Mediterr. J. Math. 13 (2016), 1263–1272
DOI 10.1007/s00009-015-0535-1
1660-5446/16/031263-10
published online February 24, 2015

Mediterranean Journal of Mathematics

Real Hypersurfaces in Complex Two-Plane Grassmannians with GTW Reeb Lie Derivative Structure Jacobi Operator

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Abstract. Using GTW connection, we considered a real hypersurface M in a complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ when the GTW Reeb Lie derivative of the structure Jacobi operator coincides with the Reeb Lie derivative. Next using the method of simultaneous diagonalization, we prove a complete classification for a real hypersurface in $G_2(\mathbb{C}^{m+2})$ satisfying such a condition. In this case, we have proved that M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.

Mathematics Subject Classification. Primary 53C40; Secondary 53C15.

Keywords. Real hypersurface, complex two-plane Grassmannian, Hopf hypersurface, GTW connection, structure Jacobi operator, GTW Lie derivative.

Introduction

For real hypersurfaces with parallel curvature tensor, many differential geometers have studied in complex projective spaces or in quaternionic projective spaces [8, 12, 13]. From a different point of view, it is attractive to classify real hypersurfaces into complex two-plane Grassmannians with certain conditions. For example, there is some result about parallel structure Jacobi operator (for more detail, see [6, 7]). It is natural to question about complex two-plane Grassmannians.

As an ambient space, a complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ consists of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. This Riemannian symmetric space is the unique compact irreducible Riemannian manifold being equipped with both a Kähler structure J and a quaternionic Kähler structure $\mathfrak{J}$ not containing J. Then, we could naturally consider two geometric conditions for hypersurfaces M in $G_2(\mathbb{C}^{m+2})$, namely, that the

This work was supported by Grant Proj. No. NRP-2012-R1A2A2A-01043023.

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one-dimensional distribution $[\xi] = \mathrm{Span}\{\xi\}$ and the three-dimensional distribution $\mathfrak{D}^\perp = \mathrm{Span}\{\xi_1, \xi_2, \xi_3\}$ are both invariant under the shape operator A of M [2], where the *Reeb* vector field ξ is defined by $\xi = -JN$, N denotes a local unit normal vector field of M in $G_2(\mathbb{C}^{m+2})$ and the *almost contact 3-structure* vector fields ξ_ν are defined by $\xi_\nu = -J_\nu N$ $(\nu = 1, 2, 3)$.

Using the result in Alekseevskii [1], Berndt and Suh [2] proved the following result about space of type (A) [sentence about (A)] and type (B) [one about (B)]:

Theorem A. *Let M be a connected orientable real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then both $[\xi]$ and $\mathfrak{D}^\perp$ are invariant under the shape operator of M if and only if*

(A) *M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or*

(B) *m is even, say $m = 2n$, and M is an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$.*

When we consider the Reeb vector field ξ in the expression of the curvature tensor R for a real hypersurface M in $G_2(\mathbb{C}^{m+2})$, the structure Jacobi operator R_ξ can be defined in such as

$$R_\xi(X) = R(X, \xi)\xi,$$

for any tangent vector field X on M.

Using the structure Jacobi operator R_ξ, they [6] considered a notion of *parallel structure Jacobi operator*, that is, $\nabla_X R_\xi = 0$ for any vector field X on M, and gave a non-existence theorem. And authors [7] considered the general notion of $\mathfrak{D}^\perp$*-parallel structure Jacobi operator* defined in such a way that $\nabla_{\xi_i} R_\xi = 0$, $i = 1, 2, 3$, which is weaker than the notion of parallel structure Jacobi operator. They also gave a non-existence theorem.

By the way, the Reeb vector field ξ is said to be *Hopf* if it is invariant under the shape operator A. The one-dimensional foliation of M by the integral manifolds of the Reeb vector field ξ is said to be the *Hopf foliation* of M. We say that M is a *Hopf hypersurface* in $G_2(\mathbb{C}^{m+2})$ if and only if the Hopf foliation of M is totally geodesic. Using the formulas in [4, section 1] it can be easily checked that M is Hopf if and only if the Reeb vector field ξ is Hopf.

Now, instead of the Levi–Civita connection for real hypersurfaces in Kähler manifolds, we consider another new connection named *generalized Tanaka–Webster connection* (in short, let us say the *GTW connection*) $\hat{\nabla}^{(k)}$ for a non-zero real number k [9]. This new connection $\hat{\nabla}^{(k)}$ can be regarded as a natural extension of Tanno's generalized Tanaka–Webster connection $\hat{\nabla}$ for contact metric manifolds. Actually, Tanno [16] introduced the generalized Tanaka–Webster connection $\hat{\nabla}$ for contact Riemannian manifolds using the canonical connection on a non-degenerate, integrable CR manifold.

On the other hand, the original *Tanaka–Webster connection* [15,17] was given as a unique affine connection on a non-degenerate, pseudo-Hermitian CR manifold associated with the almost contact structure. In particular, if a real hypersurface in a Kähler manifold satisfies $\phi A + A\phi = 2k\phi$ $(k \neq 0)$, then the GTW connection $\hat{\nabla}^{(k)}$ coincides with the Tanaka–Webster connection.

Related to GTW connection, due to Jeong et al. [4,5], the *GTW Lie derivative* was defined by

$$\hat{\mathcal{L}}^{(k)}_X Y = \hat{\nabla}^{(k)}_X Y - \hat{\nabla}^{(k)}_Y X, \tag{1}$$

where $\hat{\nabla}^{(k)}_X Y = \nabla_X Y + g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y,\ k \in \mathbb{R}\backslash\{0\}$.

In this paper, using the GTW Lie derivative, we consider a condition that the *GTW Reeb Lie derivative of the structure Jacobi operator coincides with the Reeb Lie derivative*, that is,

$$\left(\hat{\mathcal{L}}^{(k)}_\xi R_\xi\right) Y = (\mathcal{L}_\xi R_\xi)Y, \tag{2}$$

for any tangent vector field Y in M. Using above notion, we have a classification theorem as follows:

Main Theorem. *Let M be a connected orientable Hopf hypersurface in a complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. If the GTW Reeb Lie derivative of the structure Jacobi operator coincides with the Reeb Lie derivative and the Reeb curvature is non-vanishing constant along the Reeb vector field, then M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.*

As a corollary, we consider a condition stronger than the condition (2) as follows:

$$\left(\hat{\mathcal{L}}^{(k)}_X R_\xi\right) Y = (\mathcal{L}_X R_\xi)Y$$

for any tangent vector fields X, Y in M. Then we assert the following

Corollary. *There do not exist any connected orientable Hopf real hypersurfaces in $G_2(\mathbb{C}^{m+2})$, $m\geq 3$, with $(\hat{\mathcal{L}}^{(k)}_X R_\xi)Y = (\mathcal{L}_X R_\xi)Y$ when the Reeb curvature is constant along the direction of the Reeb vector field.*

In Sect. 1, we introduce basic equations in relation to the structure Jacobi operator and prove the key lemmas which will be useful to proceed our main theorem. In Sects. 2 and 3, we give a complete proof of the main theorem and corollary, respectively. In this paper, we refer to [1–3,6,10] for Riemannian geometric structures of $G_2(\mathbb{C}^{m+2})$ and its geometric quantities, respectively.

1. Key Lemmas

In this section, we introduce some fundamental equation of structure Jacobi operator and lemmas. The structure Jacobi operator is given as

$$\begin{aligned} R_\xi X = & \; R(X,\xi)\xi \\ = & \; X - \eta(X)\xi \\ & -\sum\nolimits_{\nu=1}^{3}\Big\{\eta_\nu(X)\xi_\nu - \eta(X)\eta_\nu(\xi)\xi_\nu + 3g(\phi_\nu X,\xi)\phi_\nu\xi + \eta_\nu(\xi)\phi_\nu\phi X\Big\} \\ & +\alpha AX - \alpha^2\eta(X)\xi, \end{aligned} \tag{1.1}$$

for any tangent field X on M.

In [4], they defined the GTW Lie derivative as follows:

$$\hat{\mathcal{L}}_X^{(k)}Y = \hat{\nabla}_X^{(k)}Y - \hat{\nabla}_Y^{(k)}X,$$

where $\hat{\nabla}_X^{(k)}Y = \nabla_X Y + F_X^{(k)}Y$, $F_X^{(k)}Y = g(\phi AX,Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y$. The operator $F_X^{(k)}Y$ said to be the *generalized Tanaka–Webster operator* (in short, GTW operator).

Putting $X=\xi$ and $Y=\xi$, the GTW operator is written as

$$F_\xi^{(k)}Y = -k\phi Y \quad \text{and} \quad F_X^{(k)}\xi = -\phi AX, \text{ respectively.} \tag{1.2}$$

For the (1,1) type tensor R_ξ, this condition $(\hat{\mathcal{L}}_X^{(k)}R_\xi)Y = (\mathcal{L}_X R_\xi)Y$ is equivalent to

$$F_X^{(k)}(R_\xi Y) - F_{R_\xi Y}^{(k)}X - R_\xi F_X^{(k)}Y + R_\xi F_Y^{(k)}X = 0. \tag{1.3}$$

Replacing X=ξ in (1.3) and using (1.2), we get

$$-k\phi R_\xi Y + \phi A R_\xi Y + kR_\xi\phi Y - R_\xi\phi AY = 0. \tag{1.4}$$

Since R_ξ is a symmetric tensor field, taking symmetric part of (1.4), we have

$$kR_\xi\phi Y - R_\xi A\phi Y - k\phi R_\xi Y + A\phi R_\xi Y = 0. \tag{1.5}$$

Subtracting (1.5) from (1.4), we obtain

$$(\phi A - A\phi)R_\xi Y = R_\xi(\phi A - A\phi)Y. \tag{1.6}$$

Therefore, this condition the GTW Reeb Lie derivative of the structure Jacobi operator coincides with the Reeb Lie derivative has such a geometric meaning, that is, $(\phi A - A\phi)$ and R_ξ commute with each other.

Putting $Y=\xi$ in (1.3) and using (1.2), (1.3) is replaced by

$$R_\xi(\phi AX) - kR_\xi(\phi X) = 0. \tag{1.7}$$

Taking the transpose part on (1.7), we get

$$-A\phi R_\xi X + k\phi R_\xi X = 0. \tag{1.8}$$

Using these above equations, we can give two lemmas which will contribute to prove our main theorem.

Lemma 1.1. *Let M be a Hopf hypersurface M in $G_2(\mathbb{C}^{m+2})$. If the GTW Reeb Lie derivative of the structure Jacobi operator coincides with the Reeb Lie derivative of this operator and the principal curvature α is constant along the direction of the Reeb vector field ξ, then the Reeb vector field ξ belongs to the distribution $\mathfrak{D}$ or the distribution $\mathfrak{D}^\perp$*

Proof. Let us put $\xi = \eta(X_0)X_0 + \eta_1(\xi_1)\xi_1$, for some unit vector fields $X_0 \in \mathfrak{D}$ and $\xi_1 \in \mathfrak{D}^\perp$. If $\alpha = 0$, then $\xi \in \mathfrak{D}$ or $\xi \in \mathfrak{D}^\perp$, which is proved by Pérez and Suh [14].

Therefore, we consider the other case $\alpha \neq 0$. Putting $X = \xi_1$ into (1.1) and using $A\xi_1 = \alpha\xi_1$, we have

$$R_\xi(\xi_1) = \alpha^2\xi_1 - \alpha^2\eta(\xi_1)\xi. \tag{1.9}$$

Replacing $X = \phi\xi_1$ into (1.1), (1.1) becomes

$$R_\xi(\phi\xi_1) = \left(\alpha^2 + 8\eta^2(X_0)\right)\phi_1\xi. \tag{1.10}$$

Putting $X = \xi$ into (1.3) and using (1.2), (1.1) is written as

$$-k\phi R_\xi Y + \phi A R_\xi Y + kR_\xi(\phi Y) - R_\xi(\phi AY) = 0. \tag{1.11}$$

Substituting $Y = \xi_1$ in the above equation and using (1.9), (1.10), it becomes

$$8(k-\alpha)\eta^2(X_0)\phi_1\xi = 0. \tag{1.12}$$

Taking the inner product with $\phi_1\xi$, we get

$$8(k-\alpha)\eta^4(X_0) = 0. \tag{1.13}$$

This equation induces that $k = \alpha$ or $\eta^4(X_0) = 0$. Therefore, it completes the proof of our lemma. □

In next section, we will give a complete proof of our main theorem. To do this, first we consider the case that $\xi \in \mathfrak{D}^\perp$. Without loss of generality, we may put $\xi = \xi_1$.

Lemma 1.2. *Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ when the Reeb curvature is non-vanishing. If the GTW Reeb Lie derivative of the structure Jacobi operator coincides with the Reeb Lie derivative of this operator and the Reeb vector field ξ belongs to the distribution $\mathfrak{D}^\perp$, then the shape operator A commutes with the structure tensor ϕ.*

Proof. Putting $\xi = \xi_1$ in (1.1), we get

$$R_\xi X = X - \eta(X)\xi - \phi_1\phi X + \alpha AX - \alpha^2\eta(X)\xi + 2\eta_2(X)\xi_2 + 2\eta_3(X)\xi_3. \tag{1.14}$$

Replacing X with AX in (1.14), it is written as

$$R_\xi AX = AX - \alpha\eta(X)\xi - \phi_1\phi AX + \alpha A^2X - \alpha^3\eta(X)\xi + 2\eta_2(AX)\xi_2 + 2\eta_3(AX)\xi_3. \tag{1.15}$$

And applying the shape operator A on (1.14) becomes

$$AR_\xi X = AX - \alpha\eta(X)\xi - A\phi_1\phi X + \alpha A^2X - \alpha^3\eta(X)\xi + 2\eta_2(X)A\xi_2 + 2\eta_3(X)A\xi_3. \tag{1.16}$$

On the other hand, applying the structure tensor field ϕ to the equation (1.8) in [11], we get

$$AX = \alpha\eta(X)\xi + 2\eta_2(AX)\xi_2 + 2\eta_3(AX)\xi_3 - \phi\phi_1 AX. \tag{1.17}$$

Taking the symmetric part of (1.17), we obtain

$$AX = \alpha\eta(X)\xi + 2\eta_2(X)A\xi_2 + 2\eta_3(X)A\xi_3 - A\phi_1\phi X. \tag{1.18}$$

Putting $\nu = 1$ in the first equation of (1.5) in [4], it becomes

$$\phi\phi_1 X = \phi_1 \phi X. \tag{1.19}$$

Using (1.17), (1.18), (1.19) and subtracting (1.16) from (1.15), we have

$$R_\xi AX = AR_\xi X. \tag{1.20}$$

Putting $Y = X$ in (1.6) and using (1.20), (1.6) is written as

$$A(R_\xi\phi - \phi R_\xi)X = (R_\xi\phi - \phi R_\xi)AX. \tag{1.21}$$

Putting $X = \phi X$ in (1.14), we have

$$R_\xi \phi X = \phi X - \phi_1\phi^2 X + \alpha A\phi X + 2\eta_2(\phi X)\xi_2 + 2\eta_3(\phi X)\xi_3. \tag{1.22}$$

Applying the structure tensor field ϕ to (1.14), we get

$$\phi R_\xi X = \phi X - \phi\phi_1\phi X + \alpha\phi AX + 2\eta_2(X)\phi\xi_2 + 2\eta_3(X)\phi\xi_3. \tag{1.23}$$

Subtracting (1.23) from (1.22), we obtain

$$(R_\xi\phi - \phi R_\xi)X = \alpha(A\phi - \phi A)X. \tag{1.24}$$

Using the Eq. (1.24), the equivalent condition of (1.21) is this one as

$$\alpha A(A\phi - \phi A)X = \alpha(A\phi - \phi A)AX. \tag{1.25}$$

By our assumption $\alpha \neq 0$, the above equation can be replaced by

$$A(A\phi - \phi A)X = (A\phi - \phi A)AX. \tag{1.26}$$

Because of (1.26), there is a common basis $\{e_i \mid i = 1, \ldots, 4m-1\}$ such that

$$Ae_i = \lambda_i e_i \tag{1.27}$$

and

$$(A\phi - \phi A)e_i = \gamma_i e_i. \tag{1.28}$$

Using (1.27), (1.28) becomes

$$\gamma_i e_i = A\phi e_i - \phi A e_i = A\phi e_i - \lambda_i \phi e_i. \tag{1.29}$$

Taking the inner product with e_i, we get $\gamma_i = 0$. Since the eigenvalue γ_i vanishes for all i, from (1.28) we conclude that

$$A\phi - \phi A = 0. \tag{1.30}$$

Consequently, we proved this lemma. □

2. Proof of Main Theorem

Let us consider a Hopf hypersurface M in $G_2(\mathbb{C}^{m+2})$ with $(\hat{\mathcal{L}}_\xi^{(k)} R_\xi)Y = (\mathcal{L}_\xi R_\xi)Y$.

By Lemma 1.1 in Sect. 1, we can conclude that the Reeb vector field ξ in M belongs either to the distribution $\mathfrak{D}$ or $\mathfrak{D}^\perp$.

Then, we can divide the following two cases:

- Case I: $\xi \in \mathfrak{D}^\perp$
- Case II: $\xi \in \mathfrak{D}$

Now, we check the first case in our consideration. If $\xi \in \mathfrak{D}^{\perp}$, by Theorem A and Lemma 1.2, we can assert that M is locally congruent to the model space of type (A). We have to check if the model space of type (A) satisfies the condition $(\hat{\mathcal{L}}_{\xi}^{(k)} R_{\xi})Y = (\mathcal{L}_{\xi} R_{\xi})Y$ or not. For type (A)-space, detailed information (eigenspaces, corresponding eigenvalues, and multiplicities) was given in [2].

Putting $X = \xi$ in (1.3), we get the equivalent condition of $(\hat{\mathcal{L}}_{\xi}^{(k)} R_{\xi})Y = (\mathcal{L}_{\xi} R_{\xi})Y$ as follows:

$$-k\phi R_{\xi} Y + \phi A R_{\xi} Y + k R_{\xi}\phi Y - R_{\xi}\phi A Y = 0. \tag{2.1}$$

On the other hand, putting $\xi = \xi_1$ into (1.1), we get

$$R_{\xi} X = X - \eta(X)\xi - \phi_1\phi X + \alpha A X - \alpha^2\eta(X)\xi + 2\eta_2(X)\xi_2 + 2\eta_3(X)\xi_3. \tag{2.2}$$

Using (2.1) and (2.2), we get the following result:

$$-k\phi R_{\xi} Y + \phi A R_{\xi} Y + k R_{\xi}\phi Y - R_{\xi}\phi A Y = \begin{cases} 0, & \text{if } Y \in T_{\alpha} \\ 0, & \text{if } Y \in T_{\beta} \\ 0, & \text{if } Y \in T_{\lambda} \\ 0, & \text{if } Y \in T_{\mu}. \end{cases} \tag{2.3}$$

Therefore, we can assert that if ξ in $\mathfrak{D}^{\perp}$, then M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.

If the Reeb vector field $\xi \in \mathfrak{D}$, due to [10], we can assert that M is locally congruent to space of type (B). It remains whether type (B)-space satisfies this condition $(\hat{\mathcal{L}}_{X}^{(k)} R_{\xi})Y = (\mathcal{L}_{X} R_{\xi})Y$. Also, using information of type (B)-space given in [2], we can check this problem.

We suppose that type (B)-space satisfies $(\hat{\mathcal{L}}_{\xi}^{(k)} R_{\xi})Y = (\mathcal{L}_{\xi} R_{\xi})Y$. Then, as an equivalent condition, this space must satisfy

$$-k\phi R_{\xi} Y + \phi A R_{\xi} Y + k R_{\xi}\phi Y - R_{\xi}\phi A Y = 0. \tag{2.4}$$

Since ξ belongs to $\mathfrak{D}$, the structure Jacobi operator in $G_2(\mathbb{C}^{m+2})$ can be replaced as follows:

$$R_{\xi} X = X - \eta(X)\xi - \sum_{\nu=1}^{3}\Big\{\eta_{\nu}(X)\xi_{\nu} + 3g(\phi_{\nu}X, \xi)\phi_{\nu}\xi\Big\} + \alpha A X - \alpha^2\eta(X)\xi. \tag{2.5}$$

Applying $Y = \phi_1\xi \in T_{\gamma}$ into (2.4) and using (2.5), we get

$$k(4 - \alpha\beta)\xi_1 = 0. \tag{2.6}$$

Since $k \neq 0$ and $\alpha\beta = 4$, this makes a contradiction.

Hence summing up these assertions, we have given a complete proof of our main theorem in the introduction.

3. Proof of Corollary

In this section, we consider another problem for this condition

$$(\hat{\mathcal{L}}_{X}^{(k)} R_{\xi})Y = (\mathcal{L}_{X} R_{\xi})Y, \tag{3.1}$$

for any tangent vector fields X, Y in M.

If the Reeb curvature is non-vanishing, the condition $\phi A = A\phi$ have been already proved in Lemma 1.2. Thus, we now consider only the case that α is vanishing. Under these assumptions, we give the following lemma.

Lemma 3.1. *Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with vanishing the Reeb curvature. If the GTW Reeb Lie derivative of structure Jacobi operator coincides with Reeb Lie derivative of this operator and the Reeb vector field ξ belongs to the distribution $\mathfrak{D}^{\perp}$, then shape operator A and the structure tensor ϕ commute with each other.*

Proof. Recall that (1.3) was given by

$$F_X^{(k)} R_\xi Y - F_{R_\xi Y}^{(k)} X - R_\xi F_X^{(k)} Y + R_\xi F_Y^{(k)} X = 0. \tag{3.2}$$

Putting $X = \xi$ in the above equation and using (1.7), (1.8), (3.2) is written as

$$(\phi A - A\phi) R_\xi Y = 0. \tag{3.3}$$

Applying $\alpha = 0$ in (2.2), it becomes

$$R_\xi X = X - \eta(X)\xi - \phi_1 \phi X + 2\eta_2(X)\xi_2 + 2\eta_3(X)A\xi_3. \tag{3.4}$$

On the other hand, applying ϕ and $X = \phi X$ to (1.18), respectively, we have

$$\begin{aligned} \phi A X &= 2\eta_2(X)\phi A\xi_2 + 2\eta_3(X)\phi A\xi_3 - \phi A \phi_1 \phi X, \\ A\phi X &= 2\eta_3(X)A\xi_2 - 2\eta_2(X)A\xi_3 - A\phi_1\phi^2 X. \end{aligned} \tag{3.5}$$

Combining (3.3), (3.4), (3.5) and using (1.19), we get

$$2(\phi A - A\phi)Y = 0. \tag{3.6}$$

Therefore, we also get the same conclusion in case of $\alpha = 0$. □

By Lemmas 1.2 and 3.1, we can assert that if $\xi \in \mathfrak{D}^{\perp}$, then M is the model space of type (A). Now we need to check if the space of type (A) satisfies (3.1) or not.

Then the type (A)-space must satisfy the following equivalent property

$$F_X^{(k)} R_\xi Y - F_{R_\xi Y}^{(k)} X - R_\xi F_X^{(k)} Y + R_\xi F_Y^{(k)} X = 0. \tag{3.7}$$

Putting $Y = \xi$ into (3.7), we have

$$R_\xi \phi A X - k R_\xi \phi X = 0. \tag{3.8}$$

Using (3.4), (3.8) becomes

$$\begin{aligned} &\phi A X + \phi_1 A X + \alpha A \phi A X + 2\eta_3(AX)\xi_2 - 2\eta_2(AX)\xi_3 \\ &\quad -k\phi X - k\phi_1 X - k\alpha A\phi X - 2k\eta_3(X)\xi_2 + 2k\eta_2(X)\xi_3 = 0. \end{aligned} \tag{3.9}$$

Replacing ξ_2 into X, we get

$$(\alpha\beta + 2)(k - \beta)\xi_3 = 0. \tag{3.10}$$

Taking the inner product with ξ_3, the above equation implies $\alpha\beta = -2$ or $k = \beta$. However, since $k \neq 0$, $\alpha = \sqrt{8}\cot(\sqrt{8}r)$ and $\beta = \sqrt{2}\cot(\sqrt{2}r)$, this makes a contradiction.

Hence, we can assert our corollary in the introduction.

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Received: November 9, 2014.
Revised: January 20, 2015.
Accepted: February 2, 2015.