{Hom}_R(U^\dagger,Z)\xrightarrow{\mathrm{Hom}_R(U^\dagger,\iota)}\mathrm{Hom}_R(U^\dagger,Y^\dagger)\xrightarrow{\mathrm{Hom}_R(U^\dagger,\alpha^\dagger)}\mathrm{Hom}_R(U^\dagger,X^\dagger)\,,$$

which is exact since $\iota\colon Z \to Y^\dagger$ is a pseudo-kernel of $\alpha^\dagger\colon Y^\dagger \to X^\dagger$. □

We shall find the following notation useful.

Definition 4.3. For an R-module M, we use the notation $(M,-)$ for the left MCM-module $\mathrm{Hom}_R(M,-)|_{\mathsf{MCM}}$, and $(-,M)$ for the right MCM-module $\mathrm{Hom}_R(-,M)|_{\mathsf{MCM}}$.

Theorem 3.4 and Proposition 4.2 show that MCM-mod and mod-MCM are abelian, and hence the left and right global dimensions of the category MCM are both well defined; see Definition 3.6. In fact, they are equal:

Proposition 4.4. *The left and right global dimensions of* MCM *coincide, that is,*

$$\mathrm{l.\,gldim}\,(\mathsf{MCM}) = \mathrm{r.\,gldim}\,(\mathsf{MCM}).$$

This number is called the global dimension of MCM*, and it is denoted* $\mathrm{gldim}\,(\mathsf{MCM})$. □

Proof. The equivalence in 2.2 induces an equivalence between the abelian categories of (all) left and right MCM-modules given by:

$$\mathsf{MCM}\text{-}\mathsf{Mod} \underset{G\circ(-)^{\dagger} \leftarrow\!\shortmid G}{\overset{F \mapsto F\circ(-)^{\dagger}}{\rightleftarrows}} \mathsf{Mod}\text{-}\mathsf{MCM}. \tag{4}$$

These functors preserve finitely generated projective modules. Indeed, if $P = (X, -)$ with $X \in \mathsf{MCM}$ is a finitely generated projective left MCM-module, then the right MCM-module $P \circ (-)^{\dagger} = (X, (-)^{\dagger})$ is isomorphic to $(-, X^{\dagger})$, which is finitely generated projective. Similarly, if $Q = (-, Y)$ with $Y \in \mathsf{MCM}$ is a finitely generated projective right MCM-module, then $Q \circ (-)^{\dagger} = ((-)^{\dagger}, Y)$ is isomorphic to $(Y^{\dagger}, -)$, which is finitely generated projective.

Since the functors in (4) are exact and preserve finitely generated projective modules, they restrict to an equivalence between finitely presented objects, that is, MCM-mod and mod-MCM are equivalent. It follows that MCM-mod and mod-MCM have the same global dimension, i.e., the left and right global dimensions of MCM coincide. □

We begin our study of $\mathrm{gldim}\,(\mathsf{MCM})$ with a couple of easy examples.

Example 4.5. If R is regular, in which case the global dimension of R is equal to d, then one has $\mathsf{MCM} = \mathsf{proj}$, and it follows from Example 3.7 that $\mathrm{gldim}\,(\mathsf{MCM}) = d$.

Example 4.6. Assume that R has finite CM-type and denote the Auslander algebra by E. It follows from Example 4.1 that $\mathrm{gldim}\,(\mathsf{MCM}) = \mathrm{gldim}\, E$.

We turn our attention to projective dimensions of representable right MCM-modules.

Proposition 4.7. *Let M be a finitely generated R-module. Then, $(-, M)$ is a finitely presented right* MCM*-module with projective dimension equal to* $d - \mathrm{depth}_R\, M$.

Proof. First, we argue that $(-, M)$ is finitely presented. By Theorem 2.4, there is an MCM-precover $\pi \colon X \to M$, which by definition yields an epimorphism $(-, \pi) \colon (-, X) \twoheadrightarrow (-, M)$. Hence, $(-, M)$ is finitely generated. As the Hom functor is left exact, the kernel of $(-, \pi)$ is the functor $(-, \mathrm{Ker}\, \pi)$. Since $\mathrm{Ker}\, \pi$ is a finitely generated R-module, the argument above shows that $(-, \mathrm{Ker}\, \pi)$ is finitely generated, and therefore $(-, M)$ is finitely presented.

If $M = 0$, then $(-, M)$ is the zero functor which has projective dimension $d - \operatorname{depth}_R M = -\infty$. Thus, we can assume that M is non-zero such that $m := d - \operatorname{depth}_R M$ is an integer. By successively taking MCM-precovers, whose existence is guaranteed by Theorem 2.4, we construct an exact sequence of R-modules, $0 \to K_m \to X_{m-1} \to \cdots \to X_1 \to X_0 \to M \to 0$, where $X_0, \ldots, X_{m-1}$ are maximal Cohen–Macaulay and $K_m = \operatorname{Ker}(X_{m-1} \to X_{m-2})$, such that the sequence

$$0 \longrightarrow (-, K_m) \longrightarrow (-, X_{m-1}) \longrightarrow \cdots \longrightarrow (-, X_1) \longrightarrow (-, X_0) \longrightarrow (-, M) \longrightarrow 0$$

in mod-MCM is exact. Lemma 2.2 shows that $\operatorname{depth}_R K_m \geqslant \min\{d, \operatorname{depth}_R M + m\} = d$, and hence K_m is maximal Cohen–Macaulay. Thus, exactness of the sequence displayed above shows that the projective dimension of $(-, M)$ is $\leqslant m$.

To prove that the projective dimension of $(-, M)$ is $\geqslant m$, we must show that if

$$0 \longrightarrow (-, Y_n) \xrightarrow{\tau_n} \cdots \longrightarrow (-, Y_1) \xrightarrow{\tau_1} (-, Y_0) \xrightarrow{\tau_0} (-, M) \longrightarrow 0$$

is any exact sequence in mod-MCM, where $Y_0, \ldots, Y_n$ are maximal Cohen–Macaulay, then $n \geqslant m$. By Yoneda's lemma, each τ_i has the form $\tau_i = (-, \beta_i)$ for some homomorphism $\beta_i \colon Y_i \to Y_{i-1}$ when $1 \leqslant i \leqslant n$ and $\beta_0 \colon Y_0 \to M$. By evaluating the sequence on the maximal Cohen–Macaulay module R, it follows that the sequence of R-modules,

$$0 \longrightarrow Y_n \xrightarrow{\beta_n} \cdots \longrightarrow Y_1 \xrightarrow{\beta_1} Y_0 \xrightarrow{\beta_0} M \longrightarrow 0\ ,$$

is exact. Thus, Lemma 2.1 yields $\operatorname{depth}_R M \geqslant d - n$, that is, $n \geqslant m$. □

In contrast to what is the case for representable right MCM-modules, representable left MCM-modules are "often" zero. For example, if $d > 0$ then $\operatorname{Hom}_R(k, X) = 0$ for every maximal Cohen–Macaulay R-module X, and hence $(k, -)$ is the zero functor. In particular, the projective dimension of a representable left MCM-module is typically not very interesting. Proposition 4.9 below gives concrete examples of finitely presented left MCM-modules that do have interesting projective dimension.

Lemma 4.8. *For every Cohen–Macaulay R-module M of dimension t there is the following natural isomorphism of functors* MCM $\to$ Ab*,*

$$\operatorname{Hom}_R((-)^\dagger, \operatorname{Ext}_R^{d-t}(M, \Omega)) \cong \operatorname{Ext}_R^{d-t}(M, -).$$

Proof. Since M is Cohen–Macaulay of dimension t one has $\operatorname{Ext}_R^i(M, \Omega) = 0$ for $i \neq d - t$; see [5, Thm. 3.3.10]. Thus, there is an isomorphism in the derived category of R,

$$\operatorname{Ext}_R^{d-t}(M, \Omega) \cong \Sigma^{d-t} \mathbf{R}\operatorname{Hom}_R(M, \Omega).$$

In particular, there is an isomorphism $X^\dagger = \operatorname{Hom}_R(X, \Omega) \cong \mathbf{R}\operatorname{Hom}_R(X, \Omega)$ for $X \in$ MCM. This explains the first isomorphism below. The second isomorphism is trivial, the third one is by "swap" [7, (A.4.22)], and the fourth one follows as Ω is a dualizing R-module.

$$\begin{aligned}\mathbf{R}\mathrm{Hom}_R(X^\dagger, \mathrm{Ext}_R^{d-t}(M,\Omega)) &\cong \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(X,\Omega), \Sigma^{d-t}\mathbf{R}\mathrm{Hom}_R(M,\Omega))\\ &\cong \Sigma^{d-t}\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(X,\Omega), \mathbf{R}\mathrm{Hom}_R(M,\Omega))\\ &\cong \Sigma^{d-t}\mathbf{R}\mathrm{Hom}_R(M, \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(X,\Omega),\Omega))\\ &\cong \Sigma^{d-t}\mathbf{R}\mathrm{Hom}_R(M,X).\end{aligned}$$

The assertion now follows by taking the zeroth homology group H_0. □

Proposition 4.9. *If M is any Cohen–Macaulay R-module of dimension t, then the functor $\mathrm{Ext}_R^{d-t}(M,-)|_{\mathsf{MCM}}$ is a finitely presented left MCM-module with projective dimension equal to $d-t$.*

Proof. As M is Cohen–Macaulay of dimension t, so is $\mathrm{Ext}_R^{d-t}(M,\Omega)$; see [5, Thm. 3.3.10]. Proposition 4.7 shows that $\mathrm{Hom}_R(-,\mathrm{Ext}_R^{d-t}(M,\Omega))|_{\mathsf{MCM}}$ is a finitely presented right MCM-module with projective dimension equal to $d-t$. The proof of Proposition 4.4 now gives that

$$\mathrm{Hom}_R((-)^\dagger, \mathrm{Ext}_R^{d-t}(M,\Omega))|_{\mathsf{MCM}}$$

is a finitely presented left MCM-module with projective dimension $d-t$, and Lemma 4.8 finishes the proof. □

Theorem 4.10. *The category MCM has finite global dimension. In fact, one has*

$$d \leqslant \mathrm{gldim}\,(\mathsf{MCM}) \leqslant \max\{2,d\}.$$

In particular, if $d \geqslant 2$ then there is an equality $\mathrm{gldim}\,(\mathsf{MCM}) = d$.

Proof. The residue field k of R is a finitely generated R-module with depth 0. Thus, Proposition 4.7 shows that $(-,k)$ is finitely presented right MCM-module with projective dimension d. Consequently, we must have $d \leqslant \mathrm{gldim}\,(\mathsf{MCM})$.

To prove the other inequality, set $m = \max\{2,d\}$ and let G be any finitely presented right MCM-module. Take any exact sequence in $\mathsf{mod}\text{-}\mathsf{MCM}$,

$$(-,X_{m-1}) \xrightarrow{\tau_{m-1}} \cdots \longrightarrow (-,X_1) \xrightarrow{\tau_1} (-,X_0) \xrightarrow{\varepsilon} G \longrightarrow 0\,, \qquad (5)$$

where $X_0, X_1, \ldots, X_{m-1}$ are in MCM. Note that, since $m \geqslant 2$ there is at least one "τ" in this sequence. By Yoneda's lemma, every τ_i has the form $\tau_i = (-,\alpha_i)$ for some homomorphism $\alpha_i \colon X_i \to X_{i-1}$. By evaluating (5) on R, we get an exact sequence of R-modules:

$$X_{m-1} \xrightarrow{\alpha_{m-1}} \cdots \longrightarrow X_1 \xrightarrow{\alpha_1} X_0\,.$$

Since $m \geqslant d$ it follows from Lemma 2.2 that the module $X_m = \mathrm{Ker}\,\alpha_{m-1}$ is maximal Cohen–Macaulay. As the Hom functor is left exact, we see that $0 \to (-,X_m) \to (-,X_{m-1})$ is exact. This sequence, together with (5), shows that G has projective dimension $\leqslant m$. □

In view of Example 4.6 and Theorem 4.10, we immediately get the following result due to Iyama [12, Thm. 1.4.1] and Leuschke [14, Thm. 6].

Corollary 4.11. *Assume that R has finite CM-type and let X be any representation generator of* MCM *with Auslander algebra $E = \operatorname{End}_R(X)$. There are inequalities,*

$$d \leqslant \operatorname{gldim} E \leqslant \max\{2, d\}.$$

In particular, if $d \geqslant 2$ then there is an equality $\operatorname{gldim} E = d$. □

Example 4.12. If $d = 0$ then $\mathsf{MCM} = \mathsf{mod}$ and hence $\operatorname{gldim}(\mathsf{MCM}) = \operatorname{gldim}(\mathsf{mod})$. Since mod is abelian, it is a well-known result of Auslander [1] that the latter number must be either 0 or 2 (surprisingly, it can not be 1). Thus, one of the inequalities in Theorem 4.10 is actually an equality. If, for example, $R = k[x]/(x^2)$ where k is a field, then $\operatorname{gldim}(\mathsf{mod}) = 2$.

Example 4.13. If $d = 1$ then Theorem 4.10 shows that $\operatorname{gldim}(\mathsf{MCM}) = 1, 2$. The 1-dimensional Cohen–Macaulay ring $R = k[\![x, y]\!]/(x^2)$ does not have finite CM-type,[6] and since it is not regular, it follows from Theorem 4.15 below that $\operatorname{gldim}(\mathsf{MCM}) = 2$.

Recall that in any abelian category with enough projectives (such as $\mathsf{mod}\text{-}\mathcal{A}$ in the case where $\mathcal{A}$ has pseudo-kernels) one can well-define and compute Ext in the usual way.

Lemma 4.14. *Assume that $\mathcal{A}$ is precovering in an abelian category $\mathcal{M}$ (in which case, the category* $\mathsf{mod}\text{-}\mathcal{A}$ *is abelian by Observation 3.3 and Theorem 3.4). Let*

$$0 \longrightarrow A' \xrightarrow{\alpha'} A \xrightarrow{\alpha} A''$$

be an exact sequence in $\mathcal{M}$ where A, A', A'' belong to $\mathcal{A}$. Consider the finitely presented right $\mathcal{A}$-module $G = \operatorname{Coker} \mathcal{A}(-, \alpha)$, that is, G is defined by exactness of the sequence

$$\mathcal{A}(-, A) \xrightarrow{\mathcal{A}(-,\alpha)} \mathcal{A}(-, A'') \longrightarrow G \longrightarrow 0\,.$$

For any finitely presented right $\mathcal{A}$-module H, there is an isomorphism of abelian groups,

$$\operatorname{Ext}^2_{\mathsf{mod}\text{-}\mathcal{A}}(G, H) \;\cong\; \operatorname{Coker} H(\alpha').$$

Proof. By the definition of G and left exactness of the Hom functor, the chain complex

$$0 \longrightarrow \mathcal{A}(-, A') \xrightarrow{\mathcal{A}(-,\alpha')} \mathcal{A}(-, A) \xrightarrow{\mathcal{A}(-,\alpha)} \mathcal{A}(-, A'') \longrightarrow 0\,, \tag{6}$$

is a non-augmented projective resolution in $\mathsf{mod}\text{-}\mathcal{A}$ of G. To compute $\operatorname{Ext}^2_{\mathsf{mod}\text{-}\mathcal{A}}(G, H)$, we must first apply the functor $(\mathsf{mod}\text{-}\mathcal{A})(?, H)$ to (6) and then take the second cohomology group of the resulting cochain complex. By Yoneda's lemma, there is a natural isomorphism

$$(\mathsf{mod}\text{-}\mathcal{A})(\mathcal{A}(-, B), H) \;\cong\; H(B)$$

[6] See Buchweitz et al. [6, Prop. 4.1] for a complete list of the indecomposable maximal Cohen–Macaulay modules over this ring.

for any $B \in \mathcal{A}$; hence application of $(\mathsf{mod}\text{-}\mathcal{A})(?,H)$ to (6) yields the cochain complex

$$0 \longrightarrow H(A'') \xrightarrow{H(\alpha)} H(A) \xrightarrow{H(\alpha')} H(A') \longrightarrow 0\,.$$

The second cohomology group of this cochain complex is $\operatorname{Coker} H(\alpha')$. □

Recall that a commutative ring is called a *discrete valuation ring (DVR)* if it is a principal ideal domain with exactly one non-zero maximal ideal. There are of course many other equivalent characterizations of such rings.

Theorem 4.15. *If* $\operatorname{gldim}(\mathsf{MCM}) \leqslant 1$*, then* R *is regular. In particular, one has*

$$\operatorname{gldim}(\mathsf{MCM}) = 0 \iff R \text{ is a field.}$$
$$\operatorname{gldim}(\mathsf{MCM}) = 1 \iff R \text{ is a discrete valuation ring.}$$

Proof. Assume that $\operatorname{gldim}(\mathsf{MCM}) \leqslant 1$. Let X be any maximal Cohen–Macaulay R-module and let $\pi\colon L \twoheadrightarrow X$ be an epimorphism where L is finitely generated and free. Note that $Y = \operatorname{Ker}\pi$ is also maximal Cohen–Macaulay by Lemma 2.2, so we have an exact sequence,

$$0 \longrightarrow Y \xrightarrow{\iota} L \xrightarrow{\pi} X \longrightarrow 0,$$

of maximal Cohen–Macaulay R-modules. With $G = \operatorname{Coker}(-,\pi)$ and $H = (-,Y)$, we have

$$\operatorname{Coker}(\iota,Y) \;\cong\; \operatorname{Ext}^2_{\mathsf{mod}\text{-}\mathsf{MCM}}(G,H) \;\cong\; 0;$$

here, the first isomorphism comes from Lemma 4.14, and the second isomorphism follows from the assumption that $\operatorname{gldim}(\mathsf{MCM}) \leqslant 1$. Hence, the homomorphism

$$\operatorname{Hom}_R(L,Y) \xrightarrow{\operatorname{Hom}_R(\iota,Y)} \operatorname{Hom}_R(Y,Y)$$

is surjective. Thus, ι has a left inverse and X becomes a direct summand of the free module L. Therefore, every maximal Cohen–Macaulay R-module is projective, so R is regular.

The displayed equivalences now follows in view of Example 4.5 and the fact that a regular local ring has Krull dimension 0, respectively, 1, if and only if it is a field, respectively, a discrete valuation ring. □

As as corollary, we get the following addendum to Corollary 4.11.

Corollary 4.16. *Assume that* R *has finite CM-type and let* X *be any representation generator of* MCM *with Auslander algebra* $E = \operatorname{End}_R(X)$*. If* $\operatorname{gldim} E \leqslant 1$*, then* R *is regular.* □

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