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Pseudo Almost Periodic Mild Solution of Nonautonomous Impulsive Integro-Differential Equations

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Abstract. In this paper, we investigate the existence, uniqueness and stability of pseudo almost periodic mild solution to nonautonomous impulsive integro-differential equations in Banach space. The working tools are based on the fixed point theorems and Gronwall–Bellman inequality. To illustrate our main results, we study pseudo almost periodic solution of the heat equations with Dirichlet conditions.

Mathematics Subject Classification. Primary 35B15; Secondary 35R12.

Keywords. Impulsive integro-differential equations, pseudo almost periodicity, exponential stability, Gronwall–Bellman inequality.

1. Introduction

The study of the existence of almost periodic solution is one of the most interesting and important topics in the qualitative theory of differential equations. Many authors have made important contributions to this theory. On the other hand, pseudo almost periodic function was introduced by Zhang as a natural generalization of almost periodic function in [1,2]. Since then, this notion has been attracted the attention of many researchers, the generalization of pseudo almost periodic function and its applications in ordinary differential equations, function differential equations, integral equations are studied. For more details about this topics, one can see [3–8] for more details.

The impulsive differential equations arise from the real-world problems to describe the dynamics of processes in which sudden, discontinuous jumps occur. In the past several years, the theory of impulsive differential equations has received much attention in recent years due to their wide applications in population dynamics, ecology, biological systems, neural networks, industrial robotics, and economics. The asymptotic properties of solutions of impulsive

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differential equations have been studied from different points, such as almost periodicity [9–12], pseudo almost periodicity [13,14], asymptotic stability [15], asymptotic equivalence [16] and so on.

Motivated by the above-mentioned papers, in this paper, we investigate the existence, uniqueness, and stability of pseudo almost periodic mild solution of abstract nonautonomous impulsive integro-differential equations:

$$\begin{cases} u'(t) = A(t)u(t) + f(t, u(t), (Ku)(t)), \quad t \in \mathbb{R},\, t \neq t_i,\, i \in \mathbb{Z}, \\ (Ku)(t) = \displaystyle\int_{-\infty}^{t} k(t-s)g(s, u(s))\mathrm{d}s, \\ \Delta u(t_i) = u(t_i^+) - u(t_i^-) = I_i(u(t_i)), \end{cases} \tag{1.1}$$

where $A(t) : X \to X$ are closed linear operators on Banach space X, f, I_i, t_i satisfy suitable conditions which will be established later. $u(t_i^+), u(t_i^-)$ represent the right-hand side and the left-hand side limits of $u(\cdot)$ at t_i, respectively.

In (1.1), if $A(t) = A$ is constant and without impulsive effects, some recent contributions on asymptotic properties of solutions are well studied, such as almost periodicity, almost automorphy, pseudo almost periodicity, pseudo almost automorphy [17–22]. However, for the nonautonomous case and with impulsive effects, i.e., (1.1), the study of asymptotic behavior of solution is rare, particularly for the pseudo almost periodicity of (1.1), it is an untreated topic and this is the main motivation of this paper. We will make use of the fixed point theorems and Gronwall–Bellman inequality to derive some sufficient conditions to the existence, uniqueness and exponential stability of pseudo almost periodic mild solution to (1.1).

The paper is organized as follows. In Sect. 2, we recall some fundamental results about the notion of piecewise pseudo almost periodic function including composition theorem. Section 3 is devoted to the existence, uniqueness and stability of mild solution to nonautonomous impulsive integro-differential equations in Banach space. In Sect. 4, an application to heat equations with Dirichlet conditions is given.

2. Preliminaries and Basic Results

Let $(X, \|\cdot\|)$, $(Y, \|\cdot\|)$ be Banach spaces, Ω be a subset of X and $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ stand for the set of natural numbers, integers, real numbers, and complex numbers, respectively. For A being a linear operator on X, $D(A)$, $\rho(A)$, $R(\lambda, A)$, $\sigma(A)$ stand for the domain, the resolvent set, the resolvent and spectrum of A. Let T be the set consisting of all real sequences $\{t_i\}_{i\in\mathbb{Z}}$ such that $\alpha = \inf_{i\in\mathbb{Z}}(t_{i+1} - t_i) > 0$. It is immediate that this condition implies that $\lim_{i\to\infty} t_i = \infty$ and $\lim_{i\to-\infty} t_i = -\infty$.

To facilitate the discussion below, we further introduce the following notations:

- $C(\mathbb{R}, X)$ (resp. $C(\mathbb{R} \times \Omega, X)$): the set of continuous functions from $\mathbb{R}$ to X (resp. from $\mathbb{R} \times \Omega$ to X).

- $BC(\mathbb{R}, X)$ (resp. $BC(\mathbb{R}\times\Omega, X)$): the Banach space of bounded continuous functions from $\mathbb{R}$ to X (resp. from $\mathbb{R}\times\Omega$ to X) with the supremum norm.
- $PC(\mathbb{R}, X)$: the space formed by all piecewise continuous functions $f : \mathbb{R} \to X$ such that $f(\cdot)$ is continuous at t for any $t \notin \{t_i\}_{i\in\mathbb{Z}}$, $f(t_i^+)$, $f(t_i^-)$ exists and $f(t_i^-) = f(t_i)$ for all $i \in \mathbb{Z}$.
- $PC(\mathbb{R}\times\Omega, X)$: the space formed by all piecewise continuous functions $f : \mathbb{R}\times\Omega \to X$ such that for any $x \in \Omega$, $f(\cdot, x) \in PC(\mathbb{R}, X)$ and for any $t \in \mathbb{R}$, $f(t, \cdot)$ is continuous at $x \in \Omega$.
- $L(X, Y)$: the Banach space of bounded linear operators from X to Y endowed with the operator topology. In particular, we write $L(X)$ when $X = Y$.
- $l^\infty(\mathbb{Z}, X) = \{x : \mathbb{Z} \to X : \|x\| = \sup_{n\in\mathbb{Z}} \|x(n)\| < \infty\}$.

2.1. Piecewise Pseudo Almost Periodicity

Definition 2.1 [23]. A function $f \in \mathrm{C}(\mathbb{R}, X)$ is said to be almost periodic if for each $\varepsilon > 0$, there exists an $l(\varepsilon) > 0$, such that every interval J of length $l(\varepsilon)$ contains a number τ with the property that $\|f(t+\tau) - f(t)\| < \varepsilon$ for all $t \in \mathbb{R}$. Denote by $AP(\mathbb{R}, X)$, the set of such functions.

Definition 2.2 [24]. A sequence $\{x_n\}$ is called almost periodic if for any $\varepsilon > 0$, there exists a relatively dense set of its ε-periods, i.e., there exists a natural number $l = l(\varepsilon)$, such that for $k \in \mathbb{Z}$, there is at least one number p in $[k, k+l]$, for which inequality $\|x_{n+p} - x_n\| < \varepsilon$ holds for all $n \in \mathbb{N}$. Denote by $AP(\mathbb{Z}, X)$, the set of such sequences.

Define

$$PAP_0(\mathbb{Z}, X) = \left\{x \in l^\infty(\mathbb{Z}, X) : \lim_{n\to\infty} \frac{1}{2n} \sum_{k=-n}^{n} \|x(k)\| = 0\right\}.$$

Definition 2.3 [25]. A sequence $\{x_n\}_{n\in\mathbb{Z}} \in l^\infty(\mathbb{Z}, X)$ is called pseudo almost periodic if $x_n = x_n^1 + x_n^2$, where $x_n^1 \in AP(\mathbb{Z}, X)$, $x_n^2 \in PAP_0(\mathbb{Z}, X)$. Denote by $PAP(\mathbb{Z}, X)$ the set of such sequences.

For $\{t_i\}_{i\in\mathbb{Z}} \in T$, $\left\{t_i^j\right\}$ defined by

$$\left\{t_i^j = t_{i+j} - t_i\right\}, \quad i \in \mathbb{Z}, \quad j \in \mathbb{Z}.$$

It is easy to verify that the numbers t_i^j satisfy

$$t_{i+k}^j - t_i^j = t_{i+j}^k - t_i^k,\ t_i^j - t_i^k = t_{i+k}^{j-k} \quad \text{for } i, j, k \in \mathbb{Z}.$$

Definition 2.4 [24]. A function $f \in PC(\mathbb{R}, X)$ is said to be piecewise almost periodic if the following conditions are fulfilled:

(1) $\left\{t_i^j = t_{i+j} - t_i\right\}, i, j \in \mathbb{Z}$ are equipotentially almost periodic, that is, for any $\varepsilon > 0$, there exists a relatively dense set in $\mathbb{R}$ of ε-almost periods common for all of the sequences $\{t_i^j\}$.

(2) For any $\varepsilon > 0$, there exists a positive number $\delta = \delta(\varepsilon)$ such that if the points t' and t'' belong to the same interval of continuity of f and $|t' - t''| < \delta$, then $\|f(t') - f(t'')\| < \varepsilon$.
(3) For any $\varepsilon > 0$, there exists a relatively dense set Ω_ε in $\mathbb{R}$ such that if $\tau \in \Omega_\varepsilon$, then

$$\|f(t+\tau) - f(t)\| < \varepsilon$$

for all $t \in \mathbb{R}$ which satisfy the condition $|t - t_i| > \varepsilon, i \in \mathbb{Z}$.

We denote by $AP_T(\mathbb{R}, X)$ the space of all piecewise almost periodic functions. Obviously, $AP_T(\mathbb{R}, X)$ endowed with the supremum norm is a Banach space. Throughout the rest of this paper, we always assume that $\{t_i^j\}$ are equipotentially almost periodic. Let $\mathcal{U}PC(\mathbb{R}, X)$ be the space of all functions $f \in PC(\mathbb{R}, X)$ such that f satisfies the condition (2) in Definition 2.4.

Lemma 2.1 [24]. *If the sequences $\{t_i^j\}$ are equipotentially almost periodic, then for each $j > 0$, there exists a positive integer N such that on each interval of length j, there are no more than N elements of the sequence $\{t_i\}$, i.e.,*

$$i(t, s) \le N(t - s) + N,$$

where $i(t, s)$ is the number of the points $\{t_i\}$ in the interval $[s, t]$.

Definition 2.5. $f \in PC(\mathbb{R} \times \Omega, X)$ is said to be piecewise almost periodic in t uniformly in $x \in \Omega$ if for each compact set $K \subseteq \Omega$, $\{f(\cdot, x) : x \in K\}$ is uniformly bounded, and given $\varepsilon > 0$, there exists a relatively dense set Ω_ε such that $\|f(t+\tau, x) - f(t, x)\| \le \varepsilon$ for all $x \in K, \tau \in \Omega_\varepsilon$ and $t \in \mathbb{R}$, $|t - t_i| > \varepsilon$. Denote by $AP_T(\mathbb{R} \times \Omega, X)$ the set of all such functions.

Define

$$PC_T^0(\mathbb{R}, X) = \left\{ f \in PC(\mathbb{R}, X) : \lim_{t\to\infty} \|f(t)\| = 0 \right\},$$

$$PAP_T^0(\mathbb{R}, X) = \left\{ f \in PC(\mathbb{R}, X) : \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \|f(t)\| \mathrm{d}t = 0 \right\},$$

$$PAP_T^0(\mathbb{R} \times \Omega, X) = \left\{ f \in PC(\mathbb{R} \times \Omega, X) : \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \|f(t, x)\| \mathrm{d}t = 0 \right.$$
$$\text{uniformly with respect to } x \in K, \text{ where } K \text{ is an}$$
$$\left. \text{arbitrary compact subset of } \Omega\{ \right\}.$$

Definition 2.6. A function $f \in PC(\mathbb{R}, X)$ is said to be piecewise asymptotically almost periodic if it can be decomposed as $f = g + \varphi$, where $g \in AP_T(\mathbb{R}, X)$ and $\varphi \in PC_T^0(\mathbb{R}, X)$. Denote by $AAP_T(\mathbb{R}, X)$ the set of all such functions.

Definition 2.7 [14]. A function $f \in PC(\mathbb{R}, X)$ is said to be piecewise pseudo almost periodic if it can be decomposed as $f = g + \varphi$, where $g \in AP_T(\mathbb{R}, X)$

and $\varphi \in PAP_T^0(\mathbb{R}, X)$. Denote by $PAP_T(\mathbb{R}, X)$ the set of all such functions. $PAP_T(\mathbb{R}, X)$ is a Banach space when endowed with the supremum norm.

It follows from [1,14], one has

Remark 2.1. (i) $PAP_T^0(\mathbb{R}, X)$ is a translation invariant set of $PC(\mathbb{R}, X)$.
(ii) $PC_T^0(\mathbb{R}, X) \subset PAP_T^0(\mathbb{R}, X)$ and $AAP_T(\mathbb{R}, X) \subset PAP_T(\mathbb{R}, X)$.

Similarly as the proof of [4, Lemma 2.5], one has

Lemma 2.2. *Let $\{f_n\}_{n\in\mathbb{N}} \subset PAP_T^0(\mathbb{R}, X)$ be a sequence of functions. If f_n converges uniformly to f, then $f \in PAP_T^0(\mathbb{R}, X)$.*

Theorem 2.1 [14]. *Suppose the sequence of vector-valued functions $\{I_i\}_{i\in\mathbb{Z}}$ is pseudo almost periodic, i.e, for any $x \in \Omega$, $\{I_i(x), i \in \mathbb{Z}\}$ is a pseudo almost periodic sequence. Assume that the following conditions hold:*

(i) *$\{I_i(x), i \in \mathbb{Z}, x \in K\}$ is bounded for every bounded subset $K \subset \Omega$.*
(ii) *$I_i(x)$ is uniformly continuous in $x \in \Omega$ uniformly in $i \in \mathbb{Z}$.*

If $\phi \in PAP_T(\mathbb{R}, X) \cap \mathcal{U}PC(\mathbb{R}, X)$ such that $R(\phi) \subset \Omega$, then $I_i(\phi(t_i))$ is pseudo almost periodic, where $R(\phi)$ is the range of ϕ.

Corollary 2.1 [14]. *Assume that the sequence of vector-valued functions $\{I_i\}_{i\in\mathbb{Z}}$ is pseudo almost periodic, and there exists a constant $L_1 > 0$ such that*

$$\|I_i(u) - I_i(v)\| \leq L_1\|u - v\|, \quad \text{for all}\, u, v \in \Omega,\, i \in \mathbb{Z}.$$

if $\phi \in PAP_T(\mathbb{R}, X) \cap \mathcal{U}PC(\mathbb{R}, X)$ such that $R(\phi) \subset \Omega$, then $I_i(\phi(t_i))$ is pseudo almost periodic.

Definition 2.8 [14]. Let $PAP_T(\mathbb{R}\times\Omega, X)$ consist of all functions $f \in PC(\mathbb{R}\times\Omega, X)$ such that $f = g + \varphi$, where $g \in AP_T(\mathbb{R} \times \Omega, X)$ and $\varphi \in PAP_T^0(\mathbb{R} \times \Omega, X)$.

Similarly as the Definition 2.8, one can define the space $PAP_T(\mathbb{R}\times\Omega_1\times\Omega_2, X)$. Similarly as the proof of [14, Theorem 3.1], the composition theorems hold for piecewise pseudo almost periodic function.

Theorem 2.2. *Suppose $f \in PAP_T(\mathbb{R}\times\Omega_1\times\Omega_2, X)$. Assume that the following conditions hold:*

(i) *$\{f(t, u, v) : t \in \mathbb{R}, u \in K_1, v \in K_2\}$ is bounded for every bounded subset $K_1 \times K_2 \subseteq \Omega_1 \times \Omega_2$.*
(ii) *$f(t, \cdot, \cdot)$ is uniformly continuous in each bounded subset of $\Omega_1 \times \Omega_2$ uniformly in $t \in \mathbb{R}$.*

If $\varphi_1 \in PAP_T(\mathbb{R}, X)$, $\varphi_2 \in PAP_T(\mathbb{R}, X)$ such that $R(\varphi_1)\times R(\varphi_2) \subset \Omega_1\times\Omega_2$, then $f(\cdot, \varphi_1(\cdot), \varphi_2(\cdot)) \in PAP_T(\mathbb{R}, X)$, where $R(\varphi_1)$, $R(\varphi_2)$ is the range of φ_1, φ_2, respectively.

Corollary 2.2. *Let $f \in PAP_T(\mathbb{R} \times \Omega_1 \times \Omega_2, X)$, $\varphi_1 \in PAP_T(\mathbb{R}, X)$, $\varphi_2 \in PAP_T(\mathbb{R}, X)$ and $R(\varphi_1) \times R(\varphi_2) \subset \Omega_1 \times \Omega_2$. Assume that there exists a constant $L_f > 0$ such that*

$$\begin{aligned} &\|f(t,u_1,v_1)-f(t,u_2,v_2)\| \\ &\quad \le L_f(\|u_1-u_2\|+\|v_1-v_2\|),\ t\in\mathbb{R},\ u_1,u_2\in\Omega_1,\ v_1,v_2\in\Omega_2, \end{aligned}$$

then $f(\cdot,\varphi_1(\cdot),\varphi_2(\cdot))\in PAP_T(\mathbb{R},X)$.

2.2. Gronwall–Bellman Inequality and Compactness Criterion

First, we recall the definition of strong continuous evolution family and generalized Gronwall–Bellman inequality which will be used in the later.

Definition 2.9 [26]. A family of bounded linear operators $(U(t,s))_{t\ge s}$ on a Banach space X is called a strong continuous evolution family if

- (i) $U(t,r)U(r,s)=U(t,s)$ and $U(s,s)=I$ for all $t\ge r\ge s$ and $t,r,s\in\mathbb{R}$.
- (ii) The map $(t,s)\to U(t,s)x$ is continuous for all $x\in X, t\ge s$ and $t,s\in\mathbb{R}$.

Lemma 2.3 ([24] Generalized Gronwall–Bellman inequality). *Let a nonnegative function* $u(t)\in PC(\mathbb{R},X)$ *satisfy for* $t\ge t_0$

$$u(t)\le C+\int_{t_0}^{t} v(\tau)u(\tau)\mathrm{d}\tau+\sum_{t_0<t_i<t}\beta_i u(\tau_i),$$

where $C\ge 0$, $\beta_i\ge 0$, $v(\tau)>0$ *and* τ_i *are the first-kind discontinuity points of the functions* $u(t)$. *Then, the following estimate holds for the function* $u(t)$,

$$u(t)\le C\prod_{t_0<t_i<t}(1+\beta_i)\mathrm{e}^{\int_{t_0}^{t} v(\tau)\mathrm{d}\tau}.$$

Next, we recall a useful compactness criterion on $PC(\mathbb{R},X)$.

Let $h:\mathbb{R}\to\mathbb{R}^+$ be a continuous function such that $h(t)\ge 1$ for all $t\in\mathbb{R}$ and $h(t)\to\infty$ as $|t|\to\infty$. Define

$$PC_h^0(\mathbb{R},X):=\left\{f\in PC(\mathbb{R},X):\lim_{|t|\to\infty}\frac{f(t)}{h(t)}=0\right\}$$

endowed with the norm $\|f\|_h=\sup\limits_{t\in\mathbb{R}}\dfrac{\|f(t)\|}{h(t)}$, it is a Banach space.

Lemma 2.4 [14]. *A set* $B\subseteq PC_h^0(\mathbb{R},X)$ *is relatively compact if and only if it verifies the following conditions:*

1. $\lim_{|t|\to\infty}\dfrac{\|f(t)\|}{h(t)}=0$ *uniformly for* $f\in B$.
2. $B(t)=\{f(t):f\in B\}$ *is relatively compact in* X *for every* $t\in\mathbb{R}$.
3. *The set* B *is equicontinuous on each interval* (t_i,t_{i+1}) $(i\in\mathbb{Z})$.

3. Nonautonomous Impulsive Integro-Differential Equations

In this section, we investigate the existence, uniqueness, and stability of piecewise pseudo almost periodic mild solution of (1.1).

First, we make the following assumptions:

(H$_1$) There exists constants $\lambda_0 \geq 0, \theta \in (\frac{\pi}{2}, \pi), L, \widetilde{M} \geq 0$ and $\beta, \gamma \in (0,1)$ with $\beta + \gamma > 1$ such that

$$\Sigma_\theta \cup \{0\} \subset \rho(A(t) - \lambda_0), \quad \|R(\lambda, A(t) - \lambda_0)\| \leq \frac{\widetilde{M}}{1 + |\lambda|}$$

and

$$\|(A(t)-\lambda_0)R(\lambda, A(t)-\lambda_0)[R(\lambda_0, A(t))-R(\lambda_0, A(s))]\| \leq L|t-s|^\beta |\lambda|^{-\gamma}$$

for $t, s \in \mathbb{R}, \Sigma_\theta = \{\lambda \in \mathbb{C}\backslash\{0\} : |\arg\lambda| \leq \theta\}$.

(H$_2$) $R(\lambda_0, A(\cdot)) \in AP(\mathbb{R}, L(X))$.

(H$_3$) The evolution family $(U(t,s))_{t \geq s}$ generated by $A(t)$ is exponentially stable, i.e., there exist constants $M > 0$, $\omega > 0$ such that $\|U(t,s)\| \leq M\mathrm{e}^{-\omega(t-s)}$, $t \geq s$, $t, s \in \mathbb{R}$.

(H$_4$) For each $x \in X$, $U(t+h, t)x \to x$ as $h \to 0^+$ uniformly for $t \in \mathbb{R}$.

(H$_5$) $f \in PAP_T(\mathbb{R} \times \Omega_1 \times \Omega_2, X)$ and there exists a constant $L_f > 0$ such that

$$\|f(t, u_1, v_1) - f(t, u_2, v_2)\| \\ \leq L_f(\|u_1 - u_2\| + \|v_1 - v_2\|), \ t \in \mathbb{R}, \ u_1, u_2 \in \Omega_1, \ v_1, v_2 \in \Omega_2,$$

(H$_6$) $g \in PAP_T(\mathbb{R} \times \Omega_3, X)$ and there exists a constant $L_g > 0$ such that

$$\|g(t,u) - g(t,v)\| \leq L_g\|u - v\|, \quad t \in \mathbb{R}, \quad u, v \in \Omega_3.$$

(H$_7$) $I_i \in PAP(\mathbb{Z}, X)$ and there exists a constant $L_1 > 0$ such that

$$\|I_i(u) - I_i(v)\| \leq L_1\|u - v\|, \quad t \in \mathbb{R}, \quad u, v \in \Omega_1, i \in \mathbb{Z}.$$

(H$_8$) $k \in C(\mathbb{R}^+, \mathbb{R})$ and $|k(t)| \leq C_k \mathrm{e}^{-\eta t}$ for some positive constants C_k, η.

Remark 3.1. (H$_1$) is usually called "Acquistapace-Terreni" conditions, which was first introduced in [27] and widely used to study nonautonomous differential equations in [5,26–28]. If (H$_1$) holds, there exists a unique evolution family $(U(t,s))_{t \geq s}$ on X, which governs the homogeneous version of (1.1) [28].

Before starting our main results, we recall the definition of the mild solution to (1.1).

Definition 3.1 [24]. A function $u : \mathbb{R} \to X$ is called a mild solution of (1.1) if for any $t \in \mathbb{R}$, $t > \sigma$, $\sigma \neq t_i$, $i \in \mathbb{Z}$,

$$u(t) = U(t,\sigma)u(\sigma) + \int_\sigma^t U(t,s)f(s, u(s), (Ku)(s))\mathrm{d}s + \sum_{\sigma < t_i < t} U(t, t_i)I_i(u(t_i)). \tag{3.1}$$

Note that, if (H$_3$) holds, then (3.1) can be replaced by

$$u(t) = \int_{-\infty}^t U(t,s)f(s, u(s), (Ku)(s))\mathrm{d}s + \sum_{t_i < t} U(t, t_i)I_i(u(t_i)).$$

Lemma 3.1 [29]. *Assume that* (H_1)–(H_3) *hold, then for each* $\varepsilon > 0$ *and* $h > 0$, *there is a relatively dense set* $\Omega_{\varepsilon,h}$ *such that*

$$\|U(t+\tau, s+\tau) - U(t,s)\| \leq \varepsilon \mathrm{e}^{-\frac{\omega}{2}(t-s)}, \quad t-s > h, t,\, s \in \mathbb{R},\, \tau \in \Omega_{\varepsilon,h}.$$

This property can be abbreviated by writing $U \in AP(L(X))$.

Lemma 3.2 [24]. *Assume that* $f \in AP_T(\mathbb{R}, X)$, $U \in AP(L(X))$, *the sequence* $\{x_i\}_{i\in\mathbb{Z}} \in AP(\mathbb{Z}, X)$, *and* $\{t_i^j\}$, $j \in \mathbb{Z}$ *are equipotentially almost periodic. Then, for each* $\varepsilon > 0$, *there exist relatively dense sets* Ω_ε *of* $\mathbb{R}$ *and* Q_ε *of* $\mathbb{Z}$ *such that*

- (i) $\|f(t+\tau) - f(t)\| < \varepsilon$ *for all* $t \in \mathbb{R}$, $|t - t_i| > \varepsilon$, $\tau \in \Omega_\varepsilon$ *and* $i \in \mathbb{Z}$.
- (ii) $\|U(t+\tau, s+\tau) - U(t,s)\| \leq \varepsilon \mathrm{e}^{-\frac{\omega}{2}(t-s)}$ *for all* $t, s \in \mathbb{R}$, $|t-s| > 0$, $|s - t_i| > \varepsilon$, $|t - t_i| > \varepsilon$, $\tau \in \Omega_\varepsilon$ *and* $i \in \mathbb{Z}$.
- (iii) $\|x_{i+q} - x_i\| < \varepsilon$ *for all* $q \in Q_\varepsilon$ *and* $i \in \mathbb{Z}$.
- (iv) $|t_i^q - \tau| < \varepsilon$ *for all* $q \in Q_\varepsilon$, $\tau \in \Omega_\varepsilon$ *and* $i \in \mathbb{Z}$.

3.1. Lipschitz Case

In this subsection, we study the existence and uniqueness of piecewise pseudo almost periodic mild solution of (1.1) when f, g, I_i satisfy the Lipschitz condition, i.e., (H_5), (H_6), (H_7) hold.

Lemma 3.3. *Assume that* (H_1)–(H_3), (H_6), (H_8) *hold, if* $u \in PAP_T(\mathbb{R}, X)$, *then*

$$(Ku)(t) = \int_{-\infty}^{t} k(t-s) g(s, u(s)) \mathrm{d}s \in PAP_T(\mathbb{R}, X).$$

Proof. For $u \in PAP_T(\mathbb{R}, X)$, it is not difficult to see that $\phi(\cdot) = g(\cdot, u(\cdot)) \in PAP_T(\mathbb{R}, X)$ by Corollary 2.2. Let $\phi = \phi_1 + \phi_2$, where $\phi_1 \in AP_T(\mathbb{R}, X)$, $\phi_2 \in PAP_T^0(\mathbb{R}, X)$, then

$$\begin{aligned}(Ku)(t) &= \int_{-\infty}^{t} k(t-s)\phi(s)\mathrm{d}s = \int_{-\infty}^{t} k(t-s)\phi_1(s)\mathrm{d}s + \int_{-\infty}^{t} k(t-s)\phi_2(s)\mathrm{d}s \\ &:= \Psi_1(t) + \Psi_2(t).\end{aligned}$$

(i) $\Psi_1 \in AP_T(\mathbb{R}, X)$.

It is not difficult to see that $\Psi_1 \in \mathcal{U}PC(\mathbb{R}, X)$. Let $t_i < t \leq t_{i+1}$. For $\varepsilon > 0$. let Ω_ε be a relatively dense set of $\mathbb{R}$ formed by ε-periods of ϕ_1. For $\tau \in \Omega_\varepsilon$ and $0 < h < \min\{\varepsilon, \alpha/2\}$,

$$\begin{aligned}\|\Psi_1(t+\tau) - \Psi_1(t)\| &\leq \int_{-\infty}^{t} |k(t-s)| \|\phi_1(s+\tau) - \phi_1(s)\| \mathrm{d}s \\ &\leq \sum_{j=-\infty}^{i-1} \int_{t_j+h}^{t_{j+1}-h} |k(t-s)| \|\phi_1(s+\tau) - \phi_1(s)\| \mathrm{d}s\end{aligned}$$

$$+\sum_{j=-\infty}^{i-1}\int_{t_j}^{t_j+h}|k(t-s)|\|\phi_1(s+\tau)-\phi_1(s)\|\mathrm{d}s$$

$$+\sum_{j=-\infty}^{i-1}\int_{t_{j+1}-h}^{t_{j+1}}|k(t-s)|\|\phi_1(s+\tau)-\phi_1(s)\|\mathrm{d}s$$

$$+\int_{t_i}^{t}|k(t-s)|\|\phi_1(s+\tau)-\phi_1(s)\|\mathrm{d}s.$$

Since $\phi_1 \in AP_T(\mathbb{R}, X)$, one has

$$\|\phi_1(t+\tau)-\phi_1(t)\|<\varepsilon,\quad \text{for all } t\in[t_j+h,t_{j+1}-h] \text{ and } j\in\mathbb{Z}, j\le i,$$

then,

$$\begin{aligned}
&\sum_{j=-\infty}^{i-1}\int_{t_j+h}^{t_{j+1}-h}|k(t-s)|\|\phi_1(s+\tau)-\phi_1(s)\|\mathrm{d}s\\
&\le \varepsilon\sum_{j=-\infty}^{i-1}\int_{t_j+h}^{t_{j+1}-h}|k(t-s)|\mathrm{d}s\\
&\le \varepsilon C_k\sum_{j=-\infty}^{i-1}\int_{t_j+h}^{t_{j+1}-h}\mathrm{e}^{-\eta(t-s)}\mathrm{d}s\\
&\le \frac{\varepsilon C_k}{\eta}\sum_{j=-\infty}^{i-1}\mathrm{e}^{-\eta(t-t_{j+1}+h)}\\
&\le \frac{\varepsilon C_k}{\eta}\sum_{j=-\infty}^{i-1}\mathrm{e}^{-\eta\alpha(i-j-1)}\\
&\le \frac{\varepsilon C_k}{\eta(1-\mathrm{e}^{-\eta\alpha})}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\sum_{j=-\infty}^{i-1}\int_{t_j}^{t_j+h}|k(t-s)|\|\phi_1(s+\tau)-\phi_1(s)\|\mathrm{d}s\\
&\le 2C_k\|\phi_1\|\sum_{j=-\infty}^{i-1}\int_{t_j}^{t_j+h}\mathrm{e}^{-\eta(t-s)}\mathrm{d}s\\
&\le 2C_k\|\phi_1\|\varepsilon\mathrm{e}^{\eta h}\sum_{j=-\infty}^{i-1}\mathrm{e}^{-\eta(t-t_j)}
\end{aligned}$$

$$\le 2C_k\|\phi_1\|\varepsilon e^{\eta h} e^{-\eta(t-t_i)} \sum_{j=-\infty}^{i-1} e^{-\eta\alpha(i-j)}$$
$$\le \frac{2C_k\|\phi_1\| e^{(\eta\alpha)/2}\varepsilon}{1-e^{-\eta\alpha}}.$$

Similarly, one has

$$\sum_{j=-\infty}^{i-1} \int_{t_{j+1}-h}^{t_{j+1}} |k(t-s)|\|\phi_1(s+\tau)-\phi_1(s)\| ds \le N_1\varepsilon,$$

$$\int_{t_i}^{t} |k(t-s)|\|\phi_1(s+\tau)-\phi_1(s)\| ds \le N_2\varepsilon,$$

where N_1, N_2 are some positive constants. Hence, $\Psi_1 \in AP_T(\mathbb{R}, X)$.

(ii) $\Psi_2 \in PAP_T^0(\mathbb{R}, X)$.

In fact, for $r > 0$, one has

$$\frac{1}{2r}\int_{-r}^{r} \|\Psi_2(t)\| dt = \frac{1}{2r}\int_{-r}^{r} \left\| \int_{-\infty}^{t} k(t-s)\phi_2(s)ds \right\| dt$$
$$= \frac{1}{2r}\int_{-r}^{r} \left\| \int_{0}^{\infty} k(s)\phi_2(t-s)ds \right\| dt$$
$$\le \frac{1}{2r}\int_{-r}^{r}\int_{0}^{\infty} C_k e^{-\eta s} \|\phi_2(t-s)\| ds dt$$
$$\le \int_{0}^{\infty} C_k e^{-\eta s}\Phi_r(s) ds,$$

where

$$\Phi_r(s) = \frac{1}{2r}\int_{-r}^{r} \|\phi_2(t-s)\| dt.$$

Since $\phi_2 \in PAP_T^0(\mathbb{R}, X)$, it follows that $\phi_2(\cdot - s) \in PAP_T^0(\mathbb{R}, X)$ for each $s \in \mathbb{R}$ by Remark 2.1, hence $\lim_{r\to\infty} \Phi_r(s) = 0$ for all $s \in \mathbb{R}$. Using the Lebesgue's dominated convergence theorem, we have $\Psi_2 \in PAP_T^0(\mathbb{R}, X)$. This completes the proof. □

Theorem 3.1. *Assume that* (H_1)–(H_8) *hold and if*

$$\frac{ML_f(\eta + L_g C_k)}{\omega\eta} + \frac{ML_1}{1-e^{-\omega\alpha}} < 1,$$

then (1.1) *has a unique mild solution* $u \in PAP_T(\mathbb{R}, X)$.

Proof. Let $\Gamma : PAP_T(\mathbb{R}, X) \cap \mathcal{U}PC(\mathbb{R}, X) \to PC(\mathbb{R}, X)$ be the operator defined by

$$(\Gamma u)(t) = \int_{-\infty}^{t} U(t,s)f(s,u(s),(Ku)(s))\mathrm{d}s + \sum_{t_i<t} U(t,t_i)I_i(u(t_i)). \quad (3.2)$$

We will show that Γ has a fixed point in $PAP_T(\mathbb{R},X) \cap \mathcal{U}PC(\mathbb{R},X)$ and divide the proof into several steps.

(i) $\Gamma u \in \mathcal{U}PC(\mathbb{R},X)$.

Let $t', t'' \in (t_i, t_{i+1})$, $i \in \mathbb{Z}$, $t'' < t'$, $u \in PAP_T(\mathbb{R},X) \cap \mathcal{U}PC(\mathbb{R},X)$,

$$\begin{aligned}
&(\Gamma u)(t') - (\Gamma u)(t'') \\
&= \int_{-\infty}^{t'} U(t',s)f(s,u(s),(Ku)(s))\mathrm{d}s + \sum_{t_i<t'} U(t',t_i)I_i(u(t_i)) \\
&\quad - \int_{-\infty}^{t''} U(t'',s)f(s,u(s),(Ku)(s))\mathrm{d}s - \sum_{t_i<t''} U(t'',t_i)I_i(u(t_i)) \\
&= \int_{-\infty}^{t''} [U(t',s) - U(t'',s)]f(s,u(s),(Ku)(s))\mathrm{d}s \\
&\quad + \int_{t''}^{t'} U(t',s)f(s,u(s),(Ku)(s))\mathrm{d}s + \sum_{t_i<t''} [U(t',t_i) - U(t'',t_i)]I_i(u(t_i)).
\end{aligned} \quad (3.3)$$

Moreover,

$$\begin{aligned}
&\int_{-\infty}^{t''} [U(t',s) - U(t'',s)]f(s,u(s),(Ku)(s))\mathrm{d}s \\
&= \int_0^{\infty} [U(t',t''-s) - U(t'',t''-s)]f(t''-s,u(t''-s),(Ku)(t''-s))\mathrm{d}s \\
&= \int_0^{\infty} [U(t',t'')U(t'',t''-s) - U(t'',t''-s)]f(t''-s,u(t''-s),(Ku)(t''-s))\mathrm{d}s \\
&= \int_0^{\infty} [U(t',t'') - I]U(t'',t''-s)f(t''-s,u(t''-s),(Ku)(t''-s))\mathrm{d}s.
\end{aligned}$$

By (H_4), for any $\varepsilon > 0$, there exists $0 \le \delta < \dfrac{\varepsilon}{3M\|f\|}$ such that if t', t'' belongs to a same continuity and $0 < t' - t'' < \delta$, then

$$\|U(t',t'') - I\| \le \min\left\{\frac{\omega\varepsilon}{3M\|f\|}, \frac{(1-\mathrm{e}^{-\omega\alpha})\varepsilon}{3M\|I_i\|}\right\},$$

so

$$\left\| \int_{-\infty}^{t''} [U(t',s) - U(t'',s)] f(s,u(s),(Ku)(s)) \mathrm{d}s \right\|$$

$$\leq \int_0^{\infty} \|U(t',t'') - I\| \|U(t'',t''-s)\| \|f(t''-s,u(t''-s),(Ku)(t''-s))\| \mathrm{d}s$$

$$\leq \int_0^{\infty} \frac{\omega\varepsilon}{3M\|f\|} M\mathrm{e}^{-\omega s} \|f\| \mathrm{d}s$$

$$< \frac{\varepsilon}{3}, \tag{3.4}$$

and

$$\left\| \int_{t''}^{t'} U(t',s) f(s,u(s),(Ku)(s)) \mathrm{d}s \right\| \leq \int_{t''}^{t'} \|U(t',s)\| \|f(s,u(s),(Ku)(s))\| \mathrm{d}s$$

$$< \delta M \|f\| < \frac{\varepsilon}{3}. \tag{3.5}$$

Similarly,

$$\left\| \sum_{t_i < t''} [U(t',t_i) - U(t'',t_i)] I_i(u(t_i)) \right\|$$

$$\leq \left\| \sum_{t_i < t''} [U(t',t'') - I] U(t'',t_i) I_i(u(t_i)) \right\|$$

$$\leq \sum_{t_i < t''} \|U(t',t'') - I\| \|U(t'',t_i)\| \|I_i(u(t_i))\|$$

$$\leq \sum_{t_i < t''} \frac{(1-\mathrm{e}^{-\omega\alpha})\varepsilon}{3M\|I_i\|} M\mathrm{e}^{-\omega(t''-t_i)} \|I_i\|$$

$$< \frac{\varepsilon}{3}. \tag{3.6}$$

Hence, by (3.3)–(3.6), if t', t'' belongs to a same continuity and $0 < t' - t'' < \delta$, then

$$\|(\Gamma u)(t') - (\Gamma u)(t'')\| < \varepsilon,$$

which implies that $\Gamma u \in \mathcal{UPC}(\mathbb{R}, X)$.

(ii) $\Gamma u \in PAP_T(\mathbb{R}, X)$.

For $u \in PAP_T(\mathbb{R}, X) \cap \mathcal{UPC}(\mathbb{R}, X)$, by Lemma 3.3 and Corollary 2.2, $h(\cdot) = f(\cdot, u(\cdot), (Ku)(\cdot)) \in PAP_T(\mathbb{R}, X)$. By ($H_5$)–($H_7$), it is not difficult to see that $h(\cdot)$ is bounded function and $I_i(u(\cdot))$ is a bounded sequence. Similarly as the proof of Lemma 3.3, one has

$$\int_{-\infty}^{t} U(t,s) f(s,u(s),(Ku)(s)) \mathrm{d}s \in PAP_T(\mathbb{R}, X).$$

It remains to show that

$$\sum_{t_i<t} U(t,t_i)I_i(u(t_i)) \in PAP_T(\mathbb{R},X).$$

By Corollary 2.1, $I_i(u(t_i)) \in PAP(\mathbb{Z},X)$, then let $I_i(u(t_i)) = \beta_i + \gamma_i$, where $\beta_i \in AP(\mathbb{Z},X)$ and $\gamma_i \in PAP_0(\mathbb{Z},X)$, so

$$\sum_{t_i<t} U(t,t_i)I_i(u(t_i)) = \sum_{t_i<t} U(t,t_i)\beta_i + \sum_{t_i<t} U(t,t_i)\gamma_i := \Pi_1(t) + \Pi_2(t).$$

For any $\varepsilon > 0$, by Lemma 3.2, there exists relative dense sets of real numbers Ω_ε and integers Q_ε, such that for $t_i < t \le t_{i+1}$, $\tau \in \Omega_\varepsilon$, $q \in Q_\varepsilon$, $|t-t_i| > \varepsilon$, $|t-t_{i+1}| > \varepsilon$, $j \in \mathbb{Z}$, one has

$$t+\tau > t_i + \varepsilon + \tau > t_{i+q},$$

and

$$t_{i+q+1} > t_{i+1} + \tau - \varepsilon > t + \tau,$$

that is $t_{i+q} < t+\tau < t_{i+q+1}$, then

$$\begin{aligned}
\|\Pi_1(t+\tau) - \Pi_1(t)\| &= \left\| \sum_{t_i<t+\tau} U(t+\tau,t_i)\beta_i - \sum_{t_i<t} U(t,t_i)\beta_i \right\| \\
&\le \left\| \sum_{t_i<t} U(t+\tau,t_{i+q})\beta_{i+q} - \sum_{t_i<t} U(t+\tau,t_{i+q})\beta_i \right\| \\
&\quad + \left\| \sum_{t_i<t} U(t+\tau,t_{i+q})\beta_i - \sum_{t_i<t} U(t,t_i)\beta_i \right\| \\
&\le \sum_{t_i<t} \|U(t+\tau,t_{i+q})\| \|\beta_{i+q} - \beta_i\| \\
&\quad + \sum_{t_i<t} \|U(t+\tau,t_{i+q}) - U(t,t_i)\| \|\beta_i\| \\
&\le \sum_{t_i<t} M\mathrm{e}^{-\omega(t-t_i)}\varepsilon + \sum_{t_i<t} \varepsilon M_{\beta_i}\mathrm{e}^{-\frac{\omega}{2}(t-t_i)} \\
&\le \frac{M\varepsilon}{1-\mathrm{e}^{-\omega\alpha}} + \frac{M_{\beta_i}\varepsilon}{1-\mathrm{e}^{-\frac{\omega}{2}\alpha}},
\end{aligned}$$

where $M_{\beta_i} = \sup_{i\in\mathbb{Z}} \|\beta_i\|$. So $\Pi_1 \in AP_T(\mathbb{R},X)$.

Next, we show that $\Pi_2 \in PAP_T^0(\mathbb{R},X)$. For a given $i \in \mathbb{Z}$, define the function $g(t)$ by

$$g(t) = U(t,t_i)\gamma_i, \quad t_i < t \le t_{i+1},$$

then

$$\lim_{t\to\infty} \|g(t)\| = \lim_{t\to\infty} \|U(t,t_i)\gamma_i\| \le \lim_{t\to\infty} M\mathrm{e}^{-\omega(t-t_i)}\|\gamma_i\| = 0,$$

then $g \in PC_T^0(\mathbb{R},X) \subset PAP_T^0(\mathbb{R},X)$. Define $g_k : \mathbb{R} \to X$ by

$$g_k(t) = U(t,t_{i-k})\gamma_{i-k}, \quad t_i < t \le t_{i+1}, \quad k \in \mathbb{N}.$$

So $g_k \in PAP_T^0(\mathbb{R}, X)$. Moreover,

$$\begin{aligned}\|g_k(t)\| = \|U(t, t_{i-k})\gamma_{i-k}\| &\le M \sup_{i\in\mathbb{Z}} \|\gamma_i\| \mathrm{e}^{-\omega(t-t_{i-k})}\\ &\le M \sup_{i\in\mathbb{Z}} \|\gamma_i\| \mathrm{e}^{-\omega(t-t_i)} \mathrm{e}^{-\omega\alpha k}.\end{aligned}$$

Therefore, the series $\sum_{k=0}^{\infty} g_k$ is uniformly convergent on $\mathbb{R}$. By Lemma 2.2, one has

$$\Pi_2(t) = \sum_{t_i<t} U(t, t_i)\gamma_i = \sum_{k=0}^{\infty} g_k \in PAP_T^0(\mathbb{R}, X).$$

So $\Gamma u \in PAP_T(\mathbb{R}, X)$.

(iii) Γ is a contraction.

For $u, v \in PAP_T(\mathbb{R}, X) \cap \mathcal{U}PC(\mathbb{R}, X)$,

$$\begin{aligned}&\|(\Gamma u)(t) - (\Gamma v)(t)\|\\ &\le \int_{-\infty}^{t} \|U(t,s)\| \|f(s, u(s), (Ku)(s)) - f(s, v(s), (Kv)(s))\| \mathrm{d}s\\ &\quad + \sum_{t_i<t} \|U(t, t_i)\| \|I_i(u(t_i)) - I_i(v(t_i))\|\\ &\le \int_{-\infty}^{t} M\mathrm{e}^{-\omega(t-s)} \|f(s, u(s), (Ku)(s)) - f(s, v(s), (Kv)(s))\| \mathrm{d}s\\ &\quad + \sum_{t_i<t} M\mathrm{e}^{-\omega(t-t_i)} \|I_i(u(t_i)) - I_i(v(t_i))\|\\ &\le \left(\int_{-\infty}^{t} M\mathrm{e}^{-\omega(t-s)} L_f(1 + \frac{L_g C_k}{\eta}) \mathrm{d}s + \sum_{t_i<t} L_1 M \mathrm{e}^{-\omega(t-t_i)} \right) \|u - v\|\\ &\le \left(\frac{ML_f(\eta + L_g C_k)}{\omega\eta} + \frac{ML_1}{1 - \mathrm{e}^{-\omega\alpha}} \right) \|u - v\|.\end{aligned}$$

Since $\frac{ML_f(\eta+L_gC_k)}{\omega\eta} + \frac{ML_1}{1-\mathrm{e}^{-\omega\alpha}} < 1$, Γ is a contraction.

By (i), (ii), $\Gamma(PAP_T(\mathbb{R}, X) \cap \mathcal{U}PC(\mathbb{R}, X)) \subset PAP_T(\mathbb{R}, X) \cap \mathcal{U}PC(\mathbb{R}, X)$. Since (iii) holds, by the Banach contraction mapping principle, Γ has a unique fixed point in $PAP_T(\mathbb{R}, X) \cap \mathcal{U}PC(\mathbb{R}, X)$, which is the unique piecewise pseudo almost periodic mild solution to (1.1). □

3.2. Non-Lipschitz Case

In this subsection, by the Schauder fixed point theorem, we study the existence of piecewise pseudo almost periodic mild solution of (1.1) when (H_5'), (H_6'), (H_7') hold.

Theorem 3.2. *Assume that* (H_1)–(H_4), (H_8) *hold and satisfy the following conditions:*

(H_5') *$f \in PAP_T(\mathbb{R} \times \Omega_1 \times \Omega_2, X)$ and $f(t, \cdot, \cdot)$ is uniformly continuous in each bounded subset of $\Omega_1 \times \Omega_2$ uniformly in $t \in \mathbb{R}$.*

(H_6') *$g \in PAP_T(\mathbb{R} \times \Omega_3, X)$ and $g(t, \cdot)$ is uniformly continuous in each bounded subset of Ω_3 uniformly in $t \in \mathbb{R}$.*

(H_7') *$I_i \in PAP(\mathbb{Z}, X)$ and $I_i(x)$ is uniformly continuous in $x \in \Omega_1$ uniformly in $i \in \mathbb{Z}$.*

(H_9) *For any $L, \widetilde{L} > 0$, $C_{1L} = \sup_{t\in\mathbb{R},\|u\|\le L,\|v\|\le\widetilde{L}} \|f(t, u, v)\| < \infty$, $C_{2L} = \sup_{t\in\mathbb{R},\|u\|\le L} \|g(t, u)\| < \infty$, $C_{3L} = \sup_{i\in\mathbb{Z},\|u\|\le L} \|I_i(u)\| < \infty$. Moreover, there exists a constant $L_0 > 0$ such that $\frac{MC_{1L_0}}{\omega} + \frac{MC_{3L_0}}{1-\mathrm{e}^{-\omega\alpha}} \le L_0$.*

(H_{10}) *For fixed $t, s \in \mathbb{R}, t \ge s$, the operator $U(t, s) : X \to X$ is compact.*

Then, (1.1) has a mild solution $u \in PAP_T(\mathbb{R}, X)$.

Proof. Let $D = \{u \in PAP_T(\mathbb{R}, X) \cap \mathcal{U}PC(\mathbb{R}, X) : \|u\| \le L_0\}$. Define the operator Γ as in (3.2). We next show that Γ has a fixed point in D and divide the proof into several steps.

(i) For every $u \in D$, $\Gamma u \in PAP_T(\mathbb{R}, X) \cap \mathcal{U}PC(\mathbb{R}, X)$.

Since g satisfies (H_6'), (H_9), the result of Lemma 3.3 holds by Theorem 2.2. Similarly as the proof of Theorem 3.1, one has $\Gamma u \in PAP_T(\mathbb{R}, X) \cap \mathcal{U}PC(\mathbb{R}, X)$.

(ii) For every $u \in D$, $\|\Gamma u\| \le L_0$.

For every $u \in D$, by (H_3) and (H_9), one has

$$\begin{aligned}
&\|(\Gamma u)(t)\| \\
&\le \int_{-\infty}^{t} \|U(t, s)\|\|f(s, u(s), (Ku)(s))\|\mathrm{d}s + \sum_{t_i<t} \|U(t, t_i)\|\|I_i(u(t_i))\| \\
&\le \int_{-\infty}^{t} M\mathrm{e}^{-\omega(t-s)}\|f(s, u(s), (Ku)(s))\|\mathrm{d}s + \sum_{t_i<t} M\mathrm{e}^{-\omega(t-t_i)}\|I_i(u(t_i))\| \\
&\le C_{1L_0} \int_{-\infty}^{t} M\mathrm{e}^{-\omega(t-s)}\mathrm{d}s + C_{3L_0} \sum_{t_i<t} M\mathrm{e}^{-\omega(t-t_i)} \\
&\le \frac{MC_{1L_0}}{\omega} + \frac{MC_{3L_0}}{1 - \mathrm{e}^{-\omega\alpha}} \le L_0,
\end{aligned}$$

then $\|\Gamma u\| \le L_0$.

Combing (i) and (ii), it follows that $\Gamma D \subseteq D$.

(iii) Γ is continuous.

Let $u_n \subset D$, $u_n \to u$ as $n \to \infty$, then there exists a bounded subset $\widetilde{\Omega} \subseteq \Omega$ such that $R(u) \subseteq \widetilde{\Omega}$, $R(u_n) \subseteq \widetilde{\Omega}$, $n \in \mathbb{N}$. By (H_5'), (H_7'), for any $\varepsilon > 0$, there exists $\delta' > 0$ such that $u, v \in \widetilde{\Omega}$ and $\|u - v\| < \delta'$ implies that

$$\begin{aligned}
&\|f(t, u, Ku) - f(t, v, Kv)\| < \varepsilon \quad \text{for all } t \in \mathbb{R}, \\
&\|I_i(u) - I_i(v)\| < \varepsilon \quad \text{for all } i \in \mathbb{Z}.
\end{aligned}$$

For the above $\delta' > 0$, there exists n_0 such that $\|u_n(t) - u(t)\| < \delta'$ for all $n > n_0$, $t \in \mathbb{R}$, then for $n > n_0$, one has

$$\|f(t, u_n(t), (Ku_n)(t)) - f(t, u(t), (Ku)(t))\| < \varepsilon \quad \text{for all } t \in \mathbb{R},$$
$$\|I_i(u_n(t_i)) - I_i(u(t_i))\| < \varepsilon \quad \text{for all } i \in \mathbb{Z}.$$

Hence,

$$\begin{aligned}
&\|(\Gamma u_n)(t) - (\Gamma u)(t)\| \\
&\quad \le \int_{-\infty}^{t} \|U(t,s)\| \|f(s, u_n(s), (Ku_n)(s)) - f(s, u(s), (Ku)(s))\| \mathrm{d}s \\
&\qquad + \sum_{t_i < t} \|U(t, t_i)\| \|I_i(u_n(t_i)) - I_i(u(t_i))\| \\
&\quad \le \int_{-\infty}^{t} M\mathrm{e}^{-\omega(t-s)} \|f(s, u_n(s), (Ku_n)(s)) - f(s, u(s), (Ku)(s))\| \mathrm{d}s \\
&\qquad + \sum_{t_i < t} M\mathrm{e}^{-\omega(t-t_i)} \|I_i(u_n(t_i)) - I_i(u(t_i))\| \\
&\quad \le \int_{-\infty}^{t} M\mathrm{e}^{-\omega(t-s)} \varepsilon \mathrm{d}s + \sum_{t_i < t} M\mathrm{e}^{-\omega(t-t_i)} \varepsilon \\
&\quad \le \left(\frac{M}{\omega} + \frac{M}{1 - \mathrm{e}^{-\omega\alpha}} \right) \varepsilon,
\end{aligned}$$

which implies that Γ is continuous.

(iv) $B(t) = \{(\Gamma u)(t) : u \in D\}$ is a relatively compact subset of X in each $t \in \mathbb{R}$.

For each $t \in \mathbb{R}$, $0 < \varepsilon < 1$, $u \in D$, define

$$\begin{aligned}
(\Gamma^\varepsilon u)(t) &:= \int_{-\infty}^{t-\varepsilon} U(t,s) f(s, u(s), (Ku)(s)) \mathrm{d}s + \sum_{t_i < t-\varepsilon} U(t, t_i) I_i(u(t_i)) \\
&= U(t, t-\varepsilon) \left[\int_{-\infty}^{t-\varepsilon} U(t-\varepsilon, s) f(s, u(s), (Ku)(s)) \mathrm{d}s \right. \\
&\qquad \left. + \sum_{t_i < t-\varepsilon} U(t-\varepsilon, t_i) I_i(u(t_i)) \right] \\
&= U(t, t-\varepsilon)(\Gamma u)(t-\varepsilon).
\end{aligned}$$

Since $\{(\Gamma u)(t-\varepsilon) : u \in D\}$ is bounded in X and $U(t, t-\varepsilon)$ is compact by (H_{10}), so $\{(\Gamma^\varepsilon u)(t) : u \in D\}$ is a relatively compact subset of X. Moreover,

$$\begin{aligned}
&\|(\Gamma u)(t) - (\Gamma^{\varepsilon} u)(t)\| \\
&= \left\| \int_{t-\varepsilon}^{t} U(t,s) f(s,u(s),(Ku)(s)) \mathrm{d}s + \sum_{t-\varepsilon<t_i<t} U(t,t_i) I_i(u(t_i)) \right\| \\
&\leq \int_{t-\varepsilon}^{t} \|U(t,s)\| \|f(s,u(s),(Ku)(s))\| \mathrm{d}s + \sum_{t-\varepsilon<t_i<t} \|U(t,t_i)\| \|I_i(u(t_i))\| \\
&\leq \int_{t-\varepsilon}^{t} M\mathrm{e}^{-\omega(t-s)} \|f(s,u(s),(Ku)(s))\| \mathrm{d}s + \sum_{t-\varepsilon<t_i<t} M\mathrm{e}^{-\omega(t-t_i)} \|I_i(u(t_i))\| \\
&\leq \frac{\varepsilon M C_{1L_0}}{\omega} + \frac{\varepsilon M C_{3L_0}}{\alpha}.
\end{aligned}$$

So $\{(\Gamma u)(t) : u \in D\}$ is a relatively compact subset of X in each $t \in \mathbb{R}$.

By (i), $\{\Gamma u : u \in D\}$ is equipotentially continuous at each interval (t_i, t_{i+1}) ($i \in \mathbb{Z}$). Since $\{\Gamma u : u \in D\} \subset PC_h^0(\mathbb{R}, X)$, then $\{\Gamma u : u \in D\}$ is a relatively compact set by Lemma 2.4, then Γ is a compact operator. Since D is a closed convex set, by Schauder fixed point theorem, Γ has a fixed point u in D, which is the piecewise pseudo almost periodic mild solution of (1.1). □

3.3. Stability

In this subsection, we investigate the stability of a piecewise pseudo almost periodic solution of (1.1) using the generalized Gronwall–Bellman inequality (Lemma 2.3).

Theorem 3.3. *Assume that* (H_1)–(H_8) *and* $\frac{ML_f(\eta+L_gC_k)}{\omega\eta} + \frac{ML_1}{1-\mathrm{e}^{-\omega\alpha}} < 1$ *hold, then the piecewise pseudo almost periodic mild solution of* (1.1) *is exponentially stable if* $\frac{\ln(1+ML_1)}{\alpha} + \frac{ML_f(\eta+L_gC_k)}{\eta} < \omega$.

Proof. By Theorem 3.1, (1.1) has a mild solution $u(t) \in PAP_T(\mathbb{R}, X)$, for $t \in \mathbb{R}$, $t > \sigma$, $\sigma \neq t_i$, $i \in \mathbb{Z}$,

$$u(t) = U(t,\sigma)u(\sigma) + \int_{\sigma}^{t} U(t,s) f(s,u(s),(Ku)(s)) \mathrm{d}s + \sum_{\sigma<t_i<t} U(t,t_i) I_i(u(t_i)).$$

Let $u(t) = u(t,\sigma,\varphi)$ and $v(t) = v(t,\sigma,\psi)$ be two mild solutions of (1.1), then

$$\begin{aligned}
&\|u(t) - v(t)\| \\
&\leq \|U(t,\sigma)[u(\sigma) - v(\sigma)]\| \\
&\quad + \left\| \int_{\sigma}^{t} U(t,s)[f(s,u(s),(Ku)(s))\mathrm{d}s - f(s,v(s),(Kv)(s))] \mathrm{d}s \right\| \\
&\quad + \left\| \sum_{\sigma<t_i<t} U(t,t_i)[I_i((u(t_i)) - I_i(v(t_i))] \right\|
\end{aligned}$$

$$\leq \|U(t,\sigma)\|\|u(\sigma)-v(\sigma)\| + \int_{\sigma}^{t} \|U(t,s)\|\|f(s,u(s),(Ku)(s))\mathrm{d}s$$
$$-f(s,v(s),(Kv)(s))\|\mathrm{d}s + \sum_{\sigma<t_i<t} \|U(t,t_i)\|\|I_i((u(t_i)) - I_i(v(t_i))\|$$
$$\leq M\mathrm{e}^{-\omega(t-\sigma)}\|u(\sigma)-v(\sigma)\| + \int_{\sigma}^{t} M\mathrm{e}^{-\omega(t-s)} L_f\left(1+\frac{L_g C_k}{\eta}\right)\|u(s)-v(s)\|\,\mathrm{d}s$$
$$+ \sum_{\sigma<t_i<t} M\mathrm{e}^{-\omega(t-t_i)} L_1\|u(t_i)-v(t_i)\|,$$

then

$$\mathrm{e}^{\omega t}\|u(t)-v(t)\| \leq M\mathrm{e}^{\omega\sigma}\|u(\sigma)-v(\sigma)\| + \int_{\sigma}^{t} \frac{ML_f(\eta+L_gC_k)}{\eta}\mathrm{e}^{\omega s}\,\|u(s)-v(s)\|\,\mathrm{d}s + \sum_{\sigma<t_i<t} ML_1\mathrm{e}^{\omega t_i}\|u(t_i)-v(t_i)\|.$$

Let $y(t) = \mathrm{e}^{\omega t}\|u(t)-v(t)\|$, then

$$y(t) \leq My(\sigma) + \int_{\sigma}^{t} \frac{ML_f(\eta+L_gC_k)}{\eta} y(s)\mathrm{d}s + \sum_{\sigma<t_i<t} ML_1 y(t_i).$$

By Lemma 2.3, one has

$$\begin{aligned} y(t) &\leq My(\sigma)\prod_{\sigma<t_i<t}(1+ML_1)\mathrm{e}^{\int_{\sigma}^{t}\frac{ML_f(\eta+L_gC_k)}{\eta}\mathrm{d}s} \\ &= My(\sigma)\prod_{\sigma<t_i<t}(1+ML_1)\mathrm{e}^{\frac{ML_f(\eta+L_gC_k)}{\eta}(t-\sigma)} \\ &\leq My(\sigma)(1+ML_1)^{\frac{t-\sigma}{\alpha}}\mathrm{e}^{\frac{ML_f(\eta+L_gC_k)}{\eta}(t-\sigma)} \\ &= My(\sigma)\mathrm{e}^{\left[\frac{\ln(1+ML_1)}{\alpha}+\frac{ML_f(\eta+L_gC_k)}{\eta}\right](t-\sigma)}, \end{aligned}$$

that is

$$\|u(t)-v(t)\| \leq M\|u(\sigma)-v(\sigma)\|\mathrm{e}^{\left[\frac{\ln(1+ML_1)}{\alpha}+\frac{ML_f(\eta+L_gC_k)}{\eta}-\omega\right](t-\sigma)}.$$

Since $\frac{\ln(1+ML_1)}{\alpha} + \frac{ML_f(\eta+L_gC_k)}{\eta} - \omega < 0$, then the piecewise pseudo almost periodic mild solution of (1.1) is exponentially stable. □

4. Example

Consider the heat equations with Dirichlet conditions

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = \dfrac{\partial^2 u(t,x)}{\partial x^2} - 2u(t,x) + \left(\sin t + \sin\sqrt{2}t\right) u(t,x) \\ \quad +f(t,x,u(t,x),(Ku)(t,x)), \\ (Ku)(t,x) = \displaystyle\int_{-\infty}^{t} k(t-s)g(s,x,u(s,x))\mathrm{d}s,\ t\in\mathbb{R}, t\neq t_i,\ i\in\mathbb{Z},\ x\in[0,\pi], \\ \Delta u(t_i,x) = \beta_i u(t_i,x), i\in\mathbb{Z},\ x\in[0,\pi], \\ u(t,0)=u(t,\pi)=0,\ t\in\mathbb{R}, \end{cases} \tag{4.1}$$

where $f \in PAP_T(\mathbb{R}\times[0,\pi]\times L^2[0,\pi]\times L^2[0,\pi], L^2[0,\pi])$, $g \in PAP_T(\mathbb{R}\times[0,\pi]\times L^2[0,\pi], L^2[0,\pi])$, $t_i = i+\frac{1}{4}|\sin i+\sin\sqrt{2}i|$, $\beta_i \in PAP(\mathbb{Z},\mathbb{R})$. Note that, $\{t_i^j\}$, $i\in\mathbb{Z}$, $j\in\mathbb{Z}$ are equipotentially almost periodic and $\alpha = \inf_{i\in\mathbb{Z}}(t_{i+1}-t_i) > 0$, one can see [14,24] for more details.

Take $X = L^2[0,\pi]$ is equipped with its natural topology and define

$$D(A) = \{u\in L^2[0,\pi] : u''\in L^2[0,\pi], u(0)=u(\pi)=0\},$$
$$Au = u''-2u,\ \text{for all } u\in D(A).$$

Let $\varphi_n(t) = \sqrt{\dfrac{2}{\pi}}\sin(nt)$ for all $n\in\mathbb{N}$. It is well know that A is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ on $L^2[0,\pi]$ with $\|T(t)\| \leq \mathrm{e}^{-3t}$ for $t\geq 0$ [30]. Moreover,

$$T(t)\varphi = \sum_{n=1}^{\infty} \mathrm{e}^{-(n^2+2)t}\langle\varphi,\varphi_n\rangle\varphi_n,$$

for each $\varphi\in L^2[0,\pi]$.

Define a family of linear operators $A(t)$ by

$$D(A(t)) = D(A),$$
$$A(t)\varphi(x) = \left(A+\sin t+\sin\sqrt{2}t\right)\varphi(x), \quad \forall x\in[0,\pi],\ \varphi\in D(A).$$

Then, the system

$$u'(t) = A(t)u(t), \quad t\geq s,$$
$$u(s) = \varphi\in L^2[0,\pi],$$

has an associated evolution family $(U(t,s))_{t\geq s}$ on $L^2[0,\pi]$, which can be explicitly express by

$$U(t,s)\varphi = T(t-s)\mathrm{e}^{\int_s^t(\sin\tau+\sin\sqrt{2}\tau)\mathrm{d}\tau}\varphi.$$

Moreover,

$$\|U(t,s)\| \leq \mathrm{e}^{-(t-s)} \quad \text{for every } t\geq s.$$

Note that, $\sin t + \sin\sqrt{2}t \in AP(\mathbb{R},\mathbb{R})$ and it is not difficult to verify that $A(t)$ satisfy (H_1)–(H_4) with $M = 1, \omega = 1$. One can see [4] for more details. Since $I_i(u) = \beta_i u$ and $\beta_i \in PAP(\mathbb{Z},\mathbb{R})$, then (H_7) holds with $L_1 = \sup_{i\in\mathbb{Z}} \|\beta_i\|$.

Now, the following theorem is an immediate consequence of Theorem 3.3.

Theorem 4.1. *Under the assumptions* (H_5), (H_6), (H_8), (4.1) *admits an exponentially stable mild solution* $u(t) \in PAA_T(\mathbb{R}, L^2[0,\pi])$ *if* $\Theta := \frac{\ln(1+L_1)}{\alpha} + \frac{L_f(\eta+L_gC_k)}{\eta} + \frac{L_1}{1-\mathrm{e}^{-\alpha}} < 1$.

Note that, in the above example, if

$$\begin{aligned} g(t,x,u(t,x)) &= \frac{1}{16}(\sin t + \sin\sqrt{2}t + \mathrm{e}^{-t^2\cos^2 t} + h(t))u(t,x),\\ f(t,x,u(t,x),(Ku)(t,x)) &= \frac{1}{16}(\sin t + \sin\sqrt{2}t + \mathrm{e}^{-t^2\cos^2 t} + h(t))u(t,x)\\ &\quad +(Ku)(t,x), \end{aligned}$$

where $h \in \mathcal{U}PC(\mathbb{R},\mathbb{R})$ satisfies $|h(t)| \le 1$ and $\lim_{r\to\infty} \frac{1}{2lr}\int_{-r}^{r} |h(t)|\mathrm{d}t = 0$. Then, (H_5), (H_6) hold with $L_g = \frac{1}{4}$, $L_f = \max(\frac{1}{4}, \frac{1}{4}C_k\eta^{-1})$, so the conclusion of Theorem 4.1 holds if it satisfies (H_8) and $\Theta < 1$.

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