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A Practical Method for Generating Trigonometric Polynomial Surfaces over Triangular Domains

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Abstract. A class of trigonometric polynomial basis functions over triangular domain with three shape parameters is constructed in this paper. Based on these new basis functions, a kind of trigonometric polynomial patch over triangular domain, which can be used to construct some surfaces whose boundaries are arcs of ellipse or parabola, is proposed. Without changing the control points, the shape of the trigonometric polynomial patch can be adjusted flexibly in a foreseeable way using the shape parameters. For computing the proposed trigonometric polynomial patch stably and efficiently, a practical de Casteljau-type algorithm is developed. Moveover, the conditions for G^1 continuous smooth joining two trigonometric polynomial patches are deduced.

Mathematics Subject Classification. Primary 65D07, 65D17; Secondary 41A15, 42A05, 42A10.

Keywords. Trigonometric polynomial, triangular domain, triangular patch, shape parameter, de Casteljau-type algorithm.

1. Introduction

In the last decades, lots of generalized spline bases have been constructed in various non-polynomial function spaces (typically including hyperbolic or trigonometric functions) for curves and surfaces design. For instance, in [19], a basis for the space of hyperbolic polynomials was given, using which hyperbola can be represented exactly. In [1,11,14,15,27,29,30], the study for mixing algebraic and trigonometric/hyperbolic generalized B-splines can be found. Trigonometric polynomials and splines have also attracted widespread interest within computer-aided geometric design (CAGD), particularly within curves design. In [12], the recurrence relation for the trigonometric B-splines of arbitrary order was established. Afterwards, in [22], it was further shown that the trigonometric B-splines of odd order form a partition of a constant in the case of equidistant knots and thus the associated trigonometric B-spline curves possess the important convex hull property. In [23], a family of trigonometric polynomials, which contains the trigonometric Lagrange and Bernstein polynomials, was introduced. In [3,4,6,7], some quadratic trigonometric polynomial splines with shape parameters were shown. In [5], a class of cubic trigonometric Bézier (T-Bézier, for short) curve with a shape parameters was proposed. Later, in [9], the cubic T-Bézier curve was further extended to possess two shape parameters. In [10,21], based on the theory of envelop and topological mapping, shape features of the T-Bézier curve were analyzed. There are some recent papers concerning representation of curves using trigonometric spline with shape parameters; see for example [8,13,16,24] and the references quoted therein. Since tensor product Bézier patch is the direct extension of Bézier curve, we can easily get rectangular patches with shape parameters through these new curves. However, the Bernstein–Bézier patch over the triangular domain is not a tensor product patch exactly. Therefore, we cannot get triangular surfaces with an adjustable shape using the method of tensor product. Surfaces modeling over triangular domain is important for many applications and thus it is worth studying the practical methods for generating surfaces over a triangular domain.

Recently, some researchers have made many efforts for the establishment of new bases over triangular domain with shape parameters; see [2,20,25,26,28,31] and the references quoted therein. In [2], Cao and Wang constructed a class of Bernstein–Bézier patch over the triangular domain with a shape parameter. By changing the value of the shape parameter, different surfaces under the fixed control points can be obtained. In [20], Shen and Wang proposed a kind of linear Bernstein-like trigonometric polynomial basis over the triangular domain with a shape parameter, which was a triangular domain extension of the p-Bézier basis of order three given in [17]. In [25], Wei et al. extended the C-Bézier basis of order four on the univariate domain given in [29] to a new Bézier-like basis on the triangular domain, which has a shape parameter and can be used to generate some surfaces whose boundaries are arcs of ellipse. In [26], Yang and Zeng gave a class of triangular Bézier surfaces with $3n(n+1)/2$ shape parameters. In [28], Yan and Liang constructed a set of initial basis functions of order 2 with a shape parameter and then defined the basis functions of order n using the classical recursive approach for the Bernstein–Bézier basis over the triangular domain. Based on the basis, they proposed a class of triangular Bernstein–Bézier-like surface with a shape parameter. Recently, Zhu and Han [31] have constructed a class of $\alpha\beta\gamma$-Bernstein–Bézier basis with three exponential shape parameters over the triangular domain, which includes the cubic triangular Said–Ball basis and the cubic triangular Bernstein–Bézier basis as special cases.

The purpose of this paper is to present a new class of trigonometric polynomial basis functions over the triangular domain, which has three shape parameters and is useful for generating triangular surface patch. The given basis functions are a triangular domain extension of the cubic trigonometric Bézier basis functions with two shape parameters given in [9]. The three shape parameters in the corresponding trigonometric polynomial patch have a predictable adjusting role on the patch. The proposed trigonometric polynomial patch can be used to construct some surfaces whose boundaries are arcs of ellipse or parabola. The rest of this paper is organized as follows. Section 2 gives the construction and properties of the trigonometric polynomial basis functions over the triangular domain. In Sect. 3, the definition and properties of the trigonometric polynomial patch over the triangular domain with three shape parameters are shown. A practical de Casteljau-type algorithm for computing the proposed trigonometric polynomial patch over the triangular domain is developed. Conclusions are given in Sect. 4.

2. Trigonometric Polynomial Basis Functions Over the Triangular Domain

2.1. Construction of the Trigonometric Polynomial Basis Functions

The classical cubic Bernstein–Bézier basis over the triangular domain with ten functions is widely used for surface design on the grounds of flexibility and efficiency; see [18]. However, since the cubic Bernstein–Bézier basis over the triangular domain does not possess any additional shape parameters, the shapes of the corresponding triangular Bernstein–Bézier cubic patches are fixed relatively to their control net. Although we can use weight factors to adjust the shape of the cubic rational Bernstein–Bézier patch over the triangular domain, see [18], the effect of the weight factors in modifying the shape of a patch is sometimes hard to predict. Trigonometric polynomials and splines has been widely studied for constructing curves; however, there are few publications concerning trigonometric polynomial surfaces over triangular domain with shape parameters. Therefore, we want to construct ten new trigonometric polynomial basis functions over the triangular domain, which possess three shape parameters to adjust the shape of the corresponding trigonometric polynomial surface patch.

Definition 2.1. Let $\lambda, \mu, \gamma \in [-2, 1]$, for the given triangular domain $D = \{(t, s, w)\,|t + s + w = \pi/2, t \geq 0, s \geq 0, w \geq 0\}$, the following ten functions are defined to be trigonometric polynomial basis functions, with three shape parameters λ, μ and γ, over the triangular domain D:

$$\begin{cases} T^3_{3,0,0}(t,s,w;\lambda,\mu,\gamma) = (1-\cos t)^2(1-\lambda\cos t), \\ T^3_{0,3,0}(t,s,w;\lambda,\mu,\gamma) = (1-\cos s)^2(1-\mu\cos s), \\ T^3_{0,0,3}(t,s,w;\lambda,\mu,\gamma) = (1-\cos w)^2(1-\gamma\cos w), \\ T^3_{2,1,0}(t,s,w;\lambda,\mu,\gamma) = \cos w\sin s(1-\cos t)\left[2+\lambda-\lambda\cos t\right], \\ T^3_{2,0,1}(t,s,w;\lambda,\mu,\gamma) = \cos s\sin w(1-\cos t)\left[2+\lambda-\lambda\cos t\right], \\ T^3_{1,2,0}(t,s,w;\lambda,\mu,\gamma) = \cos w\sin t(1-\cos s)\left[2+\mu-\mu\cos s\right], \\ T^3_{0,2,1}(t,s,w;\lambda,\mu,\gamma) = \cos t\sin w(1-\cos s)\left[2+\mu-\mu\cos s\right], \\ T^3_{1,0,2}(t,s,w;\lambda,\mu,\gamma) = \cos s\sin t(1-\cos w)\left[2+\gamma-\gamma\cos w\right], \\ T^3_{0,1,2}(t,s,w;\lambda,\mu,\gamma) = \cos t\sin s(1-\cos w)\left[2+\gamma-\gamma\cos w\right], \\ T^3_{1,1,1}(t,s,w;\lambda,\mu,\gamma) = 1-\sum\limits_{\substack{i+j+k=3,\\ i\cdot j\cdot k\neq 1}} T^3_{i,j,k}(t,s,w;\lambda,\mu,\gamma). \end{cases} \tag{2.1}$$

Remark 2.2. When one of the three variables w is taken as zero, the ten trigonometric polynomial basis functions $T^3_{i,j,k}(t,s,w;\lambda,\mu,\gamma)$ $(i,j,k \in \mathbb{N}, i+j+k=3)$ will degenerate to the following four cubic trigonometric Bézier (T-Bézier for short) basis functions (notice $s=\pi/2-t$) with two shape parameters λ,μ given in [9]

$$\begin{cases} T_0(t;\lambda,\mu) = (1-\sin t)^2(1-\mu\sin t), \\ T_1(t;\lambda,\mu) = \sin t(1-\sin t)(2+\mu-\mu\sin t), \\ T_2(t;\lambda,\mu) = \cos t(1-\cos t)(2+\lambda-\lambda\cos t), \\ T_3(t;\lambda,\mu) = (1-\cos t)^2(1-\lambda\cos t). \end{cases} \tag{2.2}$$

For $\lambda=\mu$, this kind of T-Bézier basis functions with a shape parameter was also proposed in [5]. Therefore, we can see that the trigonometric polynomial basis functions $T^3_{i,j,k}(t,s,w;\lambda,\mu,\gamma)$ $(i,j,k \in \mathbb{N}, i+j+k=3)$ are a triangular domain extension of the cubic T-Bézier basis functions given in [5,9].

Before further discussion, we want to prove the following lemma, which is extremely useful in the following discussion.

Lemma 2.3. *For $t+s+w=\pi/2$, we have*

$$1-\left(\sin^2 t+\sin^2 s+\sin^2 w\right) = 2\sin t\sin s\sin w. \tag{2.3}$$

Proof. For $t+s+w=\pi/2$, direct computation gives that

$$\begin{aligned} 1-\left(\sin^2 t+\sin^2 s+\sin^2 w\right) &= \frac{1}{2}\left(\cos 2t+\cos 2s+\cos 2w-1\right) \\ &= \cos(t+s)\cos(t-s)-\sin^2 w \\ &= \cos(t-s)\sin w-\cos(t+s)\sin w \\ &= [\cos(t-s)-\cos(t+s)]\sin w \\ &= 2\sin t\sin s\sin w. \end{aligned}$$

These imply the lemma. □

2.2. Properties of the Trigonometric Polynomial Basis Functions

From the definition of the trigonometric polynomial basis functions over the triangular domain, we can obtain the following important properties of the basis.

Theorem 2.4. *The trigonometric polynomial basis functions given in* (2.1) *have the following properties:*

(A) *Nonnegativity.* $T^3_{i,j,k}(t,s,w;\lambda,\mu,\gamma) \geq 0$ *for* $i,j,k \in \mathbb{N}, i+j+k=3$.
(B) *Partition of unity.* $\sum_{i+j+k=3} T^3_{i,j,k}(t,s,w;\lambda,\mu,\gamma) = 1$.
(C) *Symmetry. For all* $i,j,k \in \mathbb{N}, i+j+k=3$, *we have*

$$\begin{aligned} T^3_{i,j,k}(t,s,w;\lambda,\mu,\gamma) &= T^3_{j,i,k}(s,t,w;\mu,\lambda,\gamma) = T^3_{j,k,i}(s,w,t;\mu,\gamma,\lambda) \\ &= T^3_{i,k,j}(t,w,s;\lambda,\gamma,\mu) = T^3_{k,i,j}(w,t,s;\gamma,\lambda,\mu) \\ &= T^3_{k,j,i}(w,s,t;\gamma,\mu,\lambda). \end{aligned}$$

(D) *Boundary property. When one of the three variables* t, s, w *is set to be zero, the ten trigonometric basis functions* $T^3_{i,j,k}(t, s, w; \lambda, \mu, \gamma)$ $(i, j, k \in \mathbb{N}, i + j + k = 3)$ *with three shape parameters will degenerate to four corresponding univariate cubic T-Bézier basis functions* $T_i(\cdot\,;\,\cdot\,,\,\cdot)$ $(i = 0, 1, 2, 3)$ *with two associated shape parameters.*

(E) *Linear independence.* $\{T^3_{i,j,k}(t, s, w; \lambda, \mu, \gamma), i, j, k \in \mathbb{N}, i + j + k = 3\}$ *are linearly independent.*

Proof. We shall prove (A) and (E). The remaining cases follow obviously.

(A) Apparently, for any $\lambda, \mu, \gamma \in [-2, 1]$, $i, j, k \in \mathbb{N}, i+j+k = 3$ and $ijk \neq 1$, we have $T^3_{i,j,k}(t, s, w; \lambda, \mu, \gamma) \geq 0$. Furthermore, for $T^3_{1,1,1}(t, s, w; \lambda, \mu, \gamma)$, using Lemma 2.3, we have

$$\begin{aligned} T^3_{1,1,1}(t, s, w; \lambda, \mu, \gamma) &= 1 - \sum_{\substack{i+j+k=3,\\ i\cdot j\cdot k\neq 1}} T^3_{i,j,k}(t, s, w; \lambda, \mu, \gamma) \\ &= 1 - \left(\sin^2 t + \sin^2 s + \sin^2 w\right) \\ &= 2 \sin t \sin s \sin w \geq 0. \end{aligned}$$

(E) For any $\lambda, \mu, \gamma \in [-2, 1]$, $\alpha_{i,j,k} \in \mathbb{R}$ $(i, j, k \in \mathbb{N}, i+j+k = 3)$, we consider a linear combination

$$\sum_{i+j+k=3} \alpha_{i,j,k} T^3_{i,j,k}(t, s, w; \lambda, \mu, \gamma) = 0.$$

Let $w = 0$, we have

$$\sum_{i=0}^{3} \alpha_{i,3-i,0} T_i(t; \lambda, \mu) = 0. \tag{2.4}$$

Differentiating with respect to the variable t on both sides, we have

$$\sum_{i=0}^{3} \alpha_{i,3-i,0} T'_i(t; \lambda, \mu) = 0. \tag{2.5}$$

For $t = 0$, from (2.4) and (2.5), we get the following linear system of equations with respect to $\alpha_{0,3,0}$ and $\alpha_{1,2,0}$

$$\begin{cases} \alpha_{0,3,0} = 0, \\ (\mu + 2)\left(\alpha_{1,2,0} - \alpha_{0,3,0}\right) = 0. \end{cases}$$

Thus, we have $\alpha_{0,3,0} = \alpha_{1,2,0} = 0$. For $t = \pi/2$, from (2.4) and (2.5), we have $\alpha_{3,0,0} = \alpha_{2,1,0} = 0$. Similarly, $\alpha_{i,0,3-i} = \alpha_{0,i,3-i} = 0$ for $i = 0, 1, 2, 3$. Finally, $\alpha_{1,1,1} = 0$.

These imply the theorem. □

Figure 1 shows some plots of trigonometric polynomial basis functions over the triangular domain. The three shape parameters take values $\lambda = 1$, $\mu = 0$ and $\gamma = -1$, respectively.

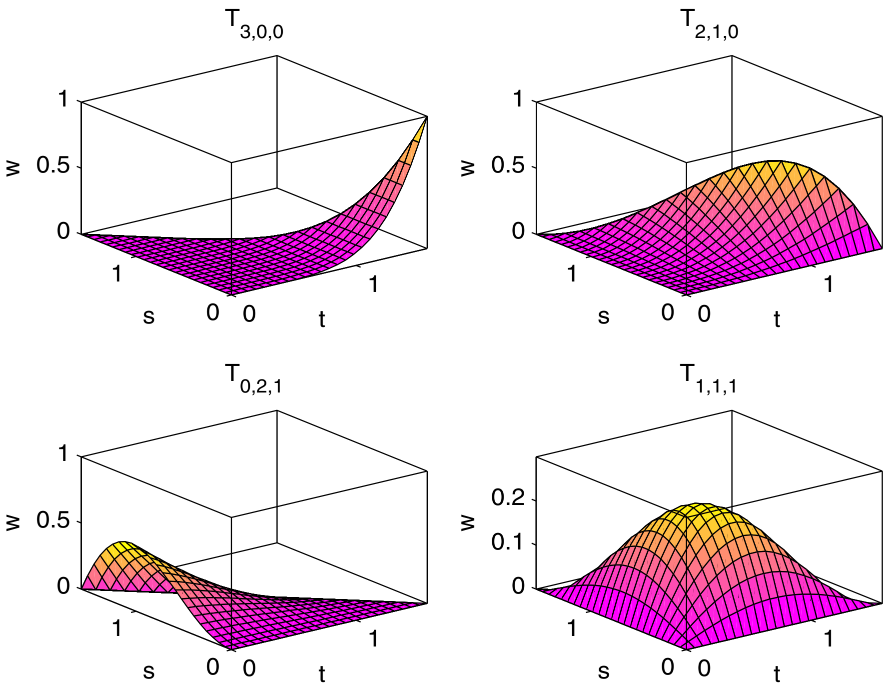

Figure 1. Some plots of trigonometric polynomial basis functions over the triangular domain

3. Trigonometric Polynomial Patch Over the Triangular Domain with Three Shape Parameters

3.1. Definition and Properties of the Trigonometric Polynomial Patch

Definition 3.1. For any real numbers $\lambda, \mu, \gamma \in [-2, 1]$, given a triangular domain $D = \{(t, s, w)\,|t + s + w = \pi/2, t \geq 0, s \geq 0, w \geq 0\}$, and ten control points $P_{i,j,k} \in \mathbb{R}^3 (i, j, k \in \mathbb{N}, i + j + k = 3)$. We call

$$R(t, s, w) = \sum_{i+j+k=3} T^3_{i,j,k}(t, s, w; \lambda, \mu, \gamma) P_{i,j,k}, \quad (t, s, w) \in D \qquad (3.1)$$

the trigonometric polynomial patch over the triangular domain with three shape parameters λ, μ and γ.

According to the properties of the trigonometric polynomial basis functions given in (2.1), some properties of the corresponding trigonometric polynomial patch given in (3.1) can be obtained as follows:

(A) Affine invariance and convex hull property. Since the trigonometric polynomial basis functions (2.1) have the properties of partition of unity and nonnegativity, these imply that the corresponding trigonometric polynomial patch (3.1) has affine invariance and convex hull property.

(B) End point interpolation property. Direct computation gives that

$$R(\pi/2, 0, 0) = P_{3,0,0}, \quad R(0, \pi/2, 0) = P_{0,3,0}, \quad R(0, 0, \pi/2) = P_{0,0,3}.$$

These indicate that the trigonometric polynomial patch interpolates at the three end points.

(C) End point tangent property. Let $w = \pi/2 - s - t$, we have

$$\left.\frac{\partial R(t,s,w)}{\partial t}\right|_{(\pi/2,0,0)} = (\lambda+2)\left(P_{3,0,0} - P_{2,0,1}\right),$$
$$\left.\frac{\partial R(t,s,w)}{\partial s}\right|_{(\pi/2,0,0)} = (\lambda+2)\left(P_{2,1,0} - P_{2,0,1}\right),$$
$$\left.\frac{\partial R(t,s,w)}{\partial t}\right|_{(0,\pi/2,0)} = (\mu+2)\left(P_{1,2,0} - P_{0,2,1}\right),$$
$$\left.\frac{\partial R(t,s,w)}{\partial s}\right|_{(0,\pi/2,0)} = (\mu+2)\left(P_{0,3,0} - P_{0,2,1}\right),$$
$$\left.\frac{\partial R(t,s,w)}{\partial t}\right|_{(0,0,\pi/2)} = (\gamma+2)\left(P_{1,0,2} - P_{0,0,3}\right),$$
$$\left.\frac{\partial R(t,s,w)}{\partial s}\right|_{(0,0,\pi/2)} = (\gamma+2)\left(P_{0,1,2} - P_{0,0,3}\right).$$

These indicate that the tangent plane at the three end points $(\pi/2, 0, 0)$, $(0, \pi/2, 0)$, $(0, 0, \pi/2)$ are the three planes spanned by the control points $P_{3,0,0}$, $P_{2,1,0}$, $P_{2,0,1}$; $P_{0,3,0}$, $P_{1,2,0}$, $P_{0,2,1}$; $P_{0,0,3}$, $P_{1,0,2}$, $P_{0,1,2}$, respectively.

(D) Boundary property. For $w = 0$, $R(t, s, w)$ is just the following cubic T-Bézier curve given in [9], with two shape parameters λ and μ.

$$R(t, \pi/2 - t, 0) = \sum_{i=0}^{3} P_{i,3-i,0} T_i(t; \lambda, \mu). \tag{3.2}$$

Similarly, $R(0, s, \pi/2 - s)$ and $R(\pi/2 - w, 0, w)$ are also T-Bézier curve with shape parameters μ, γ and λ, γ, respectively. For $\lambda = \mu = 0$, the T-Bézier curve (3.2) can represent exactly elliptic and parabolic arcs; see [9]. These imply that for $\lambda = \mu = \gamma = 0$, the three boundaries of trigonometric polynomial patch (3.1) can be arcs of ellipse or parabola, respectively. Figure 2 shows the trigonometric polynomial patches generated by setting $\lambda = \mu = \gamma = 0$. On the left, the figure shows the trigonometric polynomial patch whose boundaries are two elliptic arcs and a parabolic arc, respectively. Its control points are $\{P_{3,0,0} = (0, -4, 0)$, $P_{0,3,0} = (2, 0, 0)$, $P_{0,0,3} = (0, 0, 1)$, $P_{2,1,0} = (1, -4, 0)$, $P_{2,0,1} = (0, -4, 1/2)$, $P_{1,2,0} = (2, 0, 0)$, $P_{0,2,1} = (2, 0, 1/2)$, $P_{1,0,2} = (0, -2, 1)$, $P_{0,1,2} = (1, 0, 1)$, $P_{1,1,1} = (1, -2, 1)\}$. The corresponding parametric equations of the three boundaries are: $t = 0$, $s = -4\sin x$, $w = \cos x$; $t = 2\sin x, s = 0, w = \cos x$; and $t = 2\cos x, s = -4 + 4\cos^2 x, w = 0$, where $x \in [0, \pi/2]$. On the right, the figure shows the trigonometric polynomial patch with three same boundaries as a quarter of the unit circle. The three boundaries are fitted onto the unit sphere. The associated control points of the trigonometric polynomial patch are $\{P_{3,0,0} = (0, 1, 0)$, $P_{0,3,0} = (1, 0, 0)$, $P_{0,0,3} = (0, 0, 1)$,

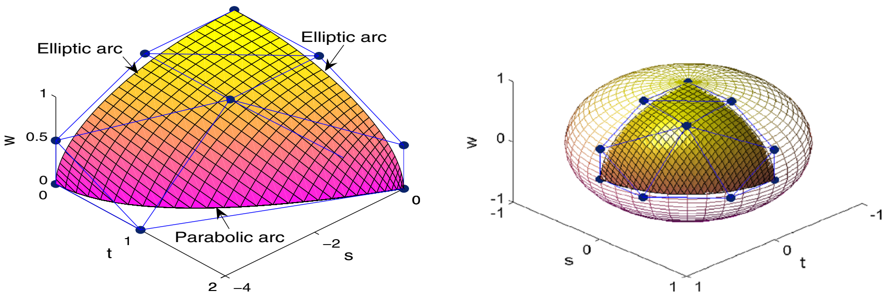

Figure 2. Trigonometric polynomial patches with *elliptic* or *parabolic boundary curves.*

$P_{2,1,0} = (1/2, 1, 0)$, $P_{2,0,1} = (0, 1, 1/2)$, $P_{1,2,0} = (1, 1/2, 0)$, $P_{0,2,1} = (1, 0, 1/2)$, $P_{1,0,2} = (0, 1/2, 1)$, $P_{0,1,2} = (1/2, 0, 1)$, $P_{1,1,1} = (1, -2, 1)\}$. The corresponding three parametric equations of the boundaries are: $t = 0, s = \sin x, w = \cos x$; $t = \sin x, s = 0, w = \cos x$; and $t = \cos x, s = \sin x, w = 0$, where $x \in [0, \pi/2]$.

(E) Shape adjustable property. Without changing the control points, we can adjust the shape of the obtained trigonometric polynomial patch conveniently using the three shape parameters λ, μ and γ. As the three shape parameters increase at the same time, the trigonometric polynomial patch will be made close to the control net. From the boundary property of the trigonometric polynomial patch, we can see that the three shape parameters λ, μ and γ have nothing to do with the boundary curves $R(0, s, w)$, $R(t, 0, w)$ and $R(t, s, 0)$, respectively. It is equivalent to say that changing the values of single one shape parameter, one corresponding boundary curve will not change. Moreover, from (3.1), differentiate with respect to the shape parameter λ, we have

$$\frac{\partial R(t,s,w)}{\partial \lambda} = (1 - \cos t)^2 \left[P_{2,1,0} \cos w \sin s + P_{2,0,1} \cos s \sin w - P_{3,0,0} \cos t\right]. \tag{3.3}$$

Therefore, there is no relationship between $\frac{\partial R(t,s,w)}{\partial \lambda}$ and λ. These imply that for the fixed control points and the given value $(t, s, w) \in D$, changing single one shape parameter λ will make the corresponding point on the trigonometric polynomial surface patch (3.1) move linearly in the direction given by (3.3). The shape parameters μ and γ have similar effect on the trigonometric polynomial surface patch.

Figure 3 shows the trigonometric polynomial patches and the effect on the patches by altering the values of the shape parameters under the same control points. Figure 4 shows the directions of the three vectors $\partial R/\partial \lambda$, $\partial R/\partial \mu$ and $\partial R/\partial \gamma$ at a fixed point $R(\pi/6, \pi/6, \pi/6)$. Here, to show the directions of the three vectors more clearly, the lengths of the three vectors are all magnified by 15 times.

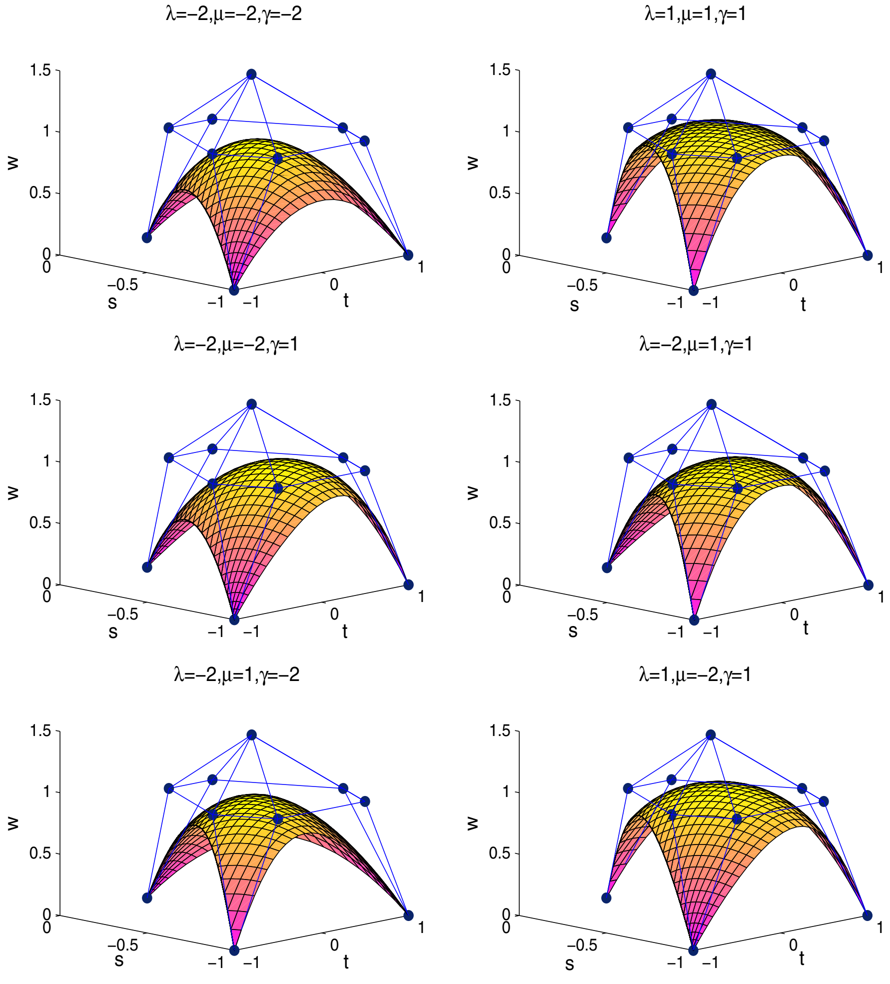

Figure 3. Trigonometric polynomial patches with different shape parameters.

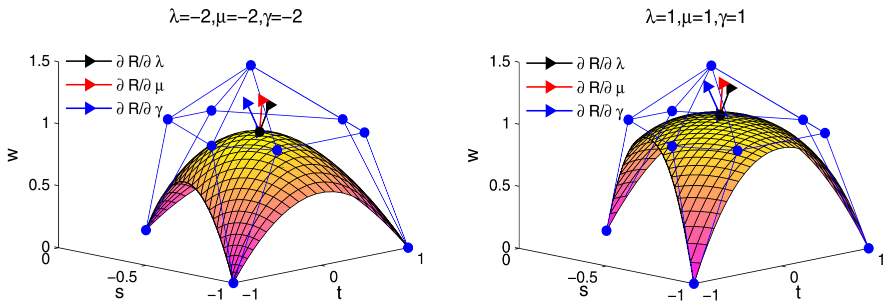

Figure 4. The directions of the three vectors $\partial R/\partial\lambda, \partial R/\partial\mu$, $\partial R/\partial\gamma$ at a fixed point $R(\pi/6, \pi/6, \pi/6)$.

3.2. De Casteljau-type Algorithm

The classical de Casteljau algorithm is a stable and efficient process for computing the triangular Bernstein–Bézier patch. Now, we want to develop a practical de Casteljau-type algorithm for computing the proposed trigonometric polynomial patch given in (3.1). For this purpose, for any $(t, s, w) \in D$, let

$$\begin{aligned}
f_1(t,s,w) &:= \frac{\sin t \cos w \left(\sin^2 t+\sin^2 s+\sin^2 w\right)}{\cos w\left(\sin t+\sin s\right)\left(\sin^2 t+\sin^2 s+\sin^2 w\right)+\sin w\left(\sin^2 t+\sin^2 s\right)},\\
f_2(t,s,w) &:= \frac{\sin s \cos w \left(\sin^2 t+\sin^2 s+\sin^2 w\right)}{\cos w\left(\sin t+\sin s\right)\left(\sin^2 t+\sin^2 s+\sin^2 w\right)+\sin w\left(\sin^2 t+\sin^2 s\right)},\\
f_3(t,s,w) &:= \frac{\sin w \left(\sin^2 t+\sin^2 s\right)}{\cos w\left(\sin t+\sin s\right)\left(\sin^2 t+\sin^2 s+\sin^2 w\right)+\sin w\left(\sin^2 t+\sin^2 s\right)},\\
g_1(t,s,w) &:= (1-\cos t)\left(\sin^2 t+\sin^2 s+\sin^2 w\right),\\
g_2(t,s,w) &:= \sin s \cos w\left(\sin^2 t+\sin^2 s+\sin^2 w\right)+\sin t \sin s \sin w,\\
g_3(t,s,w) &:= \cos s \sin w\left(\sin^2 t+\sin^2 s+\sin^2 w\right)+\sin t \sin s \sin w,
\end{aligned}$$

and

$$\begin{aligned}
P^1_{2,0,0} &:= \frac{(1-\lambda\cos t)}{1+\cos t}P_{3,0,0}+\frac{(1+\lambda)\sin s\cos w}{1+\cos t}P_{2,1,0}+\frac{(1+\lambda)\cos s\sin w}{1+\cos t}P_{2,0,1},\\
P^1_{0,2,0} &:= \frac{(1+\mu)\sin t\cos w}{1+\cos s}P_{1,2,0}+\frac{(1-\mu\cos s)}{1+\cos s}P_{0,3,0}+\frac{(1+\mu)\cos t\sin w}{1+\cos s}P_{0,2,1},\\
P^1_{0,0,2} &:= \frac{(1+\gamma)\sin t\cos s}{1+\cos w}P_{1,0,2}+\frac{(1+\gamma)\cos t\sin s}{1+\cos s}P_{0,1,2}+\frac{(1-\gamma\cos w)}{1+\cos s}P_{0,0,3},\\
P^1_{1,1,0} &:= f_1(t,s,w)P_{2,1,0}+f_2(t,s,w)P_{1,2,0}+f_3(t,s,w)P_{1,1,1},\\
P^1_{1,0,1} &:= f_1(t,w,s)P_{2,0,1}+f_3(t,w,s)P_{1,1,1}+f_2(t,w,s)P_{1,0,2},\\
P^1_{0,1,1} &:= f_3(s,w,t)P_{1,1,1}+f_1(s,w,t)P_{0,2,1}+f_2(s,w,t)P_{0,1,2}.
\end{aligned}$$

Then, we can rewrite the expression of the trigonometric polynomial patch (3.1) as follows:

$$\begin{aligned}
R(t,s,w) = {} & \frac{1-\cos^2 t}{\sin^2 t+\sin^2 s+\sin^2 w}\\
& \times\left[g_1(t,s,w)P^1_{2,0,0}+g_2(t,s,w)P^1_{1,1,0}+g_3(t,s,w)P^1_{1,0,1}\right]\\
& +\frac{1-\cos^2 s}{\sin^2 t+\sin^2 s+\sin^2 w}\\
& \times\left[g_2(s,t,w)P^1_{1,1,0}+g_1(s,t,w)P^1_{0,2,0}+g_3(s,t,w)P^1_{0,1,1}\right]\\
& +\frac{1-\cos^2 w}{\sin^2 t+\sin^2 s+\sin^2 w}\\
& \times\left[g_3(w,s,t)P^1_{1,0,1}+g_2(w,s,t)P^1_{0,1,1}+g_1(w,s,t)P^1_{0,0,2}\right].
\end{aligned} \tag{3.4}$$

Furthermore, by setting

$$\begin{aligned} P^2_{1,0,0} &:= g_1(t,s,w)P^1_{2,0,0} + g_2(t,s,w)P^1_{1,1,0} + g_3(t,s,w)P^1_{1,0,1},\\ P^2_{0,1,0} &:= g_2(s,t,w)P^1_{1,1,0} + g_1(s,t,w)P^1_{0,2,0} + g_3(s,t,w)P^1_{0,1,1},\\ P^2_{0,0,1} &:= g_3(w,s,t)P^1_{1,0,1} + g_2(w,s,t)P^1_{0,1,1} + g_1(w,s,t)P^1_{0,0,2}, \end{aligned}$$

we have

$$\begin{aligned} R(t,s,w) &= \frac{1-\cos^2 t}{\sin^2 t + \sin^2 s + \sin^2 w} P^2_{1,0,0} + \frac{1-\cos^2 s}{\sin^2 t + \sin^2 s + \sin^2 w} P^2_{0,1,0}\\ &\quad + \frac{1-\cos^2 w}{\sin^2 t + \sin^2 s + \sin^2 w} P^2_{0,0,1}\\ &:= P^3_{0,0,0}. \end{aligned} \tag{3.5}$$

For $t+s+w=\pi/2$, it is easy to check that $f_1(t,s,w)+f_2(t,s,w)+f_3(t,s,w)=1$ and $g_1(t,s,w)+g_2(t,s,w)+g_3(t,s,w)=1$ (by using Lemma 2.3). Thus (3.4) and (3.5) really indicate a de Casteljau-type algorithm for computing the proposed trigonometric polynomial patch given in (3.1).

3.3. Joining Two Trigonometric Polynomial Patches

In practical surface construction, we often need to join several patches together to generate shapes that are too complex to handle with a single patch. During the joining trigonometric polynomial patches, we need to control the smoothness of the connecting surface. Let two trigonometric polynomial patches be

$$R_1(t,s,w) = \sum_{i+j+k=3} T^3_{i,j,k}(t,s,w;\lambda_1,\mu,\gamma)P_{i,j,k}, \quad (t,s,w)\in D, \tag{3.6}$$

and

$$R_2(t,s,w) = \sum_{i+j+k=3} T^3_{i,j,k}(t,s,w;\lambda_2,\mu,\gamma)Q_{i,j,k}, \quad (t,s,w)\in D, \tag{3.7}$$

respectively.

Apparently, if the control points satisfy

$$P_{0,j,k} = Q_{0,j,k}, \quad j,k\in\mathbb{N}, j+k=3, \tag{3.8}$$

the two patches join along a common boundary curve: $R_1(0,s,w)=R_2(0,s,w)$, $s+w=\pi/2$. Thus, the two patches clearly form a surface with positional continuity, or a surface with C^0 continuity.

For the common boundary curve $R_1(0,s,\pi/2-s)$, differentiating with respect to s, we have

$$\begin{aligned} \frac{dR_1(0,s,\pi/2-s)}{ds} &= \sin s(1-\cos s)(2+\mu-3\mu\cos s)(P_{0,3,0}-P_{0,2,1})\\ &\quad +2\sin s\cos s(P_{0,2,1}-P_{0,1,2})\\ &\quad +\cos s(1-\sin s)(2+\gamma-3\gamma\sin s)(P_{0,1,2}-P_{0,0,3}). \end{aligned} \tag{3.9}$$

For $R_1(t, s, \pi/2 - t - s)$ and $R_2(t, s, \pi/2 - t - s)$, differentiating with respect to t respectively, we get

$$\left.\frac{\partial R_1(t, s, \pi/2 - t - s)}{\partial t}\right|_{t=0} = \sin s(1 - \cos s)(2 + \mu - 3\mu \cos s)(P_{1,2,0} - P_{0,2,1}) \\ +2 \sin s \cos s(P_{1,1,1} - P_{0,1,2}) \\ + \cos s(1-\sin s)(2+\gamma-3\gamma \sin s)(P_{1,0,2}-P_{0,0,3}), \tag{3.10}$$

$$\left.\frac{\partial R_2(t, s, \pi/2 - t - s)}{\partial t}\right|_{t=0} = \sin s(1 - \cos s)(2 + \mu - 3\mu \cos s)(Q_{1,2,0} - Q_{0,2,1}) \\ +2 \sin s \cos s(Q_{1,1,1} - Q_{0,1,2}) \\ + \cos s(1-\sin s)(2+\gamma-3\gamma \sin s)(Q_{1,0,2}-Q_{0,0,3}). \tag{3.11}$$

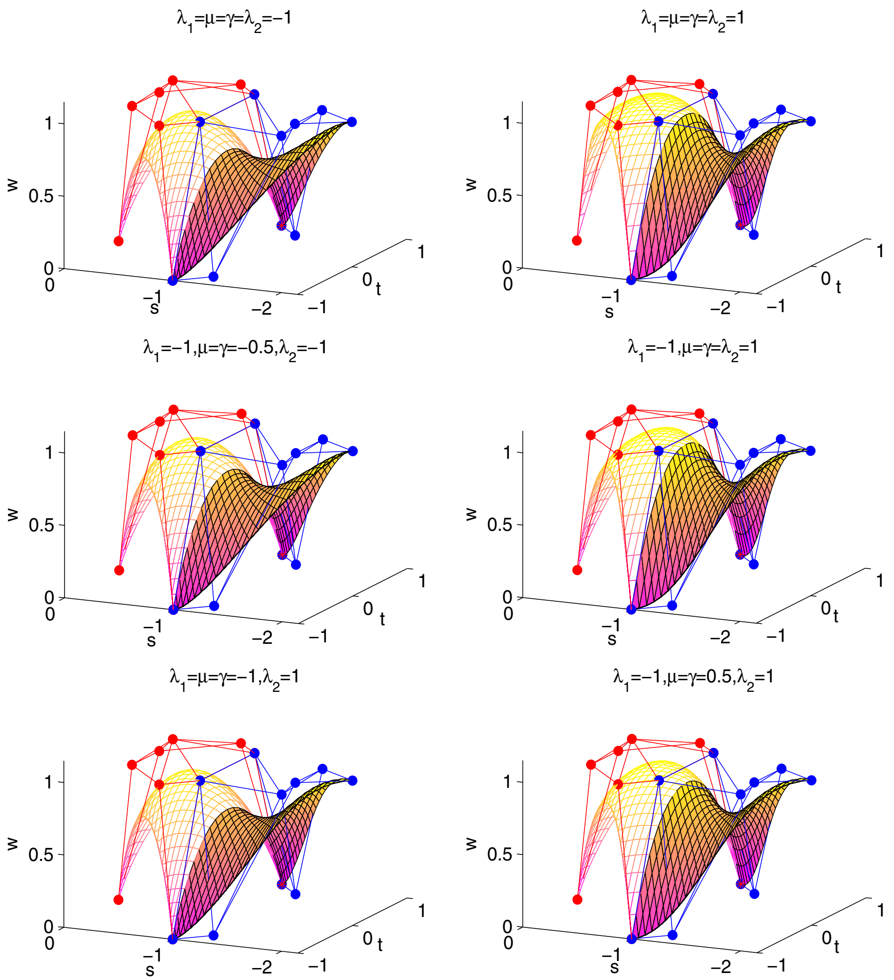

Figure 5. G^1 continuous smooth joining two trigonometric polynomial patches with different shape parameters.

The condition for smooth joining is that the vectors defined by Eqs. (3.9) through (3.11) are coplanar for any value of s, see [18], which can be expressed as follows:

$$\left.\frac{\partial R_2(t,s,\pi/2-t-s)}{\partial t}\right|_{t=0} = \phi\frac{dR_1(0,s,\pi/2-s)}{ds} + \varphi \left.\frac{\partial R_1(t,s,\pi/2-t-s)}{\partial t}\right|_{t=0},$$

where ϕ and φ both are constants. From these, we can obtain a rule

$$\begin{cases} Q_{1,2,0} - Q_{0,2,1} = \phi(P_{0,3,0} - P_{0,2,1}) + \varphi(P_{1,2,0} - P_{0,2,1}), \\ Q_{1,1,1} - Q_{0,1,2} = \phi(P_{0,2,1} - P_{0,1,2}) + \varphi(P_{1,1,1} - P_{0,1,2}), \\ Q_{1,0,2} - Q_{0,0,3} = \phi(P_{0,1,2} - P_{0,0,3}) + \varphi(P_{1,0,2} - P_{0,0,3}). \end{cases} \tag{3.12}$$

Summarizing the above discussion, we can conclude the following theorem.

Theorem 3.2. *For $\lambda_l, \mu, \gamma \in [-2, 1], l = 1, 2$, the surface connected* (3.6) *with* (3.7) *is G^1 continuous, if the control points satisfy the conditions* (3.8) *and* (3.12).

From Theorem 3.2, we can see that the conditions for smooth joining two trigonometric polynomial patches are analogous to that for joining two triangular Bernstein–Bézier cubic patches; see [18]. However, we can adjust the shape of the obtained G^1 continuous surface conveniently using the shape parameters in the trigonometric polynomial patches.

Figure 5 shows the G^1 continuous surface generated by smooth joining two trigonometric polynomial patches with different shape parameters. The parameters take fixed values $\phi = 1$ and $\varphi = -1$.

4. Conclusion

The given trigonometric polynomial basis over the triangular domain with three shape parameters is a new construction for geometric design and computing, which has properties of nonnegativity, partition of unity, symmetry, linear independence and so on. Using the new basis, we propose a class of trigonometric polynomial patch over the triangular domain, which has some properties analogous to that of the triangular Bernstein–Bézier cubic patch and is useful for generating some surfaces whose boundaries are arcs of an ellipse or parabola. The boundary curves of the trigonometric polynomial patch are precisely the T-Bézier curves given in [5,9]. The newly developed de Casteljau-type algorithm is practical for computing the proposed trigonometric polynomial patch. Without changing the control net, the shape of the obtained G^1 continuous smooth surface can be adjusted predictably and conveniently using the three shape parameters.

Even though many properties of the given trigonometric polynomial basis over the triangular domain have been discussed in detail, the approximation power of the basis has not been discussed. In practice, the extension of the given trigonometric polynomial basis over the triangular domain to higher degrees and subdivision algorithm for the proposed trigonometric polynomial patch are important considerations. More work is needed to address these problems.

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