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A Note on Derivations on the Algebra of Operators in Hilbert C^*-Modules

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Abstract. Let $\mathfrak{M}$ be a Hilbert C^*-module on a C^*-algebra $\mathfrak{A}$ and let $End_{\mathfrak{A}}(\mathfrak{M})$ be the algebra of all operators on $\mathfrak{M}$. In this paper, first the continuity of $\mathfrak{A}$-module homomorphism derivations on $End_{\mathfrak{A}}(\mathfrak{M})$ is investigated. We give some sufficient conditions on which every derivation on $End_{\mathfrak{A}}(\mathfrak{M})$ is inner. Next, we study approximately innerness of derivations on $End_{\mathfrak{A}}(\mathfrak{M})$ for a σ-unital C^*-algebra $\mathfrak{A}$ and full Hilbert $\mathfrak{A}$-module $\mathfrak{M}$. Finally, we show that every bounded linear mapping on $End_{\mathfrak{A}}(\mathfrak{M})$ which behave like a derivation when acting on pairs of elements with unit product, is a Jordan derivation.

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1. Introduction and Preliminaries

Let $\mathfrak{A}$ be a complex algebra, a linear mapping $d : \mathfrak{A} \longrightarrow \mathfrak{A}$ is called a derivation, if for all $a, b \in \mathfrak{A}$,

$$d(ab) = d(a)b + ad(b).$$

Moreover, if there exists $b_0 \in \mathfrak{A}$ such that $d(a) = ab_0 - b_0a$ for all $a \in \mathfrak{A}$ then d is a derivation which is called inner derivation.

Derivations are very important maps both in theory and applications and are studied on different spaces. An important question is that on which spaces and under what conditions, a derivation is inner or automatically continuous. Kaplansky [9] conjectured that every derivation on a C^*-algebra or a semisimple Banach algebra is continuous. Sakai confirmed Kaplansky's conjecture for the C^*-algebra case in [16]. In [14], Ringrose generalized these results to derivations from C^*-algebra into their Banach modules. Johnson and Sinclair [7] proved continuity of derivations on semisimple Banach algebras. In addition, Sakai [15] showed that every derivation on $\mathfrak{W}^*$-algebra is inner. In [8], Kadison proved that every derivation of a C^*-algebra $\mathfrak{A}$ on a Hilbert space $\mathfrak{H}$ is spatial and every derivation of a von Neumann algebra is

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inner. Also, innerness of every derivation on a nest algebra was proved by Christensen [2]. For more information on derivations of Banach algebras, we refer to [3].

Similar to [5], we define approximately inner derivation. A derivation $d : \mathfrak{A} \longrightarrow \mathfrak{A}$ is called approximately inner if there exists a net $\{a_\lambda\} \subset \mathfrak{A}$ such that for every $b \in \mathfrak{A}$, $d(b) = \lim_\lambda b.a_\lambda - a_\lambda.b$, the limit being in norm.

Hilbert C^*-module is a natural generalization of a Hilbert space in which the field of scalars $\mathbb{C}$ is replaced by a C^*-algebra. Firstly, the theory of Hilbert C^*-modules was described in work of Kaplansky [9] which has noteworthy role in theory of operator algebras, operator K-theory, theory of operator spaces and so on. For the convenience of the reader, we recall the definition and some basics of a Hilbert C^*-module. More information about Hilbert C^*-modules may be found in [12].

Definition 1.1. Let $\mathfrak{M}$ be a Hilbert C*-module over a C^*-algebra $\mathfrak{A}$ with action of $a \in \mathfrak{A}$ on $x \in \mathfrak{M}$ which is denoted by $a.x$. $\mathfrak{A}$-module $\mathfrak{M}$ is called pre-Hilbert $\mathfrak{A}$-module if it is equipped with a sesquilinear form $\langle ., . \rangle : \mathfrak{M} \times \mathfrak{M} \longrightarrow \mathfrak{A}$ (which is called $\mathfrak{A}$-valued inner product) satisfying the following properties:

1. $\langle \lambda x + y, z \rangle = \lambda \langle x, z \rangle + \langle y, z \rangle$, for all $x, y, z \in \mathfrak{M}$ and $\lambda \in \mathbb{C}$;
2. $\langle x, x \rangle \geq 0$, and $\langle x, x \rangle = 0$ if and only if $x = 0$ for $x \in \mathfrak{M}$;
3. $\langle ax, y \rangle = a \langle x, y \rangle$, for every $x, y \in \mathfrak{M}$ and $a \in \mathfrak{A}$;
4. $\langle x, y \rangle^* = \langle y, x \rangle$, for each $x, y \in \mathfrak{M}$.

A pre-Hilbert $\mathfrak{A}$-module $\mathfrak{M}$ which is complete with respect to the norm $\|x\| = \|\langle x, x \rangle\|^{1/2}$ is called a Hilbert C^*-module. For example, each C^*-algebra $\mathfrak{A}$, each ideal of $\mathfrak{A}$ and $H_{\mathfrak{A}}$, the direct sum of a countable number of copies of a Hilbert module $\mathfrak{M}$ are favorites Hilbert C^*-modules. Denote by $\langle \mathfrak{M}, \mathfrak{M} \rangle$, the closure of the linear span of all $\{\langle x, y \rangle | x, y \in \mathfrak{M}\}$. We call $\mathfrak{M}$ full if $\langle \mathfrak{M}, \mathfrak{M} \rangle = \mathfrak{A}$. Also, $\mathfrak{M}$ is called self-dual if $\mathfrak{M}' = \mathfrak{M}$. A C^*-algebra $\mathfrak{A}$ is called σ-unital if it possesses a countable approximate unit. To prove our results, we need the following lemma. For a direct proof, see [12, Lemma 2.4.3].

Lemma 1.2. *Let $\mathfrak{A}$ be a σ-unital C^*-algebra, and let $\mathfrak{M}$ be a full Hilbert $\mathfrak{A}$-module. Then there exists a sequence $\{x_i\}$ in $\mathfrak{M}$, such that the sequence of partial sums of the series $\sum_i \langle x_i, x_i \rangle$ is a countable approximate unit of the algebra $\mathfrak{A}$.*

A bounded $\mathbb{C}$-linear $\mathfrak{A}$-module homomorphism [i.e., $T(ax) = aT(x)$] on the module $\mathfrak{M}$ is called an operator on $\mathfrak{M}$. The set of all operators on $\mathfrak{M}$ is denoted by $End_{\mathfrak{A}}(\mathfrak{M})$ which is obviously a Banach algebra. However, there is no natural involution on this algebra. Denote by $End^*_{\mathfrak{A}}(\mathfrak{M})$ the set of all adjointable operators in $End_{\mathfrak{A}}(\mathfrak{M})$ which is a C^*-algebra. Let $\mathfrak{M}'$ be the dual module, the set of all $\mathfrak{A}$-linear maps from $\mathfrak{M}$ to $\mathfrak{A}$. By introducing $(\lambda.f) := \overline{\lambda} f(x)$ and $(a.f)(x) := a^* f(x)$, where $\lambda \in \mathbb{C}$ and $a \in \mathfrak{A}$; $\mathfrak{M}'$ becomes a $\mathfrak{A}$-module. This module with respect to the norm $\|f\| = \sup_{\|x\| \leq 1} \|f(x)\|$ is complete.

Remark 1.3. Clearly, functionals of the form $\widehat{x} : y \longmapsto \langle y, x \rangle$ belong to $\mathfrak{M}'$.

Consider the set $BK(\mathfrak{M})$ in $End_{\mathfrak{A}}(\mathfrak{M})$ of the closure of the linear span of operators of the form

$$\theta_{y,f}(x) = f(x).y, \quad x, y \in \mathfrak{M}, f \in \mathfrak{M}^{'}.$$

The elements of the set $BK(M)$ are called Banach-compact operators. The following lemma, obviously is proved in [12].

Lemma 1.4. *Let $\mathfrak{A}$ be a C^*-algebra and $\mathfrak{M}$ be a Hilbert $\mathfrak{A}$-module, let $T \in End_{\mathfrak{A}}(\mathfrak{M})$ and $x, y \in \mathfrak{M}$ and $f \in \mathfrak{M}^{'}$. The following equalities satisfy:*

1. $\theta_{y,f}T = \theta_{y,foT}$
2. $T\theta_{y,f} = \theta_{Ty,f}$
3. *If C^*-algebra $\mathfrak{A}$ is commutative, $\theta_{ay,f} = a\theta_{y,f}$ and $\theta_{y,f}\theta_{x,g} = f(x)\theta_{y,g}$*
4. *$BK(\mathfrak{M})$ is a two-sided ideal in $End_{\mathfrak{A}}(\mathfrak{M})$.*

Li et al. proved that if $\mathfrak{A}$ is a unital and commutative C^*-algebra and $\mathfrak{M}$ is a full Hilbert $\mathfrak{A}$-module then every derivation of the C^*-algebra of adjointable operators on $\mathfrak{M}$ is inner (Theorem 2.1 in [11]). In this paper, we are going to show that if $\mathfrak{A}$ is a unital C^*-algebra and $\mathfrak{M}$ is a Hilbert $\mathfrak{A}$-module with the property that there exist $x_0 \in \mathfrak{M}$ and $f_0 \in \mathfrak{M}^{'}$ such that $f_0(x_0) = e_{\mathfrak{A}}$ then every derivation on the algebra of operators (no adjointability is assumed) on $\mathfrak{M}$ is inner, (Theorem 2.6). Also, we study (weakly) approximately innerness of derivations on $End_{\mathfrak{A}}(\mathfrak{M})$. Furthermore, we characterize the bounded linear mappings on $End_{\mathfrak{A}}(\mathfrak{M})$ which behave like derivations when acting on pairs of elements with unit product.

2. Main Results

The following lemmas are essential for our main results.

Lemma 2.1. *Let $\mathfrak{M}$ be a Hilbert $\mathfrak{A}$-module and suppose that $\langle w, w\rangle w = 0$ for $w \in \mathfrak{M}$, then $w = 0$.*

Proof. By assumption we get $\|\langle w, w\rangle w\|^2 = 0$. For each $w \in \mathfrak{M}$, we have $\langle w, w\rangle$ is a self-adjoint element in $\mathfrak{A}$ and $\|w\|^2 = \|\langle w, w\rangle\|$, so we obtain that

$$\begin{aligned} 0 &= \|\langle\langle w, w\rangle w, \langle w, w\rangle w\rangle\| = \|\langle w, w\rangle\langle w, w\rangle\langle w, w\rangle\| \\ &= \|\langle w, w\rangle^3\|. \end{aligned}$$

Therefore, $\langle w, w\rangle^3 = 0$, and so, $\langle w, w\rangle^4 = 0$. Hence $\|\langle w, w\rangle\|^4 = \|\langle w, w\rangle^4\| = 0$. Consequently, $\langle w, w\rangle = 0$ which implies that $w = 0$. □

If $\mathfrak{A}$ is commutative C^*-algebra then using the proof of Theorem 2.1 in [11], one may find that the center of $End_{\mathfrak{A}}(\mathfrak{M})$ is $\{T_a | a \in \mathfrak{A}\}$ where $T_a x = ax$.

Lemma 2.2. *If $\mathfrak{A}$ is a commutative C^*-algebra then the center of $End_{\mathfrak{A}}(\mathfrak{M})$ is $\{T_a | a \in \mathfrak{A}\}$.*

Proof. Similar to proof of Theorem 2.1 in [11]. □

Lemma 2.3. *Let $\mathfrak{A}$ is a commutative C^*-algebra and let derivation d on $End_{\mathfrak{A}}(\mathfrak{M})$ annihilates its center, then d is $\mathfrak{A}$-module homomorphism.*

Proof. By Lemma 2.2, $d(T_a) = 0$, for $a \in \mathfrak{A}$, so we have

$$d(aS) = d(T_a S) = d(T_a)S + T_a d(S) = ad(S).$$

Hence d is $\mathfrak{A}$-module homomorphism. □

Theorem 2.4. *Let $\mathfrak{A}$ be a unital and commutative C^*-algebra and $\mathfrak{M}$ be a Hilbert $\mathfrak{A}$-module with the property that there exist $x_0 \in \mathfrak{M}$ and $f_0 \in \mathfrak{M}'$ such that $f_0(x_0) = e_{\mathfrak{A}}$. Then every $\mathfrak{A}$-module homomorphism derivation on $End_{\mathfrak{A}}(\mathfrak{M})$ is automatically continuous.*

Proof. Let $\{T_n\}$ be a sequence in $End_{\mathfrak{A}}(\mathfrak{M})$ such that converges to 0 and $d(T_n) \longrightarrow T$. We are going to show that $T = 0$ which by the closed graph theorem, d is continuous. By hypothesis, there exist $x_0 \in \mathfrak{M}$ and $f_0 \in \mathfrak{M}'$ such that $f_0(x_0) = e_{\mathfrak{A}}$. Now for every $f \in \mathfrak{M}'$ and $x, y \in \mathfrak{M}$ we have

$$\begin{aligned} d(\theta_{x,f} T_n \theta_{y,f_0}) &= d(\theta_{f(T_n y)x, f_0}) \\ &= d(f(T_n y)\theta_{x,f_0}) \\ &= f(T_n y) d(\theta_{x,f_0}). \end{aligned}$$

The third equality is true, since d is $\mathfrak{A}$-module homomorphism. Also, from $f(T_n y) \longrightarrow 0$ we get

$$0 = \lim_{n\to\infty} d(\theta_{x,f} T_n \theta_{y,f_0}) = \lim_{n\to\infty} (d(\theta_{x,f}) T_n \theta_{y,f_0} + \theta_{x,f} d(T_n \theta_{y,f_0})).$$

So convergence of $\{T_n\}$ to zero implies that

$$0 = \lim_{n\to\infty} \theta_{x,f} d(T_n \theta_{y,f_0}) = \lim_{n\to\infty} \theta_{x,f} d(T_n) \theta_{y,f_0} + \theta_{x,f} T_n d(\theta_{y,f_0}).$$

Hence, by the assumption that $d(T_n) \longrightarrow T$, we obtain $\theta_{x,f} T \theta_{y,f_0} = 0$ for every $x, y \in \mathfrak{M}$ and $f \in \mathfrak{M}'$. By acting the left side of the current equality on x_0, we have $f(Ty)x = 0$. This is true for each $f \in \mathfrak{M}'$ and also, it is well known that for each $z \in \mathfrak{M}$ the operator $\widehat{z}$ belongs to $\mathfrak{M}'$, so we find that for all $x, y, z \in \mathfrak{M}$,

$$0 = \widehat{z}(Ty)x = \langle Ty, z\rangle x.$$

In particular, taking $x = z = Ty$ in this relation we conclude that $\langle Ty, Ty\rangle Ty = 0$, for all $y \in \mathfrak{M}$. Hence by Lemma 2.1, $Ty = 0$. It means that $T = 0$ and we have done. □

As a consequence of Lemma 2.3 and Theorem 2.4, the following result immediately follows.

Corollary 2.5. *Let $\mathfrak{A}$ and $\mathfrak{M}$ be with the conditions of Theorem 2.4 and let d be a derivation on $End_{\mathfrak{A}}(\mathfrak{M})$ such that annihilates its center. Then d is automatically continuous.*

Our next goal is to show that a derivation on operator algebra (no adjointability is assumed) on a Hilbert $\mathfrak{A}$-module $\mathfrak{M}$ is inner if $\mathfrak{A}$ is a unital C^*-algebra and $\mathfrak{M}$ is a Hilbert $\mathfrak{A}$-module with the property that there exist $x_0 \in \mathfrak{M}$ and $f_0 \in \mathfrak{M}'$ such that $f_0(x_0) = e_{\mathfrak{A}}$.

Theorem 2.6. *Let $\mathfrak{A}$ be a unital C^*-algebra, let $\mathfrak{M}$ a Hilbert $\mathfrak{A}$-module with the property that there exist $x_0 \in \mathfrak{M}$ and $f_0 \in \mathfrak{M}^{'}$ such that $f_0(x_0) = e_{\mathfrak{A}}$ and let d be a derivation on $End_{\mathfrak{A}}(\mathfrak{M})$. Then d is an inner derivation.*

Proof. Since there exist $f_0 \in \mathfrak{M}^{'}$ and $x_0 \in \mathfrak{M}$ which $f_0(x_0) = e_{\mathfrak{A}}$, we define the mapping $T : \mathfrak{M} \longrightarrow \mathfrak{M}$ by

$$f(Tx) = f_0(d(\theta_{x_0,f})x), \quad x \in \mathfrak{M}, f \in \mathfrak{M}^{'}. \tag{2.1}$$

Clearly, T is well defined and $\mathbb{C}$-linearity of f_0 implies $\mathbb{C}$-linearity of T. It follows from Eq. 2.1, for each $x \in \mathfrak{M}$ and $f \in \mathfrak{M}^{'}$,

$$\begin{aligned} f((AT - TA)x) &= foA(Tx) - f(TAx) \\ &= f_0(d(\theta_{x_0,foA})x) - f_0(d(\theta_{x_0,f})Ax) \\ &= f_0(d(\theta_{x_0,f}A)x) - f_0(d(\theta_{x_0,f})Ax) \\ &= f_0(\theta_{x_0,f}d(A)x) \\ &= f_0(f(d(A)x)x_0) \\ &= f(d(A)x). \end{aligned}$$

Since $\widehat{z} \in \mathfrak{M}^{'}$ for each $z \in \mathfrak{M}$, we obtain that $\widehat{z}((AT - TA)x) = \widehat{z}(d(A)x)$. So for all $x, z \in \mathfrak{M}$ we get

$$\langle (AT - TA)x, z\rangle = \langle d(A)x, z\rangle.$$

Hence $d(A) = AT - TA$ for each $A \in End_{\mathfrak{A}}(\mathfrak{M})$. Thus, it suffices to show that $T \in End_{\mathfrak{A}}(\mathfrak{M})$. We have $f(T(ax)) = af(T(x)) = f(aT(x))$ for each $f \in \mathfrak{M}^{'}$ and $x \in \mathfrak{M}$ and $a \in \mathfrak{A}$. So $T(ax) = aT(x)$. Therefore, T is $\mathfrak{A}$-module homomorphism. Finally,

$$\begin{aligned} \|Tx\|^2 &= \|\langle Tx, Tx\rangle\|^2 \\ &= \|\widehat{Tx}(Tx)\|^2 = \|f_0(d(\theta_{x_0,\widehat{Tx}})x)\|^2 \leq K\|x\|^2. \end{aligned}$$

It means that $T \in End_{\mathfrak{A}}(\mathfrak{M})$, and so, d is inner. □

Following [4], let $\mathfrak{M}$ be a pre-Hilbert module over a C^*-algebra $\mathfrak{A}$. The $\mathfrak{M}^{'}$-weak module topology $\tau_{\mathfrak{M}'}$ on $\mathfrak{M}$ is generated by the family of semi-norms $\{\nu_f\}_{f\in\mathfrak{M}}$ where $\nu_f(x) = \|f(x)\|$ for $x \in \mathfrak{M}$ and the $\widehat{\mathfrak{M}}$-weak module topology $\tau_{\widehat{\mathfrak{M}}}$ on $\mathfrak{M}$ is generated by the family of semi-norms $\{\mu_z\}_{z\in\mathfrak{M}}$ where $\mu_z(x) = \|\langle z, x\rangle\|$ for $x \in \mathfrak{M}$. Obviously, $\tau_{\widehat{\mathfrak{M}}}$ is not stronger than $\tau_{\mathfrak{M}'}$, and these topologies coincide whenever $\mathfrak{M}$ is self-dual. For more details about these topologies we refer the reader to [4].

Motivated by definition of approximately inner, a derivation $d : End_{\mathfrak{A}}(\mathfrak{M}) \rightarrow End_{\mathfrak{A}}(\mathfrak{M})$ is called approximately $\tau_{\mathfrak{M}'}$-inner (resp. approximately $\tau_{\widehat{\mathfrak{M}}}$-inner) if there exists a net $\{T_\lambda\} \subset End_{\mathfrak{A}}(\mathfrak{M})$ such that for every $A \in End_{\mathfrak{A}}(\mathfrak{M})$ and $x \in \mathfrak{M}$, $d(A)x = w - \lim_\lambda(AT_\lambda - T_\lambda A)x$ where the w-limit being in the $\mathfrak{M}^{'}$-weak module topology (resp. $\widehat{\mathfrak{M}}$-weak module topology). Clearly, if d is approximately $\tau_{\mathfrak{M}'}$-inner then it is approximately $\tau_{\widehat{\mathfrak{M}}}$-inner.

Theorem 2.7. *Let $\mathfrak{A}$ be a σ-unital C^*-algebra, let $\mathfrak{M}$ be a full Hilbert $\mathfrak{A}$-module and let d be a derivation on $End_{\mathfrak{A}}(\mathfrak{M})$. Then d is approximately $\tau_{\mathfrak{M}'}$-inner.*

Proof. By Lemma 1.2, there exists a sequence $\{x_i\}$ in $\mathfrak{M}$, such that the sequence of partial sums of the series $\sum_i \langle x_i, x_i\rangle$ is a countable approximate unit of the algebra $\mathfrak{A}$. For every positive integer n, we define the mapping $T_n : \mathfrak{M} \to \mathfrak{M}$ by

$$f(T_n x) = \sum_{i=1}^{n} \langle d(\theta_{x_i,f})x, x_i\rangle, \quad x \in \mathfrak{M}, f \in \mathfrak{M}^{'}.$$

Obviously, T_n is a well-defined bounded $\mathbb{C}$-linear $\mathfrak{A}$-module homomorphism mapping of $\mathfrak{M}$ and so, belongs to $End_{\mathfrak{A}}(\mathfrak{M})$. For each $x \in \mathfrak{M}, f \in \mathfrak{M}^{'}$ and $A \in End_{\mathfrak{A}}(\mathfrak{M})$, we have

$$\begin{aligned}\lim_n f((AT_n - T_n A)x) &= \lim_n (\sum_{i=1}^{n} \langle d(\theta_{x_i,foA})x, x_i\rangle - \sum_{i=1}^{n} \langle d(\theta_{x_i,f})Ax, x_i\rangle) \\ &= \lim_n \sum_{i=1}^{n} \langle \theta_{x_i,f} d(A)x, x_i\rangle \\ &= \lim_n \sum_{i=1}^{n} \langle f(d(A)x)x_i, x_i\rangle \\ &= f(d(A)x).\end{aligned}$$

Consequently, for each $A \in End_{\mathfrak{A}}(\mathfrak{M})$ and $x \in \mathfrak{M}$ we get $d(A)x = w - \lim_n (AT_n - T_n A)x$ where the w-limit being in the $\mathfrak{M}^{'}$-weak module topology. Therefore, d is approximately $\tau_{\mathfrak{M}'}$-inner. □

Remark 2.8. A Hilbert C^*-module $\mathfrak{M}$ has the module Schur property if every $\tau_{\widehat{\mathfrak{M}}}$-convergent sequence in $\mathfrak{M}$ converges in norm. Frank and Povlov [4] proved that important cases as finitely generated projective modules over unital C^*-algebras, the standard Hilbert $\mathfrak{A}$-modules $\mathfrak{A}^n$ for $n \in \mathbb{N}$ and separable unital C^*-algebras possess the module Schur property. Also, let $\mathfrak{A}$ be a σ-unital C^*-algebra with the module Schur property, then finitely generated projective Hilbert modules over $\mathfrak{A}$ have the module Schur property. However, the standard Hilbert module $H_{\mathfrak{A}}$ for any C^*-algebra $\mathfrak{A}$ and the C^*-algebra $C(0, 1]$ do not have the module Schur property. For results and, in particular, characterizations of this property we refer the reader to [4]. We now assume $\mathfrak{A}$ is a σ-unital C^*-algebra and $\mathfrak{M}$ a full Hilbert $\mathfrak{A}$-module with the module Schur property then the proof of Theorem 2.7 shows that $d(A)x = \lim_n (AT_n - T_n A)x$ for each $A \in End_{\mathfrak{A}}(\mathfrak{M})$ and $x \in \mathfrak{M}$ which the limit is in norm. Similar to [1], in this case, d is called an (weakly) approximately inner derivation of $End_{\mathfrak{A}}(\mathfrak{M})$.

In recent years, several authors studied the linear maps that behave like derivations when acting on special products. Zhu and Xiong [17,18] proved that every norm-continuous generalized mapping which behaves like generalized derivations at zero on finite CSL algebras is a generalized derivation, and every strongly operator topology continuous mapping which behaves like derivations at unit operator in a nest algebra is an inner derivation. For more information, we refer the reader to [10] and the references there in. A linear mapping d on an algebra $\mathfrak{A}$ is called a Jordan derivation if

$d(a^2) = d(a)a + ad(a)$ for every $a \in \mathfrak{A}$. In the next theorem, we characterize the bounded linear mappings on $End_{\mathfrak{A}}(\mathfrak{M})$ which behave like derivations at unit operator.

Theorem 2.9. *Let $\mathfrak{A}$ be a unital C^*-algebra, $\mathfrak{M}$ be a Hilbert C^*-module and let d be a bounded linear mapping on $End_{\mathfrak{A}}(\mathfrak{M})$ with the property that $d(TS) = d(T)S + Td(S)$ for $T, S \in End_{\mathfrak{A}}(\mathfrak{M})$ with $TS = I$. Then d is a Jordan derivation. Here, I denote the identity operator on $\mathfrak{M}$.*

Proof. First we have $d(I) = d(I.I) = 2d(I)$ so $d(I) = 0$. Let $T \in End_{\mathfrak{A}}(\mathfrak{M})$. The operator $I - \lambda T$ is invertible in $End_{\mathfrak{A}}(\mathfrak{M})$ for all scalars λ with $|\lambda| < \frac{1}{\|T\|}$. Since we have $(I - \lambda T)^{-1} = \sum_{n=0}^{\infty} \lambda^n T^n$ and d is linear and bounded, we obtain that

$$\begin{aligned} 0 = d(I) &= d((I - \lambda T)(I - \lambda T)^{-1}) \\ &= d((I - \lambda T))(I - \lambda T)^{-1} + (I - \lambda T)d((I - \lambda T)^{-1}) \\ &= -\lambda d(T) \sum_{n=0}^{\infty} \lambda^n T^n + (I - \lambda T)d\left(\sum_{n=0}^{\infty} \lambda^n T^n\right) \\ &= -\sum_{n=1}^{\infty} \lambda^n d(T)T^{n-1} + \sum_{n=1}^{\infty} \lambda^{n-1} d(T^{n-1}) - \sum_{n=1}^{\infty} \lambda^n T d(T^{n-1}) \\ &= \sum_{n=1}^{\infty} \lambda^n (-d(T)T^{n-1} + d(T^n) - Td(T^{n-1})). \end{aligned}$$

The last equations are true for all λ with $|\lambda| < \frac{1}{\|T\|}$. Hence

$$d(T^n) - d(T)T^{n-1} - Td(T^{n-1}) = 0 \tag{2.2}$$

for every $n \in \mathbb{N}$. Put $n = 2$ in Eq. (2.2) to obtain $d(T^2) = d(T)T + Td(T)$ for each $T \in End_{\mathfrak{A}}(\mathfrak{M})$. Consequently, d is a Jordan derivation. $\square$

Johnson [6] proved that if $\mathfrak{A}$ is a C^*-algebra and $\mathfrak{M}$ is a Banach $\mathfrak{A}$-module, then each Jordan derivation $d : \mathfrak{A} \longrightarrow \mathfrak{M}$ is a derivation. In [11], Li et al. showed innerness of derivations of $End^*_{\mathfrak{A}}(\mathfrak{M})$ for a unital and commutative C^*-algebra $\mathfrak{A}$ and a full Hilbert module $\mathfrak{M}$. So we may apply Theorem 2.9 to obtain the following result.

Corollary 2.10. *Suppose $\mathfrak{A}$ is a unital and commutative C^*-algebra, $\mathfrak{M}$ is a full Hilbert C^*-module. Let d be a bounded linear mapping on $End^*_{\mathfrak{A}}(\mathfrak{M})$ with the property that $d(TS) = d(T)S + Td(S)$ for $T, S \in End^*_{\mathfrak{A}}(\mathfrak{M})$ with $TS = I$. Then d is an inner derivation.*

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