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Generalized Limits and Statistical Convergence

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Abstract. Consider the Banach space m of real bounded sequences, x, with $\|x\| = \sup_k |x_k|$. A positive linear functional L on m is called an S-limit if $L(\chi_K) = 0$ for every characteristic sequence χ_K of sets, K, of natural density zero. We provide regular sublinear functionals that both generate as well as dominate S-limits. The paper also shows that the set of S-limits and the collection of Banach limits are distinct but their intersection is not empty. Furthermore, we show that the generalized limits generated by translative regular methods is equal to the set of Banach limits. Some applications are also provided.

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1. Introduction

Let m and c be the spaces of all bounded and convergent real sequences $x = (x_k)$ normed by $\|x\| = \sup_n |x_n|$, respectively. Let $\mathcal{B}$ be the class of (necessarily continuous) linear functionals β on m which are nonnegative and regular, that is, if $x \geq 0$, (i.e., $x_k \geq 0$ for all $k \in \mathbb{N} := \{1, 2, \ldots\}$) then $\beta(x) \geq 0$, and $\beta(x) = \lim_k x_k$, for each $x \in c$. If β has the additional property that $\beta(\sigma(x)) = \beta(x)$ for all $x \in m$, where σ is the left shift operator, defined by $\sigma(x_1, x_2, \ldots) = (x_2, x_3, \ldots)$ then β is called a Banach limit. The existence of Banach limits has been shown by Banach [2,17,19], and another proof may be found in [3]. It is well known [21] that the space of all almost convergent sequences can be represented as the set of all $x \in m$ which have the same value under any Banach limit. In the paper, we study some generalized limits so that the space of all bounded statistically convergent sequences can be represented as the set of all bounded sequences which have the same value

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under any such limit. It is proved that the set of such limits and the set of Banach limits are distinct but their intersection is not empty. The sublinear functionals that generate or dominate these limits are also examined.

We pause to collect some notation. Let $A = [a_{nk}]$ be an infinite summability matrix. Given a sequence x the A-transform of x, denoted as $Ax = ((Ax)_n)$, is given by $(Ax)_n = \sum_k a_{nk}\, x_k$ provided that the series converges for each n. Let $\lim_A x := \lim_n (Ax)_n$ whenever the limit exists. By c_A we denote the summability domain of A, i. e., $c_A = \{x : \lim_A x \text{ exists}\}$. We say that A is regular [4,25] if $\lim_n (Ax)_n = \lim_k x_k$ for each $x \in c$. For any nonnegative such matrix A we define the A-density of a set $K \subseteq \mathbb{N}$, denoted as $\delta_A(K)$ as

$$\delta_A(K) = \lim_n \sum_k a_{nk}\, \chi_K(k) = \lim_n (A\chi_K)_n,$$

provided that the limit exists, where χ_K denotes the characteristic sequence of the set K. When A is the Cesàro matrix, C_1, the resulting C_1-density is called the natural density, which we will denote by $\delta(K)$. Throughout the paper the statement $\delta(K) \neq 0$ will mean either $\delta(K) > 0$ or that the natural density of K does not exist.

Using a density, we say that a sequence $x = (x_k)$ is A-statistically convergent to a number ℓ if, for every $\epsilon > 0$,

$$\delta_A(\{k \in \mathbb{N} : \ |x_k - \ell| \geq \epsilon\}) = 0.$$

We denote this limit by $st_A - \lim x = \ell$. In particular, when $A = C_1$, the resulting notation is simply $st - \lim x = \ell$ [7,12,14,22,27]. It is well known [6] that the space of all bounded A-statistically convergent sequences is the same as A-strongly convergent sequences, namely

$$\lim_n \sum_k a_{nk} |x_k - \ell| = 0.$$

Motivated by that of Freedman [13], we introduce the following:

Definition 1.1. Let L be a linear functional on m that satisfies the following properties:

1. $L(x) \geq 0$, if $x \geq 0$, (positivity of L),
2. $L(x) = \lim_k x_k$ for $x \in c$, (regularity of L),
3. For every $E \subseteq \mathbb{N}$ such that $\delta_A(E) = 0$ implies that $L(\chi_E) = 0$.

Every such L will be called an S_A-limit, and denote their collection by SL_A. In the particular case when $A = C_1$ is the Cesàro matrix, any such L will be called an S-limit and their collection denoted by SL . Freedman [13] proved that the space of all bounded statistically convergent sequences can be represented as the set of all $x \in m$ which have the same value under any S-limit.

Banach [2] showed that there exist positive linear regular functionals, L, such that $L(\sigma(x)) = L(x)$ for all $x \in m$, where σ is the left shift operator. Such functionals will be called Banach limits, and their collection will be denoted by BL. Along this line we introduce:

Definition 1.2. (*A-Banach limits*) Let L be a bounded linear functional on m that satisfies the following conditions:

1. $L(x) \geq 0$ if $x_k \geq 0$ for all k,
2. $L(x) = \lim_k x_k$ if $x \in c$,
3. $L(x) \leq \limsup_n \sup_j \sum_k a_{nk} x_{k+j}$ for every $x \in m$.

Any such L will be called an A-Banach limit, and the collection of all such functionals will be denoted by BL_A.

When $A = C_1$, one gets $BL_{C_1} = BL$. Lorentz [21] proved that all $L \in BL$ agree on precisely the space of almost convergent sequences.

Recall that almost convergence and statistical convergence methods are incompatible [23].

In the following sections, we will provide results concerning the properties of these generalized limits.

2. Existence of Generalized Limits

A matrix $A = [a_{nk}]$ is called translative [25] if for any $x \in m$ with $\lim_A x = \ell$ we also get $\lim_A \sigma(x) = \ell$. A necessary and sufficient condition for a regular matrix A to be (boundedly) translative [25] is that

$$\lim_{n \to \infty} \sum_k |a_{n,k+1} - a_{nk}| \ = \ 0.$$

It is known that the bounded convergence field of any regular summability method cannot be equal to the set of almost convergent sequences. A regular matrix, A, is (boundedly) translative if and only if A sums all almost convergent sequences and equals their Banach limits [25]. Such methods are called strongly regular.

Theorem 2.1. *When A is a nonnegative regular matrix, both A-Banach limits and S_A-limits exist. Furthermore, the following results hold:*

1. $BL_A \cap SL_A \neq \emptyset$.
2. $BL \cap SL_A \neq \emptyset$ when A is boundedly translative. In particular, when an almost convergent sequence is also A-statistically convergent then the two limits must be the same.
3. $BL = BL_A$ if and only if A is strongly regular.

Proof. Consider the sublinear functional

$$Q_A(x) = \limsup_n \sum_k a_{nk}\, x_k, \quad x \in m.$$

By the regularity of A, we see that $Q_A(x) = \lim_k x_k$ for each $x \in c$. By the Hahn–Banach theorem, there exist bounded linear functionals T over m so that

$$-Q_A(-x) \leq T(x) \leq Q_A(x), \quad x \in m. \tag{2.1}$$

Denote the set of all such T by $\mathcal{L}_A$. We will prove a bit more than the theorem's statement, by showing that $\mathcal{L}_A \subseteq SL_A$.

It is clear that $T(x) \geq 0$ for every $x \geq 0$ and $T(x) = \ell(x) = \lim_k x_k$ for every $x \in c$. Next if E is a set with A-density zero, then by (2.1) we see that $0 \leq T(\chi_E) \leq Q_A(\chi_E) = 0$. Hence, T is an S_A-limit. It is also a member of BL_A since $Q_A(x) \leq \limsup_n \sup_j \sum_k a_{nk} x_{k+j}$, for all $x \in m$. In fact, $\mathcal{L}_A \subseteq BL_A \cap SL_A$.

Now we show that this T is also left shift invariant when A is strongly regular. Indeed,

$$\begin{aligned} |T(\sigma x - x)| &\leq |Q_A(\sigma x - x)| \\ &\leq \limsup_n \left| \sum_k a_{nk}(x_{k+1} - x_k) \right| \\ &\leq \|x\| \limsup_n \sum_k |a_{n,k} - a_{n,k+1}| \\ &= 0, \text{ when } A \text{ is strongly regular.} \end{aligned}$$

This gives $\mathcal{L}_A \subseteq BL$ as well as $\mathcal{L}_A \subseteq BL \cap SL_A$. The consistency of these generalized limits over the common convergence fields, therefore, follows.

(3) If $L \in BL_A$, and A is strongly regular, then we get

$$|L(\sigma x - x)| \leq \|x\| \lim_n \sum_k |a_{n,k+1} - a_{n,k}| \ = \ 0,$$

giving $BL_A \subseteq BL$. Since the matrix A is regular it follows from [26], Theorem 19 (c), that $BL \subseteq BL_A$.

Conversely, if $L \in BL_A = BL$, then for any almost convergent sequence x, with limit ℓ, we must have $L(x) = \ell$. This being so for every $L \in BL_A$, this implies that

$$0 = \liminf_n \inf_j \sum_k a_{nk}(x_{k+j} - \ell) = \limsup_n \sup_j \sum_k a_{nk}(x_{k+j} - \ell) = 0.$$

This implies that $\lim_n \sum_k a_{nk}(x_k - \ell) = 0$. Therefore, A is strongly regular.

Note that in this case f_A-convergence is equivalent to f-convergence (cf. [21], Theorems 2 and 3). □

As we shall show later, when A is not strongly regular it is possible to have $BL \cap SL_A = \emptyset$. As examples of translative methods, the regular Euler and Borel matrix methods are well known. Necessary and sufficient conditions are also known for regular Hausdorff methods to be translative (cf. [25]).

To explore further relationships between various generalized limits, we recall the concepts of statistical limit superior and statistical limit inferior from [8,9,16]

$$st_A - \limsup x = \begin{cases} \sup B_x, & \text{if } B_x \neq \emptyset \\ -\infty, & \text{if } B_x = \emptyset, \end{cases}$$

where $B_x = \{b \in \Re : \delta_A(\{k \in \mathbb{N} : \ x_k > b\}) \neq 0\}$. We should point out that this concept is closely related to the concept of essential supremum of a collection of random variables. Also the A-statistical limit inferior of x is given by

$$st_A - \liminf x = \begin{cases} \inf A_x, & \text{if } A_x \neq \emptyset \\ +\infty, & \text{if } A_x = \emptyset, \end{cases}$$

where $A_x = \{a \in \Re : \delta_A(\{k \in \mathbb{N} : \ x_k < a\}) \neq 0\}$.

Proposition 2.2. *When A is a nonnegative regular matrix, and $P_A(x) := st_A - \limsup x$, the following results hold.*

(a) $-P_A(-x) = st_A - \liminf x$, *for all $x \in m$.*
(b) $P_A(x + y) \leq P_A(x) + P_A(y)$, *for any $x, y \in m$.*
(c) $P_A(\alpha x) = \alpha P_A(x)$, *for any $\alpha \geq 0$, and $x \in m$.*

Proof. Part (a) follows by the fact that A is a nonnegative matrix, and

$$P_A(-x) = \sup\left\{b : \limsup_n \sum_{k:-x_k>b} a_{nk} > 0\right\}$$
$$= -\inf\left\{-b : \limsup_n \sum_{k:x_k<-b} a_{nk} > 0\right\} = -st_A - \liminf x.$$

Similarly, for part (c), when $\alpha > 0$, we have

$$P_A(\alpha x) = \sup\left\{b : \limsup_n \sum_{k:\alpha x_k>b} a_{nk} > 0\right\}$$
$$= \alpha \sup\left\{\frac{b}{\alpha} : \limsup_n \sum_{k:x_k>b/\alpha} a_{nk} > 0\right\} = \alpha P_A(x).$$

This gives $\alpha = 0$ case as well.

For part (b), let $P_A(x) = \ell_x$ and $P_A(y) = \ell_y$. By the definition, for any $\epsilon > 0$, we therefore have that

$$\lim_n \sum_{k:x_k>\ell_x+\frac{\epsilon}{2}} a_{nk} = 0, \quad \lim_n \sum_{k:y_k>\ell_y+\frac{\epsilon}{2}} a_{nk} = 0.$$

Therefore, we have

$$\lim_n \sum_{k:x_k+y_k>\ell_x+\ell_y+\epsilon} a_{nk} \leq \lim_n \sum_{k:x_k>\ell_x+\frac{\epsilon}{2}} a_{nk} + \lim_n \sum_{k:y_k>\ell_y+\frac{\epsilon}{2}} a_{nk} = 0.$$

This gives that $P_A(x + y) \leq \ell_x + \ell_y + \epsilon$ for all $\epsilon > 0$. □

The connection of the above sublinear functional, P_A, with SL_A is made clearer in the following section.

3. Functionals that Dominate or Generate Generalized Limits

Following Simons [26], we recall the definitions of functionals that generate and/or dominate generalized limits. By m^*, we denote the algebraic dual of m.

Definition 3.1. Let R and T be sublinear functionals on m and let $\mathcal{L}$ be a collection of bounded linear functionals on m.

(i) We say that R generates $\mathcal{L}$ if for any $L \in m^*$ and $L(x) \leq R(x)$ for all $x \in m$ together imply that $L \in \mathcal{L}$.

(ii) We say that T dominates $\mathcal{L}$ if for every $L \in \mathcal{L}$ we have $L(x) \leq T(x)$ for all $x \in m$.

A sublinear functional, R, on m generates $\mathcal{L}$ if and only if $R(x) \leq W(x)$ for all x, where

$$W(x) \ := \ \sup\{L(x) : \ L \in \mathcal{L}\}, \text{ for all } x \in m.$$

Trivially a sublinear functional, R, dominates $\mathcal{L}$ if and only if $R(x) \geq W(x)$ for all $x \in m$. Combining these two statements, a sublinear functional R on m generates as well as dominates $\mathcal{L}$-limits if and only if it equals W. The following theorem shows that P_A both generates and dominates SL_A-limits.

Theorem 3.2. *Let $A = [a_{nk}]$ be a nonnegative regular matrix. Then the following results hold.*

(i) *P_A both generates and dominates SL_A. Therefore,*

$$P_A(x) = \sup\{L(x) : \ L \in SL_A\}, \ \textit{for all } x \in m.$$

(ii) *Q_A generates SL_A. However, Q_A cannot dominate SL_A when A sums a divergent* 0, 1 *sequence to a number $\ell \in (0, 1)$.*

(iii) *When I is the identity matrix, Q_I dominates SL_A. However, Q_I cannot generate SL_A when A sums a divergent* 0, 1 *sequence to zero.*

Proof. Let $L \in SL_A$. If there exists a sequence $x \in m$ so that $L(x) > P_A(x)$, then without loss of generality we may assume that $x_k \geq 0$ for all k. Then take $p \in (P_A(x), L(x))$ and take $E = \{k : x_k > p\}$. This implies that $\limsup_n \sum_{k:k\in E} a_{nk} = 0$. Hence, $\delta_A(E) = 0$. Therefore, we have

$$\begin{aligned} L(x) &= L(x\chi_E) + L(x\chi_{E^c}) \\ &\leq \|x\| L(\chi_E) + pL(\chi_{E^c}) \\ &\leq pL(e) \ = \ p \ < \ L(x) \end{aligned}$$

where $e = (1, 1, 1, \ldots)$. This contradiction shows that $L(x) \leq P_A(x)$ for all $x \in m$. Hence, P_A dominates SL_A. The fact that P_A generates SL_A follows by an identical proof as that of part (1) of Theorem 2.1. We omit the details here.

(ii) We already know that Q_A generates SL_A, by part (1) of Theorem 2.1. To show that Q_A need not dominate SL_A, find a sequence, x, of zeros and ones which A sums to a number $\ell \in (0, 1)$. Let $E \subseteq \mathbb{N}$ so that $\chi_E = x$. Note that $P_A(\chi_E) = 1$. Furthermore,

$$Q_A(\chi_E) = \lim_n \sum_{k:k\in E} a_{nk} = \ell < 1 = P_A(\chi_E) = \sup\{L(\chi_E) : \ L \in SL_A\}.$$

The last equality by part (i). Therefore, Q_A cannot dominate SL_A.

(iii) Since $P_A(x) \leq Q_I(x)$ for all $x \in m$, and P_A dominates SL_A, it must be that Q_I dominates SL_A. To show that Q_I cannot generate SL_A, we produce a positive regular functional T so that $T(x) \leq Q_I(x)$ for all $x \in m$

but $T \notin SL_A$. In this regard, let $E \subseteq \mathbb{N}$ be an infinite set so that $\delta_A(E) = 0$. Denote the members of E as $j_1 < j_2 < \cdots$. Define a new nonnegative regular matrix $B = [b_{nk}]$ where $b_{nk} = 1$ when $k = j_n$ and $b_{nk} = 0$ for other values of k. Using the resulting Q_B, and the linear functional $\lim_B$ on $m \cap c_B$, by the Hahn–Banach theorem, we obtain a bounded linear functional, T, on m so that $T(x) = \lim_B x$ on $m \cap c_B$. Certainly, $Q_B(x) \leq Q_I(x)$, and hence $T(x) \leq Q_I(x)$ for all $x \in m$. However, $T(\chi_E) = \lim_B \chi_E = 1$. On the other hand, $\delta_A(E) = 0$ which implies that for every $L \in SL_A$ we must have $L(\chi_E) = 0$. Hence, $T \notin SL_A$. □

Several sufficient conditions are known for a matrix to sum a divergent sequence of 0, 1 to a number $\ell \in (0, 1)$. The most famous being those methods that have the Borel property ([18]). Also a sufficient condition for a regular matrix in order to sum a divergent 0, 1 sequence to 0 is $\lim_n \sup_k a_{nk} = 0$ ([1]).

The next example shows how we can estimate the generalized limits that are bounded by certain sublinear functionals.

Example. Following Osikiewicz [24] and Unver et. al. [28], we recall the definition of a splice. Let $K_1, K_2, \ldots$ be a countable partition of $\mathbb{N}$. Denote the elements of K_i by $\theta_i(1) < \theta_i(2) < \cdots$. Let $x^1, x^2, \ldots$ be a sequence of sequences such that $\lim_j x^i_j = \alpha_i$ for each $i = 1, 2, \ldots$. A splice x, made by this sequence of sequences and the partition $K_1, K_2, \ldots$, is the sequence for which $x_k = x^i_j$ where $k = \theta_i(j)$. By definition, here a splice is taken to be a bounded sequence. When A is nonnegative regular matrix with row sums being equal to 1, Unver et. al. [28] show that $\lim_A x = \int_X t dF(t)$ for any splice x of sequences taking values in a Banach space X, where the integral is in the sense of Bochner with respect to a probability measure F and can be written as

$$\lim_A x = \sum_{i=1}^{\infty} \alpha_i \, \delta_A(K_i),$$

provided $\delta_A(K_i)$ exist and $\sum_i \delta_A(K_i) = 1$.

As a special case of this result we see that, when x is a splice and $\delta_A(K_i)$ exist with $\sum_i \delta_A(K_i) = 1$, then

$$L(x) = \sum_i \alpha_i \, \delta_A(K_i),$$

for all positive linear functionals L for which $L(x) \leq Q_A(x)$ for all $x \in m$. The issue is what can be said for those situations when either $\delta_A(K_i)$ do not exist or that $\sum_i \delta_A(K_i) \neq 1$. When x is a finite splice (i.e., made up of a finite partition, $K_1, K_2, \ldots, K_m$), we can estimate the value of $L(x)$ as follows,

$$L(x) \leq \sum_{i:\alpha_i>0} \alpha_i \, Q_A(\chi_{K_i}) - \sum_{i:\alpha_i<0} \alpha_i \, Q_A(-\chi_{K_i}),$$

$$L(x) \geq -\sum_{i:\alpha_i>0} \alpha_i \, Q_A(-\chi_{K_i}) + \sum_{i:\alpha_i<0} \alpha_i \, Q_A(\chi_{K_i}).$$

When an infinite splice is a strong limit (i.e., the limit with respect to the norm topology) of a sequence of finite splices, then the above expression will continue to hold. Similarly estimates can be made for all generalized limits that are bounded by the sublinear functional $\limsup_n \sup_j \sum_k a_{nk}x_{k+j}$. However, when $L(x) \leq P_A(x)$, due to the fact that $P_A(\chi_K)$ is either 0 or 1, we can estimate the value of $L(x)$ as follows,

$$L(x) \leq \sum_{i:\alpha_i>0} \alpha_i\, P_A(\chi_{K_i}) - \sum_{i:\alpha_i<0} \alpha_i\, P_A(-\chi_{K_i}) = \sum_{i:\delta_A(K_i)\neq 0} \alpha_i,$$
$$L(x) \geq - \sum_{i:\delta_A(K_i)\neq 0} \alpha_i.$$

Since statistical convergence has been characterized by a collection of summability matrices, it is natural to ask how their sublinear functionals relate. First we recall a few definitions from Fridy and Miller [15].

Let A be a nonnegative regular summability matrix for which each row adds up to one. The collection, τ_A, consists of those $B = [b_{nk}]$ such that

- (i) B is lower triangular nonnegative.
- (ii) $\sum_k b_{nk} = 1$ for each n.
- (iii) For every $E \subseteq \mathbb{N}$ with $\delta_A(E) = 0$ implies that $\delta_B(E) = 0$.

The class of Fridy and Miller, denoted as τ, is obtained when A is taken to be the Cesàro matrix C_1, namely $\tau = \tau_{C_1}$.

A larger class, τ_A^*, consists of those $B = [b_{nk}]$ such that

- (i) B is nonnegative.
- (ii) $\lim_n \sum_k b_{nk} = 1$.
- (iii) For every $E \subseteq \mathbb{N}$ with $\delta_A(E) = 0$ implies that $\delta_B(E) = 0$.

Theorem 3.3. *Let A be a nonnegative regular matrix and let $B = [b_{nk}]$ be a nonnegative summability matrix with $\sup_n \sum_k b_{nk} < \infty$. Then the following results hold.*

- (i) *Q_B generates SL_A if and only if $B \in \tau_A^*$.*
- (ii) *Q_B does not dominate SL for any $B \in \tau_A^*$.*
- (iii) *If Q_B dominates SL_A then $\liminf_n \sum_k b_{nk} \leq 1 \leq \limsup_n \sum_k b_{nk}$.*

Proof. (i) Assume $B \in \tau_A^*$, and let $L \in m^*$ so that $L(x) \leq Q_B(x)$ for all $x \in m$. Since B is nonnegative, L is positive. Since $Q_B(e) = 1$, we have $L(e) = 1$ as well. Also, for every $E \subseteq \mathbb{N}$ for which $\delta_A(E) = 0$, we have $\delta_B(E) = 0$. This gives that $L(\chi_E) = 0$. Note that when E is a finite set then $\delta_A(E) = 0$, since A is regular. Therefore, $Q_B(\chi_E) = 0$, making B a regular matrix. This makes L is regular, and hence for $x \in c$ with $\ell = \lim_k x_k$ we have $L(x) = \lim_k x_k$. That is, Q_B generates SL_A. Conversely, assume that Q_B generates SL_A. Hence, it must be that $Q_B(x) \leq P_A(x)$ for all $x \in m$. This gives that $\lim_n \sum_k b_{nk} = 1$. Also, if $E \subseteq \mathbb{N}$ such that $\delta_A(E) = 0$, then

$$0 = -P_A(-\chi_E) \leq -Q_B(-\chi_E) \leq Q_B(\chi_E) \leq P_A(\chi_E) = 0.$$

That is, $\delta_B(E) = 0$. Hence, $B \in \tau_A^*$.

(ii) If for some $B \in \tau_A^*$ the functional Q_B did dominate SL then, by part (i), it will be that $Q_B(x) = P_A(x)$ for all $x \in m$, where $A = C_1$. By a result of Connor et. al. [8], this is impossible.

(iii) This follows easily from $P_A(e) = 1$ and $P_A(x) \leq Q_B(x)$ for all $x \in m$. □

At the moment, we should remark, that it remains open if part (ii) of the above theorem the set SL could be replaced by SL_A for some A.

It is shown in [10] (see also [5]) that among two nonnegative regular matrices, A, B, the matrix B is statistically stronger than A if and only if for every $E \subseteq \mathbb{N}$ with $\delta_A(E) = 0$ implies that $\delta_B(E) = 0$. By the last theorem this is equivalent to $SL_B \subseteq SL_A$. This is further equivalent to $P_B(x) \leq P_A(x)$ for all $x \in m$. Hence, A and B are statistically equivalent on m if and only if $P_A(x) = P_B(x)$ for all $x \in m$. We should point out, however, that by no means this implies that A should be equal to B.

4. Comparison of SL_A and BL_A

In this section, we investigate the distinctive features of the two types of generalized limits. We have already shown earlier that, when A is a nonnegative regular matrix,

$$\mathcal{L}_A \subseteq SL_A \cap BL_A$$

regardless of A being translative or not, implying that if a sequence is both A-statistically convergent and A-almost convergent then their respective limits must equal $\lim_A x$.

Theorem 4.1. *When A is a nonnegative strongly regular matrix then neither SL_A nor BL_A contains the other.*

Proof. Define $x = (x_k)$ as follows:

$$x_k = \begin{cases} 1; & \text{k even} \\ 0; & \text{k odd} \end{cases}.$$

Observe that x and σx are almost convergent to $\frac{1}{2}$, and also $y = x - \sigma x$ is almost convergent to 0. Since

$$\lim_n \sum_k a_{nk} \; |y_k - 0| = \lim_n \sum_k a_{nk} \; = \; 1,$$

y is not strongly A-summable to 0 therefore y is not A -statistically convergent to 0. This implies that there exists a functional Ψ in SL_A such that $\Psi(y) \neq 0$, implying that this $\Psi \notin BL_A = BL$. Hence BL does not contain SL_A.

To show that SL_A does not contain BL_A, consider a sequence $y = (y_n)$ of zeros and ones of the following form

$$(y_n) = (1, 0, \ldots, 0, 1, 1, 0, \ldots, 0, 1, 1, 1, 0, \ldots, 0, 1, 1, 1, 1, 0, \ldots)$$

in which larger and larger sized batches of zeros are inserted so that the resulting counting function of the sequence remains below the counting function of the first kind for the summability matrix A. Such a sequence is nonnegative, and hence automatically A-strongly summable to zero, making it A-statistically convergent to 0. However, it is easy to see that this sequence is not almost convergent, which implies that it is not A-almost convergent either. Hence, SL_A does not contain BL_A. □

Corollary 4.2. *For any strongly regular matrix we have the following results.*

(i) *There exists a sublinear functional that generates S_A-limits but does not generate Banach limits. There exists a sublinear functional which generates Banach limits but does not generate S_A-limits.*

(ii) *There exists a sublinear functional that dominates S_A-limits but does not dominate Banach limits. There exists a sublinear functional which dominates Banach limits but does not dominate S_A-limits.*

Example. To show that BL and SL_A sets can be mutually exclusive, consider $A = [a_{nk}]$ for which the first row is $(\frac{1}{2}, 0, \frac{1}{2}, 0, \ldots)$. The n-th row of A has first $4(n-1)$ entries equal to 0. The $4(n-1)+1$-th to $4(n-1)+3$-rd entries are $\frac{1}{2}, 0, \frac{1}{2}$, respectively. The rest of the row is zero. Clearly the matrix is regular. When $x = (1, 0, 1, 0, 1, 0, \ldots)$ then $\lim_A x = 1$. However, $\lim_A \sigma(x) = 0$, showing that A is not strongly regular. Note that x is A-statistically convergent to 1 since, for small ϵ, the set $\{k : |x_k - 1| > \epsilon\}$ consists of even numbers whose A-density is zero. Similarly, $\sigma(x)$ is A-statistically convergent to 0, since, for small ϵ, the set $\{k : |\sigma(x)_k - 0| > \epsilon\}$ consists of even numbers whose A-density is zero.

Note that for any $L \in BL$ we have $L(1, -1, 1, -1, \ldots) = 0$. Now to see that no $L \in SL_A$ can be in BL, if there exists an $L \in SL_A$ as well as $L \in BL$ then

$$1 - 0 \; = \; L(x - \sigma x) \; = \; L(1, -1, 1, -1, \ldots) = 0.$$

Hence, in this case

$$BL \cap SL_A \; = \; \emptyset.$$

As Theorem 2.1 shows $BL_A \cap SL_A$ contains $\mathcal{L}_A$. However, the above argument also shows that the sets BL_A and SL_A are distinct, since the A-statistically convergent sequence $(1, 0, 1, 0, \ldots)$ is not A-almost convergent. Hence, there must exist $L_1, L_2 \in BL_A$ for which $L_1(x) \neq L_2(x)$, making at least one of them not in SL_A.

By Theorem 2.1, we automatically get that $BL_A \neq BL$, since A is not strongly regular. However, A being a regular matrix, we have $BL \subset BL_A$ where the inclusion must, therefore, be strict.

A simple example can also be constructed for strict inclusion. Take $x_k = 0$ for k even, $x_k = 1$ for k odd and $\frac{k+1}{2}$ odd, and $x_k = 2$ for k odd and $\frac{k+1}{2}$ even. This sequence is almost convergent to $\frac{3}{4}$. However, this sequence is not A-almost convergent. Hence, there must exist $L_1, L_2 \in BL_A$ for which $L_1(x) \neq L_2(x)$, making at least one of them not in BL.

It is interesting to note that Duran [11] gave a fundamental lemma by proving that, for any given $x \in m$, there exist strongly regular positive matrices B_1 and B_2 such that

$$\sup\{T(x),\ T \in BL\} = \lim_n (B_2 x)_n, \quad \inf\{T(x),\ T \in BL\} = \lim_n (B_1 x)_n.$$

Without loss of generality we may further assume that each row sum of these matrices, A, B, is one. Note that $BL = BL_{C_1}$, and the intersection of the convergence fields of all strongly regular summability matrices gives the space of almost convergent sequences. Similarly, the intersection of the bounded convergence fields of all the members of τ_{C_1} gives the space of all bounded statistically convergent sequences. In light of Duran's lemma, one may raise the question if one could find matrices, B_1, B_2, in τ_{C_1} so that

$$\sup\{L(x),\ L \in SL\} = \lim_n (B_2 x)_n, \quad \inf\{L(x),\ L \in SL\} = \lim_n (B_1 x)_n.$$

The following proposition shows that this is not possible, even though it is known [20] that there exists a single matrix in τ_{C_1} whose bounded convergence field is precisely the set of all bounded statistically convergent sequences.

Theorem 4.3. *There are no matrices A and B in τ such that*

$$\sup\{G(x) : G \in SL\} = \lim (Ax)_n$$

and

$$\inf\{G(x) : G \in SL\} = \lim (Bx)_n.$$

Proof. Consider the following sequence $x = (x_n)$,

$$x_n = \begin{cases} 0, & n \text{ is odd} \\ 1/2, & n = 2, 6, 10, \dots \\ 3/4, & n = 4, 12, 20, \dots \\ 7/8, & n = 8, 24, 40, \dots \\ 15/16, & n = 16, 48, 80, \dots \\ \dots & \dots \end{cases}.$$

Concretely, $x = (x_n)$ is given by

$$x_n = 0,\ n \text{ is odd and}$$
$$x_{2^n + k2^{n+1}} = \frac{2^n - 1}{2^n},\ n = 1, 2, 3, \dots \text{ and } k = 1, 2, 3, \dots.$$

For this sequence st-$\limsup x = 1$ and recall that $\sup\{G(x) : G \in SL\} = st\text{-}\limsup x$. Now we show that if A is a nonnegative, lower triangular matrix summability method satisfying (ii) and Ax converges to 1, then A does not satisfy the condition (iii) in τ. To show this, we find a set K with $\delta(K) = 0$ such that $\sum_{k \in K} a_{nk}$ does not converge to 0. Pick m_1 large enough so that

$$\sum_{i=1}^{m_1} a_{m_1,i} x_i \geq 0.99.$$

Therefore

$$\sum_{\substack{i=1 \\ x_i \leq 0.98}}^{m_1} a_{m_1,i} x_i + \sum_{\substack{i=1 \\ x_i > 0.98}}^{m_1} a_{m_1,i} x_i \geq 0.99.$$

Let

$$I_1 = \{i : i \leq m_1 \text{ and } x_i \leq 0.98\},$$
$$II_1 = \{i : i \leq m_1 \text{ and } x_i > 0.98\}$$

and

$$t_1 = \sum_{i \in I_1} a_{m_1,i} \quad \text{and} \quad s_1 = \sum_{i \in II_1} a_{m_1,i}.$$

Then

$$t_1 + s_1 = 1$$

and

$$0.98t_1 + 1s_1 \geq \sum_{\substack{i=1\\ x_i \leq 0.98}}^{m_1} a_{m_1,i}x_i + \sum_{\substack{i=1\\ x_i > 0.98}}^{m_1} a_{m_1,i}x_i \geq 0.99$$

which implies

$$t_1 \leq \frac{1}{2} \quad \text{and} \quad s_1 \geq \frac{1}{2}.$$

Now pick m_2, much larger than m_1 such that

$$x_i < 0.999\ldots98 \text{ for every } i \in II_1$$

and

$$\sum_{i=1}^{m_2} a_{m_2,i}x_i \geq 0.999\ldots99.$$

Note that the numbers $0.999\ldots98$ and $0.999\ldots99$ have the same number of digits. Then as above

$$\sum_{\substack{i=1\\ x_i \leq 0.999\ldots98}}^{m_2} a_{m_2,i}x_i + \sum_{\substack{i=1\\ x_i > 0.999\ldots98}}^{m_2} a_{m_2,i}x_i \geq 0.999\ldots99.$$

Let

$$I_2 = \{i : i \leq m_2,\ x_i \leq 0.999\ldots98\},$$
$$II_2 = \{i : i \leq m_2,\ \ x_i > 0.999\ldots98\},$$
$$t_2 = \sum_{i \in I_2} a_{m_2,i} \text{ and } s_2 = \sum_{i \in II_2} a_{m_2,i}.$$

Then

$$t_2 + s_2 = 1$$

and

$$0.999\ldots98t_2 + 1s_2 \geq \sum_{\substack{i=1\\ x_i \leq 0.999\ldots98}}^{m_2} a_{m_2,i}x_i + \sum_{\substack{i=1\\ x_i > 0.999\ldots98}}^{m_2} a_{m_2,i}x_i \geq 0.999\ldots99$$

which implies

$$t_2 \leq \frac{1}{2} \text{ and } s_2 \geq \frac{1}{2}.$$

Now pick m_3, much larger than m_2 such that

$$x_i < 0.9999\ldots998 \text{ for every } i \in II_1 \cup II_2$$

and

$$\sum_{i=1}^{m_3} a_{m_3,i}x_i \geq 0.9999\ldots999.$$

Proceed as in steps 1 and 2 and noting that the number of digits in this step is taken to be much larger than the number of digits in step 2. Let

$$I_3 = \{i : i \leq m_3,\ x_i \leq 0.9999\ldots 998\},$$
$$II_3 = \{i : i \leq m_3,\ \ x_i > 0.9999\ldots 998\}.$$

Define t_3 and s_3 as before. Then we can argue again that $t_3 \leq \frac{1}{2}$ and $s_3 \geq \frac{1}{2}$. By continuing this process we obtain $\{II_i\}_{i=1}^{\infty}$. Let

$$K := II_1 \cup II_2 \cup II_3 \cup \ldots .$$

Notice that, by construction, these sets are pairwise disjoint. Observe that for each n, $s_n \geq \frac{1}{2}$ so $\left\{\sum_{k\in K} a_{nk}\right\}$ does not converge to 0. To see that $\delta(K) = 0$ suppose $m \in [m_n, m_{n+1})$ and let $S_m := \frac{1}{m}\sum_{i=1}^{m} \chi_K(i)$. We have that

$$\begin{aligned} S_m &= \frac{1}{m}\sum_{i=1}^{m_{n-1}} \chi_K(i) + \frac{1}{m}\sum_{i=m_{n-1}+1}^{m} \chi_K(i) \\ &\leq \frac{m_{n-1}}{m_n} + \frac{1}{m}\sum_{i=m_{n-1}+1}^{m} \chi_K(i). \end{aligned} \tag{4.1}$$

By the choice of $m_1, m_2, m_3, \ldots$ we can make the term $\dfrac{m_{n-1}}{m_n}$ small enough, i.e., this term goes to zero. Let $\bar{x}_1 := 0.98$, $\bar{x}_2 := 0.999\ldots 98$, $\bar{x}_3 := 0.9999\ldots 998, \ldots$. Note that each $\bar{x}_n$ has more digits than $\bar{x}_{n-1}$. If $i \in II_1$ then we have $x_i > \bar{x}_1$ similarly if $i \in II_2$ then we have $x_i > \bar{x}_2, \ldots$. By the construction of (x_i) and $\{II_i\}_{i=1}^{\infty}$, it is clear that the percentage of $i \in K$ with $i \in (m_{n-1}, m]$ is very small, that is $\sum_{i=m_{n-1}+1}^{m} \chi_K(i) < mp_n$ where p_n goes to zero and $\bar{x}_n$ goes to one as n goes to infinity. Hence, it follows from (4.1) that $S_m \to 0$ as $m \to \infty$. This proves that $\delta(K) = 0$. □

If one restricts the attention only on the convergence fields associated with the functional, P_A, then several characterizations are known. If τ is the Fridy–Miller collection then it is known that bounded statistically convergent sequences are the same as B summable sequences for all $B \in \tau$ and to the same limit.

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