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A New Result on Multiplicity of Nontrivial Solutions for the Nonhomogenous Schrödinger–Kirchhoff Type Problem in R^N

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Abstract. In this paper, we consider the following nonhomogenous Schrödinger–Kirchhoff type problem

$$\begin{cases} -\left(a+b\int_{R^N}|\nabla u|^2dx\right)\triangle u+V(x)u=f(x,u)+g(x), & \text{for } x\in R^N,\\ u(x)\to 0, & \text{as } |x|\to\infty, \end{cases} \tag{0.1}$$

where constants $a>0, b\geq 0$, $N=1,2$ or 3, $V\in C(R^N,R)$, $f\in C(R^N\times R,R)$ and $g\in L^2(R^N)$. Under more relaxed assumptions on the nonlinear term f that are much weaker than those in Chen and Li (Nonlinear Anal RWA 14:1477–1486, 2013), using some new proof techniques especially the verification of the boundedness of Palais–Smale sequence, a new result on multiplicity of nontrivial solutions for the problem (1.1) is obtained, which sharply improves the known result of Theorem 1.1 in Chen and Li (Nonlinear Anal RWA 14:1477–1486, 2013).

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Keywords. Nonhomogenous Schrödinger–Kirchhoff type problem, Ekeland's variational principle, mountain Pass Theorem.

1. Introduction and Main Results

In this paper, we consider the following nonhomogenous Schrödinger–Kirchhoff type problem

$$\begin{cases} -\left(a+b\int_{R^N}|\nabla u|^2dx\right)\triangle u+V(x)u=f(x,u)+g(x), & \text{for } x\in R^N,\\ u(x)\to 0, & \text{as } |x|\to\infty, \end{cases} \tag{1.1}$$

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where constants $a > 0, b \geq 0, N = 1, 2$ or 3, $V \in C(R^N, R)$, $f \in C(R^N \times R, R)$ and $g \in L^2(R^N)$ satisfy some further conditions.

We note that when $a = 1,\ b = 0$, and $g \equiv 0$, the problem (1.1) reduces to the following semilinear Schrödinger equation

$$\begin{cases} -\triangle u + V(x)u = f(x,u), & \text{for } x \in R^N, \\ u(x) \to 0, & \text{as } |x| \to \infty, \end{cases} \tag{1.2}$$

which has been studied extensively by many authors, and there is a large literature on the existence and multiplicity of solutions for the Eq. (1.2), for example, we refer the reader to [1–3] and references therein.

When $V \equiv 0,\ g \equiv 0$ and R^N is replaced by a bounded domain $\Omega \subset R^N$, the problem (1.1) reduces to the following nonlocal Kirchhoff type problem

$$\begin{cases} -(a + b\int_\Omega |\nabla u|^2 dx)\triangle u = f(x,u), & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

The problem (1.3) is related to the stationary analog of the Kirchhoff equation

$$u_{tt} - \left(a + b \int_{R^N} |\nabla u|^2 dx \right) \triangle u = g(x,t), \tag{1.4}$$

which was proposed by Kirchhoff in [4] as a model given by the equation of elastic strings

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L} \int_0^L |\frac{\partial u}{\partial x}|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0. \tag{1.5}$$

The Eq. (1.5) is an extension of the classical D'Alembert's wave equation by taking into account the changes in the length of the string during the transverse vibrationa.

It was pointed out in [5] that Kirchhoff type problem (1.3) models several physical and biological systems, where u describes a process which depends on the average of itself (for example, population density). Moreover, a lot of interesting studies by variational methods can be found in [6–15] for Kirchhoff type problem (1.3) on bounded domain with several growth conditions on f.

In the recently years, Kirchhoff type problems setting on the unbounded domain or whole space R^N have also been attracted a lot of attention. In [16], by means of Fountain Theorem, Jin and Wu obtained the existence of infinitely many radial solutions for the problem (1.3) in R^N. In [17], Wu got a sequence of high-energy solutions for the problem (1.3) in R^N via Symmetric Mountain Pass Theorem. These results had been subsequently unified and shapely improved by Ye and Tang in [18] using minimax methods in critical point theory, and they also obtained infinitely many small energy solutions for the problem (1.3). In [19], the authors established an abstract theorem concerning multiple critical points of a class of functionals involving local and nonlocal nonlinearity. As an application, they studied the problem (1.3) in R^N assuming on the local nonlinearity the general hypotheses introduced by

Berestycki and Lions in [20]. In [21], Nie and Wu considered the following Schrödinger–Kirchhoff type problem with radial potential

$$\begin{cases} -(a+b\int_\Omega |\nabla u|^2dx)\Delta u + V(|x|)u = Q(|x|)f(u), & \text{in } R^N; \\ u(x)\to 0, & \text{as } |x|\to\infty. \end{cases} \tag{1.6}$$

Four existence theorems of nontrivial solutions and a sequence of high-energy solutions for the problem (1.6) were obtained by Mountain Pass Theorem and Symmetric Mountain Pass Theorem. These results had been subsequently generalized by Wang in [22] to the p-Schrödinger–Kirchhoff type problem case. In [23], Alves and Figueiredo studied a periodic Kirchhoff equation in R^N with nonlinear perturbations, and the existence of positive solutions was obtained for the subcritical nonlinearity case and critical nonlinearity case. In [24], using minimax theorems and Lusternik–Schnirelmann theory, He and Zou studied the multiplicity and concentration behavior of positive solutions for the following Kirchhoff type equation

$$\begin{cases} -(\epsilon^2 a+\epsilon b\int_\Omega |\nabla u|^2dx)\Delta u + V(x)u = f(u), & \text{in } R^3; \\ u\in\ H^1(R^3), \quad u>0, & \text{in } R^3, \end{cases} \tag{1.7}$$

where $\epsilon > 0$ is a parameter, a and b are positive constants, and f is a continuous superlinear and subcritical nonlinearity. Soon after, with the aid of the Nehari manifold methods and minimax methods, Wang et al. generalized this result to the critical nonlinearity case in [25]. Recently, without the usual compactness conditions, the authors studied the existence of a positive solution for the following Kirchhoff type problem in [26]:

$$\left(a+\lambda b\int_{R^N}|\nabla u|^2+\lambda b\int_{R^N}u^2\right)[-\triangle u+bu]=f(u), \quad \text{in } R^N. \tag{1.8}$$

To overcome the lack of compactness, they used the "monotonicity trick" introduced by Jeanjean in [27] to construct a cut-off function to obtain the bounded Palais–Smale sequences for the problem (1.8).

For the nonhomogenous Schrödinger–Kirchhoff type problem (1.1), there is little known result of the existence and multiplicity of solutions except for [28]. In [28], by applying Ekeland's variational principle and Mountain Pass Theorem, Chen and Li studied the problem (1.1) with the nonlinearity f satisfying the Ambrosetti–Rabinowitz type condition, and the existence of two solutions was obtained. Precisely, they assumed the following assumptions.

(V_1) $V \in C(R^N, R)$ satisfies $\inf V(x) \geq V_0 > 0$ and for each $M > 0$, $meas\{x \in R^N : V(x) \leq M\} < +\infty$, where V_0 is a constant and $meas$ denote the Lebesgue measure in R^N.

(f_1') $f \in C(R^N \times R, R)$ and

$$|f(x,t)|\leq C(1+|t|^{p-1}) \text{ for some } 2<p<2^*=\begin{cases}\frac{2N}{N-2}, & N>2;\\ +\infty, & N\leq 2,\end{cases}$$

where C is a positive constant.

(f_2) $f(x,t) = o(|t|)$ as $|t| \to 0$ uniformly in $x \in R^N$.

(f_3')

$$\inf_{x\in R^N,|t|=1} F(x,t) > 0,$$

where $F(x,t) = \int_0^t f(x,s)ds$.

(f_4') There exists $\mu > 4$ such that

$$\mu F(x,t) - f(x,t)t \le 0, \qquad \forall (x,t) \in R^N \times R.$$

We restate the corresponding result in [28] as in the following.

Theorem A (see [28], Theorem 1.1). *Assume that* $g \in L^2(R^N)$, $g \not\equiv 0$, (V_1), (f_1'), (f_2) *and* (f_3')–(f_4') *hold. Then there exists a constant* $m_0 > 0$ *such that the Eq.* (1.1) *has at least two different solutions in* E *whenever* $\|g\|_{L^2} < m_0$.

Remark 1. (i) Since the problem (1.1) is defined in R^N which is unbounded, the lack of compactness of the Sobolev embedding becomes more delicate using variational techniques. To overcome the lack of compactness, the condition (V_1), which was firstly introduced by Bartsch and Wang in [29], is always assumed to preserve the compactness of embedding of the working space.

(ii) It is worth pointing out that the combination of (f_3')–(f_4') implies the range of p in condition (f_1') should be $4 < p < 2^*$. Precisely, for any $x \in R^N$, $z \in R$, define

$$h(t) := F(x, t^{-1}z)t^{\mu}, \qquad \forall\, t \in [1, +\infty).$$

Then, for $|z| \ge 1$ and $t \in [1, |z|]$, (f_4') implies that

$$h'(t) = t^{\mu-1}\left[\mu F(x, t^{-1}z) - t^{-1}zf(x, t^{-1}z)\right] \le 0.$$

Hence, $h(1) \ge h(|z|)$. Therefore, (f_3') implies that

$$F(x,z) \ge F(x, \frac{z}{|z|})|z|^{\mu} \ge c|z|^{\mu}, \qquad \forall\, x \in R^N \text{ and } |z| \ge 1, \tag{1.9}$$

where $c = \inf_{x\in R^N,|t|=1} F(x,t) > 0$. If $p \le 4$, by (f_1'), we have

$$|F(x,t)| \le \int_0^1 |f(x,st)t|ds \le C\int_0^1 (1 + |st|^{p-1})|t|ds \le C(|t| + |t|^p)$$

for all $(x,t) \in R^N \times R$, which implies that

$$\limsup_{|t|\to+\infty} \frac{F(x,t)}{t^4} \le C \quad \text{uniformly in } x \in R^N.$$

This contradicts (1.9). Hence $4 < p < 2^*$.

Motivated by the works mentioned above, in the present paper, we shall consider the nonhomogeneous Schrödinger–Kirchhoff type problem, and we are interested in looking for multiple solutions of the problem (1.1). Under more relaxed assumptions on the nonlinear term f that are much weaker than those in [28], using some new proof techniques especially the verification of the boundedness of Palais–Smale sequence, a new result on multiplicity of nontrivial solutions for the problem (1.1) is obtained, which sharply improves the result of [28].

To obtain the multiplicity of solutions for the nonhomogeneous Schrödinger–Kirchhoff type problem (1.1) in R^N, we make the following assumptions.

(f_1) $f \in C(R^N \times R, R)$ and

$$|f(x,t)| \leq C(1+|t|^{p-1}) \texttt{ for some } 4 < p < 2^* = \begin{cases} \frac{2N}{N-2}, & N > 2; \\ +\infty, & N \leq 2, \end{cases}$$

where C is a positive constant.

(f_3) $\frac{F(x,t)}{t^4} \to +\infty$ as $|t| \to +\infty$ uniformly in $x \in R^N$.

(f_4) There exist $L > 0$ and $d \in [0, \frac{V_0}{2}]$ such that

$$4F(x,t) - f(x,t)t \leq d|t|^2, \quad \text{for a.e. } x \in R^N \text{ and } \forall |t| \geq L.$$

Next, we give some notations. Define the function space

$$H^1(R^N) = \{u \in L^2(R^N) : \nabla u \in L^2(R^N)\}$$

with the norm

$$\|u\|_{H^1} = \left(\int_{R^N} (|\nabla u|^2 + u^2)dx \right)^{\frac{1}{2}}.$$

Denote

$$E = \left\{ u \in H^1(R^N) : \int_{R^N} (|\nabla u|^2 + V(x)u^2)dx < +\infty \right\}$$

with the inner product and the norm

$$\langle u, v\rangle_E = \int_{R^N} (\nabla u \cdot \nabla v + V(x)uv)dx, \quad \|u\|_E = \langle u, u\rangle_E^{\frac{1}{2}}.$$

Obviously, the following embedding

$$E \hookrightarrow L^s(R^N), \quad 2 \leq s \leq 2^*$$

is continuous. Hence, for any $s \in [2, 2^*]$, there is a constant $a_s > 0$ such that

$$\|u\|_{L^s} \leq a_s \|u\|_E. \tag{1.10}$$

It is well known that a weak solution for the problem (1.1) is a critical point of the following functional I defined on E by

$$\begin{aligned} I(u) = {} & \frac{a}{2} \int_{R^N} |\nabla u|^2 dx + \frac{b}{4} \left(\int_{R^N} |\nabla u|^2 dx \right)^2 \\ & + \frac{1}{2} \int_{R^N} V(x)u^2 dx - \int_{R^N} F(x,u)dx - \int_{R^N} g(x)udx \end{aligned} \tag{1.11}$$

for all $u \in E$. We say that a weak solution $u \in E$ for the problem (1.1) is a negative energy solution if the energy $I(u) < 0$, and a weak solution $v \in E$ for the problem (1.1) is a positive energy solution if the energy $I(v) > 0$.

Now, we can state our result as follows.

Theorem 1.1. *Assume that $g \in L^2(R^N)$, $g \not\equiv 0$, (V_1) and (f_1)–(f_4) hold. Then, there exists a constant $g_0 > 0$ such that the problem (1.1) has at least two different solutions in E whenever $\|g\|_{L^2} < g_0$, one is negative energy solution, and the other is positive energy solution.*

Remark 2. Theorem 1.1 sharply improves Theorem A. In fact, (f_3)–(f_4) are much weaker than (f_3')–(f_4'). To be precise, by (f_3')–(f_4'), the inequality (1.9) in Remark 1 holds. Hence,

$$\frac{F(x,t)}{t^4} \geq c|t|^{\mu-4}, \qquad \forall x \in \ R^N \ \text{and} \ \ |t| \geq 1,$$

which implies (f_3). Moreover, note that $\mu > 4$, then (f_4') and (1.9) imply

$$\begin{aligned} 4F(x,t) - f(x,t)t &= \mu F(x,t) - f(x,t)t + (4-\mu)F(x,t) \leq (4-\mu)F(x,t) \\ &\leq (4-\mu)c|t|^{\mu} < 0 \leq d|t|^2 \end{aligned}$$

for all $x \in R^N$ and $|t| \geq 1$. This shows (f_4) holds by taking $L = 1$. Consequently, (f_3')–(f_4') imply (f_3)–(f_4). Thus, Theorem 1.1 sharply improves Theorem A.

In Theorem 1.1, we consider the case $\mu = 4$. For the case $\mu > 4$, we also have the following result about the existence of one negative energy solution, one positive energy solution for the nonhomogeneous Schrödinger–Kirchhoff type problem (1.1) in R^N, which is a corollary of Theorem 1.1 and more general than Theorem A. To begin with, we need the following assumptions.

(f_3'') There exists $L' > 0$ such that

$$c' = \inf_{x \in R^N, |t| = L'} F(x,t) > 0.$$

(f_4'') There exist $\mu > 4$ and $d' \in [0, \frac{c'(\mu-2)}{L'^2})$ such that

$$\mu F(x,t) - f(x,t)t \leq d'|t|^2, \quad \text{for a.e. } x \in R^N \text{ and } \forall |t| \geq L'.$$

Now, we can state the Corollary as follows.

Corollary 1. *If we replace (f_3)–(f_4) with (f_3'')–(f_4'') in Theorem 1.1, then the conclusion of Theorem 1.1 remains valid.*

The paper is organized as follows. In Sect. 2, we present some lemmas, which are bases of Sects. 3 and 4. In Sect. 3, we study the existence of the negative energy solution for problem (1.1). In Sect. 4, we obtain the existence of the positive energy solution and prove Theorem 1.1 and Corollary 1.

2. Some Lemmas

To apply variational techniques, we first state the key compactness result.

Lemma 2.1 (Lemma 3.4 in [30]). *Under the assumption* (V_1)*, the embedding*

$$E \hookrightarrow L^s(R^N), \quad 2 \le s < 2^*$$

is compact.

The following lemma can be obtained by a similar argument as Lemma 1 in [17] or Lemma 1 in [18].

Lemma 2.2. *Assume that* $g \in L^2(R^N)$, (V_1) *and* (f_1)*–*(f_2) *hold. Then* I *is well defined on* E, $I \in C^1(E, R)$ *and for any* $u,\ v \in E$,

$$\langle I'(u), v\rangle = \left(a + b\int_{R^N} |\nabla u|^2 dx\right)\int_{R^N} \nabla u \nabla v dx + \int_{R^N} V(x)uvdx - \int_{R^N} f(x,u)vdx - \int_{R^N} g(x)vdx. \tag{2.1}$$

Moreover, $\Psi' : E \to E^*$ *is compact, where* $\Psi(u) = \int_{R^N} F(x,u)dx - \int_{R^N} g(x) udx$.

Lemma 2.3. *Assume that* $g \in L^2(R^N)$ *and* (f_1)*–*(f_2) *hold. Then there exist some constants* ρ, α *and* $\beta > 0$ *such that* $I(u) \ge \alpha$ *whenever* $\|u\|_E = \rho$ *and* $\|g\|_{L^2} < \beta$.

Proof. For any $\epsilon > 0$, by (f_1)–(f_2), there exists $C_\epsilon > 0$ such that

$$|f(x,t)| \le \epsilon|t| + C(\epsilon)|t|^{p-1}, \quad \forall (x,t) \in R^N \times R, \tag{2.2}$$

$$|F(x,t)| \le \int_0^1 |f(x,st)t|ds \le \epsilon|t|^2 + C_\epsilon|t|^p, \quad \forall (x,t) \in R^N \times R. \tag{2.3}$$

By (1.10), (1.11), (2.3) and Hölder inequality,

$$\begin{aligned} I(u) &= \frac{a}{2}\int_{R^N} |\nabla u|^2 dx + \frac{b}{4}\left(\int_{R^N} |\nabla u|^2 dx\right)^2 + \frac{1}{2}\int_{R^N} V(x)u^2 dx \\ &\quad - \int_{R^N} F(x,u)dx - \int_{R^N} g(x)udx \\ &\ge \frac{\min\{a,1\}}{2}\|u\|_E^2 - \int_{R^N} F(x,u)dx - \int_{R^N} g(x)udx \\ &\ge \frac{\min\{a,1\}}{2}\|u\|_E^2 - (\epsilon\|u\|_{L^2}^2 + C_\epsilon\|u\|_{L^p}^p) - \|g\|_{L^2}\|u\|_{L^2} \\ &\ge \frac{\min\{a,1\}}{2}\|u\|_E^2 - a_2^2\epsilon\|u\|_E^2 - a_p^pC_\epsilon\|u\|_E^p - a_2\|g\|_{L^2}\|u\|_E \\ &= \|u\|_E\Big[\Big(\frac{\min\{a,1\}}{2} - a_2^2\epsilon\Big)\|u\|_E - a_p^pC_\epsilon\|u\|_E^{p-1} - a_2\|g\|_{L^2}\Big]. \end{aligned}$$

Choose $\epsilon = \frac{\min\{a,1\}}{4a_2^2} > 0$, and take

$$h(t) = \frac{\min\{a,1\}}{4}t - a_p^p C_\epsilon t^{p-1}, \quad \forall t \geq 0.$$

Note that $4 < p < 2^*$, we can conclude that there exists a constant $\rho > 0$ such that

$$h(\rho) = \max_{t\geq 0} h(t) > 0.$$

Therefore, take $\beta := \frac{1}{2a_2}h(\rho) > 0$, it has

$$I(u) \geq \frac{1}{2}\rho h(\rho) =: \alpha > 0$$

whenever $\|u\|_E = \rho$ and $\|g\|_{L^2} < \beta$. This completes the proof. □

Lemma 2.4. *Let assumptions (f_1)–(f_3) hold. Then there exists a function $e \in E$ with $\|e\|_E > \rho$ such that $I(e) < 0$.*

Proof. For every $M > 0$, by (f_1)–(f_3), there exists $C(M) > 0$ such that

$$F(x,t) \geq M|t|^4 - C(M)|t|^2, \qquad \forall\ (x,t) \in R^N \times R. \tag{2.4}$$

Choose $\phi \in E$ with $\|\phi\|_{L^4} = 1$, then (1.11), (2.4) and Hölder inequality imply that

$$\begin{aligned} I(t\phi) &= \frac{at^2}{2}\int_{R^N} |\nabla\phi|^2 dx + \frac{bt^4}{4}\left(\int_{R^N} |\nabla\phi|^2 dx\right)^2 + \frac{t^2}{2}\int_{R^N} V(x)\phi^2 dx \\ &\quad - \int_{R^N} F(x,t\phi)dx - t\int_{R^N} g(x)\phi dx \\ &\leq \frac{\max\{a,1\}t^2}{2}\|\phi\|_E^2 + \frac{b}{4}\|\phi\|_E^4 t^4 - M\|\phi\|_{L^4}^4 t^4 + C(M)\|\phi\|_{L^2}^2 t^2 \\ &\quad -t\int_{R^N} g(x)\phi dx \\ &\leq -\Big(M - \frac{b}{4}\|\phi\|_E^4\Big)t^4 + \Big[\frac{\max\{a,1\}}{2}\|\phi\|_E^2 + C(M)\|\phi\|_{L^2}^2\Big]t^2 \\ &\quad +\|g\|_{L^2}\|\phi\|_{L^2}t, \end{aligned}$$

which implies $I(t\phi) \to -\infty$ as $t \to +\infty$ by taking $M > \frac{b}{4}\|\phi\|_E^4$. Hence, there exists $e = t_0\phi$ with t_0 large enough such that $\|e\|_E > \rho$ and $I(e) < 0$. The proof is completed. □

Recall that, we say I satisfies the (PS) condition at the level $c \in R$ ($(PS)_c$ condition for short) if any sequence $\{u_n\} \subset E$ along with $I(u_n) \to c$ *and* $I'(u_n) \to 0$ *as* $n \to \infty$ possesses a convergent subsequence. If I satisfies $(PS)_c$ condition for each $c \in R$, then we say that I satisfies the (PS) condition.

Lemma 2.5. *Let assumption (V_1) and (f_1)–(f_2) hold. Then any bounded Palais–Smale sequence of I has a strongly convergent subsequence in E.*

Proof. Let $\{u_n\} \subset E$ be any bounded Palais–Smale sequence of I. Then, up to a subsequence, there exists $c_1 \in R$ such that

$$I(u_n) \to c_1, \quad I'(u_n) \to 0 \ \texttt{and} \ \sup_n \|u_n\|_E < +\infty. \tag{2.5}$$

Since the embedding

$$E \hookrightarrow L^s(R^N), \qquad 2 \leq s < 2^*$$

is compact, going if necessary to a subsequence, we can assume that there is a $u \in E$ such that

$$\begin{cases} u_n \rightharpoonup u, & \text{weakly in } E; \\ u_n \to u, & \text{strongly in } L^s(R^N); \\ u_n(x) \to u(x), & \text{a.e. in } R^N. \end{cases} \tag{2.6}$$

In view of (2.1), it has

$$\begin{aligned}
&\langle I'(u_n) - I'(u), u_n - u \rangle \\
&\quad = \left(a + b \int_{R^N} |\nabla u_n|^2 dx \right) \int_{R^N} \nabla u_n \cdot \nabla(u_n - u) dx + \int_{R^N} V(x)|u_n - u|^2 dx \\
&\qquad - \left(a + b \int_{R^N} |\nabla u|^2 dx \right) \int_{R^N} \nabla u \cdot \nabla(u_n - u) dx - \int_{R^N} [f(x, u_n) - f(x, u)](u_n - u) dx \\
&\quad = \left(a + b \int_{R^N} |\nabla u_n|^2 dx \right) \int_{R^N} |\nabla(u_n - u)|^2 dx + \int_{R^N} V(x)|u_n - u|^2 dx \\
&\qquad - \left(\int_{R^N} |\nabla u|^2 dx - \int_{R^N} |\nabla u_n|^2 dx \right) \int_{R^N} \nabla u \cdot \nabla(u_n - u) dx \\
&\qquad - \int_{R^N} [f(x, u_n) - f(x, u)](u_n - u) dx \\
&\quad \geq \min\{a, 1\} \|u_n - u\|_E^2 - bl; \left(\int_{R^N} |\nabla u|^2 dx - \int_{R^N} |\nabla u_n|^2 dx \right) \\
&\qquad - \int_{R^N} \nabla u \cdot \nabla(u_n - u) dx - \int_{R^N} [f(x, u_n) - f(x, u)](u_n - u) dx.
\end{aligned} \tag{2.7}$$

Then, (2.7) implies that

$$\min\{a,1\}\|u_n-u\|_E^2 \le \langle I'(u_n)-I'(u_n), u_n-u\rangle + b\left(\int_{R^N}|\nabla u|^2dx - \int_{R^N}|\nabla u_n|^2dx\right)\int_{R^N}\nabla u\cdot\nabla(u_n-u)dx + \int_{R^N}[f(x,u_n)-f(x,u)](u_n-u)dx. \quad (2.8)$$

Define the functional h_u: $E\to R$ by

$$h_u(v)=\int_{R^N}\nabla u\cdot\nabla v dx, \qquad \forall\ v\in E.$$

Obviously, h_u is a linear functional on E. Furthermore,

$$|h_u(v)|\le\int_{R^N}|\nabla u\cdot\nabla v|dx\le\|u\|_E\|v\|_E,$$

which implies h_u is bounded on E. Hence $h_u\in E^*$. Since $u_n\rightharpoonup u$ in E, it has $\lim_{n\to\infty}h_u(u_n)=h_u(u)$, that is, $\int_{R^N}\nabla u\cdot\nabla(u_n-u)dx\to 0$ as $n\to\infty$. Consequently, by (2.6) and the boundedness of $\{u_n\}$, it has

$$b\left(\int_{R^N}|\nabla u|^2dx-\int_{R^N}|\nabla u_n|^2dx\right)\int_{R^N}\nabla u\cdot\nabla(u_n-u)dx\to 0\ , \quad n\to+\infty. \quad (2.9)$$

By (2.2), using the Hölder inequality, we can conclude

$$\left|\int_{R^N}[f(x,u_n)-f(x,u)](u_n-u)dx\right| \le [\epsilon+C(\epsilon)]\int_{R^N}[|u_n|+|u|+|u_n|^{p-1}+|u|^{p-1}]|u_n-u|dx \le [\epsilon+C(\epsilon)](\|u_n\|_{L^2}+\|u\|_{L^2})\|u_n-u\|_{L^2} + [\epsilon+C(\epsilon)](\|u_n\|_{L^p}^{p-1}+\|u\|_{L^p}^{p-1})\|u_n-u\|_{L^p}.$$

Therefore, it follows from (2.6) that

$$\int_{R^N}[f(x,u_n)-f(x,u)](u_n-u)dx\to 0, \qquad \text{as } n\to\infty. \quad (2.10)$$

Moreover, combining (2.5) with (2.6), then

$$< I'(u_n)-I'(u), u_n-u >\to 0, \qquad \text{as } n\to\infty. \quad (2.11)$$

Consequently, (2.8–2.11) imply that

$$u_n\to u, \qquad \text{strongly in } E \text{ as } \ n\to\infty.$$

This completes the proof. □

3. Negative Energy Solution

In this section, we will get a negative energy solution for the problem (1.1) using the Ekeland's variational principle. We consider a minimization of I constrained in a neighborhood of zero and find a critical point of I which achieves the local minimum of I. Furthermore, the level of this local minimum is negative.

Lemma 3.1. *Assume that $g \in L^2(R^N)$, $g \not\equiv 0$ and (f_1)–(f_3) hold. Then*

$$-\infty < \inf\{I(u) : u \in \overline{B}_\rho\} < 0,$$

where $\overline{B}_r := \{u \in E : \|u\|_E \le r\}$.

Proof. By (f_1)–(f_3), it follows from the proof of Lemma 2.4 that

$$F(x,t) \ge C_1|t|^4 - C_2|t|^2, \qquad \forall\ (x,t) \in R^N \times R,$$

where C_1 and C_2 are positive constants. Since $g(x) \in L^2(R^N)$ and $g \not\equiv 0$, we can choose a function $v \in E$ such that

$$\int\limits_{R^N} g(x)v(x)dx > 0.$$

Thus,

$$\begin{aligned} I(tv) &= \frac{at^2}{2}\int\limits_{R^N} |\nabla v|^2dx + \frac{bt^4}{4}\left(\int\limits_{R^N} |\nabla v|^2dx\right)^2 + \frac{t^2}{2}\int\limits_{R^N} V(x)v^2dx \\ &\quad - \int\limits_{R^N} F(x,tv)dx - t\int\limits_{R^N} g(x)vdx \\ &\le \frac{\max\{a,1\}t^2}{2}\|v\|_E^2 + \frac{b}{4}\|v\|_E^4t^4 - C_1\|v\|_{L^4}^4t^4 + C_2\|v\|_{L^2}^2t^2 \\ &\quad -t\int\limits_{R^N} g(x)vdx < 0 \end{aligned}$$

for $t > 0$ small enough, which implies $\inf\{I(u) : u \in \overline{B}_\rho\} < 0$. In addition, by (1.10), (1.11), (2.3) and Hölder inequality,

$$\begin{aligned} I(u) &= \frac{a}{2}\int\limits_{R^N} |\nabla u|^2dx + \frac{b}{4}\left(\int\limits_{R^N} |\nabla u|^2dx\right)^2 + \frac{1}{2}\int\limits_{R^N} V(x)u^2dx \\ &\quad - \int\limits_{R^N} F(x,u)dx - \int\limits_{R^N} g(x)udx \\ &\ge -\int\limits_{R^N} F(x,u)dx - \int\limits_{R^N} g(x)udx \\ &\ge -(\epsilon\|u\|_{L^2}^2 + C_\epsilon\|u\|_{L^p}^p) - \|g\|_{L^2}\|u\|_{L^2} \\ &\ge -a_2^2\epsilon\|u\|_E^2 - a_p^pC_\epsilon\|u\|_E^p - a_2\|g\|_{L^2}\|u\|_E, \end{aligned}$$

which implies I is bounded below in $\overline{B}_\rho$. Therefore, we obtain

$$-\infty < \inf\{I(u) : u \in \overline{B}_\rho\} < 0.$$

This completes the proof. □

Ekeland's variational principle is the tool to obtain a negative energy solution, we give it here for readers' convenience.

Theorem 3.2 ([31], Theorem 4.1). *Let M be a complete metric space with metric d and let $I : M \mapsto (-\infty, +\infty]$ be a lower semicontinuous function, bounded from below and not identical to $+\infty$. Let $\epsilon > 0$ be given and $u \in M$ be such that*

$$I(u) \le \inf_M I + \epsilon.$$

Then there exists $v \in M$ such that

$$I(v) \le I(u), \quad d(u,v) \le 1,$$

and for each $w \in M$, one has

$$I(v) \le I(w) + \epsilon d(v,w).$$

Now, we could give the result of negative energy solution for the problem (1.1).

Theorem 3.3. *Assume that $g \in L^2(R^N)$, $g \not\equiv 0$, (V_1) and (f_1)–(f_3) hold. Then there exists a constant $g_0 > 0$ such that the problem (1.1) has a negative energy solution whenever $\|g\|_{L^2} < g_0$, that is, there exists a function $u_0 \in E$ such that $I'(u_0) = 0$ and $I(u_0) < 0$.*

Proof. The proof is almost the same as ([8], pp. 534–535), we give it here for the completeness. By Lemmas 2.3 and 3.1, taking $g_0 = \beta > 0$, we know that

$$-\infty < \inf_{\overline{B}_\rho} I < 0 < \alpha \le \inf_{\partial B_\rho} I$$

whenever $\|g\|_{L^2} < g_0$. Set

$$\frac{1}{n} \in \left(0, \inf_{\partial B_\rho} I - \inf_{\overline{B}_\rho} I\right), \ n \in \mathbf{N}.$$

Then, there is $u_n \in \overline{B}_\rho$ such that

$$I(u_n) \le \inf_{\overline{B}_\rho} I + \frac{1}{n}. \tag{3.1}$$

By Theorem 3.2 (Ekeland's variational principle), then

$$I(u_n) \le I(u) + \frac{1}{n}\|u - u_n\|_E, \qquad \forall u \in \overline{B}_\rho. \tag{3.2}$$

Note that

$$I(u_n) \le \inf_{\overline{B}_\rho} I + \frac{1}{n} < \inf_{\partial B_\rho} I.$$

Thus $u_n \in B_\rho$. Define $M_n : E \to R$ by

$$M_n(u) = I(u) + \frac{1}{n}\|u - u_n\|_E.$$

By (3.2), we have $u_n \in B_\rho$ minimizes M_n on $\overline{B}_\rho$. Therefore, for all $\phi \in E$ with $\|\phi\|_E = 1$, taking $t > 0$ small enough such that $u_n + t\phi \in \overline{B}_\rho$, then

$$\frac{M_n(u_n + t\phi) - M_n(u_n)}{t} \geq 0,$$

which implies that

$$\frac{I(u_n + t\phi) - I(u_n)}{t} + \frac{1}{n} \geq 0.$$

Thus,

$$< I'(u_n), \phi > \geq -\frac{1}{n}.$$

Hence,

$$\|I'(u_n)\|_E \leq \frac{1}{n}. \tag{3.3}$$

Passing to the limit in (3.1) and (3.3), we conclude that

$$I(u_n) \to \inf_{\overline{B}_\rho} I \quad \text{and} \quad \|I'(u_n)\|_E \to 0, \quad \text{as } n \to \infty. \tag{3.4}$$

Note that $\|u_n\|_E \leq \rho$, hence $\{u_n\} \subset E$ is a bounded Palais–Smale sequence of I. By Lemma 2.5, $\{u_n\}$ has a strongly convergent subsequence, still denoted by $\{u_n\}$ and $u_n \to u_0 \in \overline{B}_\rho$, as $n \to \infty$. Consequently, it follows from (3.4) that

$$I(u_0) = \inf_{\overline{B}_\rho} I < 0 \quad \text{and} \quad I'(u_0) = 0.$$

This completes the proof. □

4. Positive Energy Solution

The aim of this section is to get a positive energy solution for the problem (1.1) with the aid of Mountain Pass Theorem. Hence, we recall the classical Mountain Pass Theorem due to Ambrosetti–Rabinowitz.

Theorem 4.1 ([32], Theorem 2.2). *Let X be a real Banach space and $I \in C^1(X, R)$ satisfying (PS) condition. Suppose $I(0) = 0$ and*

(I_1) there are constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho} \geq \alpha$, and
(I_2) there is $u_1 \in X \backslash \overline{B}_\rho$ such that $I(u_1) \leq 0$.

Then I possesses a critical value $c \geq \alpha$. Moreover, c can be characterized as

$$c = \inf_{\gamma \in \Gamma} \max_{u \in \gamma([0,1])} I(u),$$

where

$$\Gamma = \{\gamma \in C([0,1], X) : \ \gamma(0) = 0, \gamma(1) = u_1\}.$$

Lemma 4.2. *Let assumptions (V_1) and (f_1)–(f_4) hold. Then any Palais–Smale sequence of I is bounded.*

Proof. Let $\{u_n\} \subset E$ be any Palais–Smale sequence of I. Then, up to a subsequence, there exists $c_1 \in R$ such that

$$I(u_n) \to c_1, \quad \text{and} \quad I'(u_n) \to 0. \tag{4.1}$$

The combination of (1.10), (1.11), (2.1), (4.1), (V_1) with (f_4) implies

$$\begin{aligned}
c_1 + 1 + \|u_n\|_E &\geq I(u_n) - \frac{1}{4}\langle I'(u_n), u_n\rangle \\
&= \frac{a}{4}\int_{R^N} |\nabla u_n|^2 dx + \frac{1}{4}\int_{R^N} V(x)u_n^2 dx + \int_{R^N} \widetilde{F}(x,u_n)dx - \frac{3}{4}\int_{R^N} g(x)u_n dx \\
&\geq \frac{a}{4}\int_{R^N} |\nabla u_n|^2 dx + \frac{1}{4}\int_{R^N} V(x)u_n^2 dx - \frac{d}{4}\int_{R^N} u_n^2 dx \\
&\quad + \int_{A_n} \widetilde{F}(x,u_n)dx - \frac{3}{4}\|g\|_{L^2}\|u_n\|_{L^2} \\
&\geq \frac{a}{4}\int_{R^N} |\nabla u_n|^2 dx + \frac{1}{4}\int_{R^N} V(x)u_n^2 dx - \frac{1}{8}\int_{R^N} V_0 u_n^2 dx \\
&\quad + \int_{A_n} \widetilde{F}(x,u_n)dx - \frac{3}{4}a_2\|g\|_{L^2}\|u_n\|_E \\
&\geq \frac{a}{4}\int_{R^N} |\nabla u_n|^2 dx + \frac{1}{4}\int_{R^N} V(x)u_n^2 dx - \frac{1}{8}\int_{R^N} V(x)u_n^2 dx \\
&\quad + \int_{A_n} \widetilde{F}(x,u_n)dx - \frac{3}{4}a_2\|g\|_{L^2}\|u_n\|_E \\
&\geq \frac{1}{16}\min\{a,1\}\|u_n\|_E^2 + \frac{1}{16}\int_{R^N} V(x)u_n^2 dx + \int_{A_n} \widetilde{F}(x,u_n)dx - \frac{3}{4}a_2\|g\|_{L^2}\|u_n\|_E,
\end{aligned}$$

where $\widetilde{F}(x,u_n) = \frac{1}{4}f(x,u_n)u_n - F(x,u_n)$ and $A_n = \{x \in R^N : \ |u_n| \leq L\}$. Hence,

$$\begin{aligned}
&c_1 + 1 + \left(1 + \frac{3}{4}a_2\|g\|_{L^2}\right)\|u_n\|_E \\
&\quad \geq \frac{1}{16}\min\{a,1\}\|u_n\|_E^2 + \frac{1}{16}\int_{R^N} V(x)u_n^2 dx + \int_{A_n} \widetilde{F}(x,u_n)dx.
\end{aligned} \tag{4.2}$$

For $x \in R^N$ and $|u_n| \leq L$, by (2.2) and (2.3), it has

$$\begin{aligned}
|\widetilde{F}(x,u_n)| &\leq \frac{1}{4}|f(x,u_n)||u_n| + |F(x,u_n)| \\
&\leq \frac{5}{4}\epsilon|u_n|^2 + \frac{5}{4}C(\epsilon)|u_n|^p \\
&= \frac{5}{4}\Big[\epsilon + C(\epsilon)|u_n|^{p-2}\Big]|u_n|^2 \\
&\leq \frac{5}{4}\Big[\epsilon + C(\epsilon)L^{p-2}\Big]|u_n|^2.
\end{aligned}$$

Take $A > \max\{20[\epsilon + C(\epsilon)L^{p-2}], V_0\}$, then

$$\widetilde{F}(x, u_n) \geq -\frac{A}{16}|u_n|^2, \quad \forall x \in R^N, \ |u_n| \leq L. \tag{4.3}$$

Let $\widetilde{A} = \{x \in R^N : V(x) \leq A\}$. By (V_1) and (4.3), we can conclude

$$\begin{aligned}\frac{1}{16}\int_{R^N} V(x)u_n^2 dx + \int_{A_n} \widetilde{F}(x, u_n)dx &\geq \frac{1}{16}\int_{|u_n|\leq L} (V(x) - A)|u_n|^2 dx \\ &\geq \frac{1}{16}\int_{\widetilde{A}\cap A_n} (V(x) - A)L^2 dx \\ &\geq \frac{1}{16}(V_0 - A)L^2 meas(\widetilde{A} \cap A_n) \\ &\geq \frac{1}{16}(V_0 - A)L^2 meas(\widetilde{A}). \end{aligned} \tag{4.4}$$

Note that $meas(\widetilde{A}) < +\infty$ due to (V_1), it follows from (4.2) and (4.4) that

$$\begin{aligned} &c_1 + 1 + \left(1 + \frac{3}{4}a_2\|g\|_{L^2}\right)\|u_n\|_E \\ &\quad \geq \frac{1}{16}\min\{a, 1\}\|u_n\|_E^2 + \frac{1}{16}(V_0 - A)L^2 meas(\widetilde{A}), \end{aligned}$$

which implies $\{u_n\} \subset E$ is bounded in E. Hence, the proof is completed. □

Theorem 4.3. *Assume that $g(x) \in L^2(R^N)$, $g \not\equiv 0$, (V_1) and (f_1)–(f_4) hold. Then the problem (1.1) has a positive energy solution whenever $\|g\|_{L^2} < g_0$, that is, there exists a function $u_1 \in E$ such that $I'(u_1) = 0$ and $I(u_1) > 0$.*

Proof. We will apply Theorem 4.1 to prove Theorem 4.3. Next, we shall verify I satisfies all the conditions of Theorem 4.1. By Theorem 3.3, we know $g_0 = \beta > 0$. Then I satisfies (I_1) whenever $\|g\|_{L^2} < g_0$ by Lemma 2.3. Lemma 2.4 implies that I satisfies (I_2), and I satisfies (PS) condition by virtue of Lemmas 2.5 and 4.2. Evidently, $I \in C^1(E, R)$ and $I(0) = 0$. Hence, applying Theorem 4.1, there exists a function $u_1 \in E$ such that $I'(u_1) = 0$ and $I(u_1) \geq \alpha > 0$. The proof is completed. □

Proof of Theorem 1.1. The desired conclusion directly follows from Theorems 3.3 and 4.3.

Proof of Corollary 1. It is sufficient to prove that (f_3'')–(f_4'') imply (f_3)–(f_4) by applying Theorem 1.1. In fact, For any $(x, z) \in R^N \times R$, define

$$k(t) := F(x, \frac{z}{t})t^\mu, \quad \forall t \in [1, +\infty].$$

Then for $|z| \geq L'$ and $t \in [1, \frac{|z|}{L'}]$, (f_4'') implies that

$$\begin{aligned} k'(t) &= f(x, \frac{z}{t})(-\frac{z}{t^2})t^{\mu} + \mu F(x, \frac{z}{t})t^{\mu-1} \\ &= t^{\mu-1}[\mu F(x, \frac{z}{t}) - f(x, \frac{z}{t})\frac{z}{t}] \\ &\leq d' t^{\mu-1}|\frac{z}{t}|^2 \\ &= d' t^{\mu-3}|z|^2. \end{aligned}$$

Thus,

$$\begin{aligned} k\left(\frac{|z|}{L'}\right) - k(1) &= \int_1^{\frac{|z|}{L'}} k'(t)dt \\ &\leq \int_1^{\frac{|z|}{L'}} d' t^{\mu-3}|z|^2 dt \\ &= \frac{d'|z|^{\mu}}{(\mu-2)L'^{\mu-2}} - \frac{d'|z|^2}{\mu-2}. \end{aligned}$$

Hence, for any $x \in R^N$ and $|z| \geq L'$, by (f_3''), one has

$$\begin{aligned} F(x,z) = k(1) &\geq k\left(\frac{|z|}{L'}\right) + \frac{d'|z|^2}{\mu-2} - \frac{d'|z|^{\mu}}{(\mu-2)L'^{\mu-2}} \\ &\geq \left[\inf_{x\in R^N, |t|=L'} F(x,t)\right](\frac{|z|}{L'})^{\mu} + \frac{d'|z|^2}{\mu-2} - \frac{d'|z|^{\mu}}{(\mu-2)L'^{\mu-2}} \\ &\geq \left(\frac{c'}{L'^{\mu}} - \frac{d'}{(\mu-2)L'^{\mu-2}}\right)|z|^{\mu}. \end{aligned}$$

By $d' \in [0, \frac{c'(\mu-2)}{L'^2})$, set $C_4 = \frac{c'}{L'^{\mu}} - \frac{d'}{(\mu-2)L'^{\mu-2}} > 0$, it has

$$F(x,z) \geq C_4|z|^{\mu}, \qquad \forall x \in R^N \texttt{ and } |z| \geq L'.$$

Hence,

$$\frac{F(x,z)}{z^4} \geq C_4|z|^{\mu-4}, \qquad \forall x \in R^N \texttt{ and } |z| \geq L'. \tag{4.5}$$

Note that $\mu > 4$, then (4.5) implies (f_3). Furthermore, it follows from (4.5) and (f_4'') that

$$\begin{aligned} 4F(x,z) - f(x,z)z &= \mu F(x,z) - f(x,z)z + (4-\mu)F(x,z) \\ &\leq d'|z|^2 - (\mu-4)C_4|z|^{\mu} \end{aligned}$$

for all $x \in R^N$ and $|z| \geq L'$. This, together with $\mu > 4$, shows there exists $L > 0$ such that

$$4F(x,z) - f(x,z)z < 0, \qquad \forall x \in R^N \texttt{ and } \quad |z| \geq L,$$

which implies (f_4). Hence, the proof is completed. □

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