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Continuity and Schatten–von Neumann Properties for Localization Operators on Modulation Spaces

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Abstract. We use sharp convolution estimates for weighted Lebesgue and modulation spaces to obtain an extension of the celebrated Cordero-Gröchenig theorems on boundedness and Schatten–von Neumann properties of localization operators on modulation spaces. We also give a new proof of the Weyl connection based on the kernel theorem for Gelfand–Shilov spaces.

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1. Introduction

We use sharp convolution estimates from [37] to extend the celebrated Cordero–Gröchenig theorems on boundedness and Schatten–von Neumann properties of localization operators on modulation spaces [6]. Related results are given by Toft in [32–35]. See also [6–12] for different aspects on localization operators.

General results on products and convolution from [37] include six Lebesgue and six weight parameters. This generality is used here to prove a refinement of certain known results from the above-mentioned references. For example, in [6] window functions (see Definition 1.1 below) are chosen to be in the same space, while here we allow different Lebesgue and weight parameters for different windows.

To define a localization operator we start with the short-time Fourier transform, a time–frequency representation related to Feichtinger's modulation spaces cf. [14].

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Let $\mathcal{S}^{(1)}(\mathbb{R}^d)$ be the Gelfand–Shilov space of smooth functions given by:

$$f \in \mathcal{S}^{(1)}(\mathbb{R}^d) \Longleftrightarrow \|f(x)e^{h\cdot|x|}\|_{L^\infty} < \infty \ \text{ and } \|\hat{f}(\omega)e^{h\cdot|\omega|}\|_{L^\infty} < \infty, \ \ \forall h > 0.$$

Any $f \in \mathcal{S}^{(1)}(\mathbb{R}^d)$ can be extended to the complex domain as holomorphic functions in a strip [17]. The dual space of $\mathcal{S}^{(1)}(\mathbb{R}^d)$ will be denoted by $(\mathcal{S}^{(1)})'(\mathbb{R}^d)$.

The short-time Fourier transform (STFT in the sequel) of $f \in \mathcal{S}^{(1)}(\mathbb{R}^d)$ with respect to the window $g \in \mathcal{S}^{(1)}(\mathbb{R}^d)\backslash 0$ is given by

$$V_g f(x,\omega) = \langle f, M_\omega T_x g\rangle = \int_{\mathbb{R}^d} f(t)\,\overline{g(t-x)}\,e^{-2\pi i\omega t}\,dt. \tag{1.1}$$

The map $(f,g) \mapsto V_g f$ from $\mathcal{S}^{(1)}(\mathbb{R}^d)\times\mathcal{S}^{(1)}(\mathbb{R}^d)$ to $\mathcal{S}^{(1)}(\mathbb{R}^{2d})$ extends uniquely to a continuous mapping from $(\mathcal{S}^{(1)})'(\mathbb{R}^d) \times (\mathcal{S}^{(1)})'(\mathbb{R}^d)$ to $(\mathcal{S}^{(1)})'(\mathbb{R}^{2d})$ by duality.

Moreover, for a fixed $g \in \mathcal{S}^{(1)}(\mathbb{R}^d)\backslash 0$, the following characterization holds:

$$f \in \mathcal{S}^{(1)}(\mathbb{R}^d) \Longleftrightarrow V_g f \in \mathcal{S}^{(1)}(\mathbb{R}^{2d}). \tag{1.2}$$

We refer to [22,31,36] for the proof and more details on STFT in more general Gelfand–Shilov type spaces. For our purposes, the duality between $\mathcal{S}^{(1)}(\mathbb{R}^d)$ and $(\mathcal{S}^{(1)})'(\mathbb{R}^d)$ will suffice, and we use it here for the simplicity and the clarity of exposition.

Definition 1.1. *Let $f \in \mathcal{S}^{(1)}(\mathbb{R}^d)$. The* localization operator $A_a^{\varphi_1,\varphi_2}$ *with* symbol $a \in \mathcal{S}^{(1)'}(\mathbb{R}^{2d})$ *and* windows $\varphi_1,\varphi_2 \in \mathcal{S}^{(1)}(\mathbb{R}^d)$ *is given by*

$$A_a^{\varphi_1,\varphi_2} f(t) = \int_{\mathbb{R}^{2d}} a(x,\omega) V_{\varphi_1} f(x,\omega) M_\omega T_x \varphi_2(t)\,dxd\omega. \tag{1.3}$$

In the weak sense,

$$\langle A_a^{\varphi_1,\varphi_2} f, g\rangle = \langle a V_{\varphi_1} f, V_{\varphi_2} g\rangle = \langle a, \overline{V_{\varphi_1} f}\, V_{\varphi_2} g\rangle, \quad f,g \in \mathcal{S}^{(1)}(\mathbb{R}^d), \tag{1.4}$$

where the brackets express a suitable duality between a pair of dual spaces. Indeed, $A_a^{\varphi_1,\varphi_2}$ is a well-defined continuous operator from $\mathcal{S}^{(1)}(\mathbb{R}^d)$ to $(\mathcal{S}^{(1)})'$ $(\mathbb{R}^d)$.

Localization operators were introduced by Berezin in the study of general Hamiltonians satisfying the so-called Feynman inequality, within a quantization problem in quantum mechanics [2].

In signal analysis, they are related to the localization technique developed by Slepian–Polak–Landau; we refer to [30] for an overview.

Localization in phase space and basic facts on localization operators, together with references to applications in optics and signal analysis are given in [13], which initiated further study of the topic. More precisely, in [13], Daubechies studied localization operators $A_a^{\varphi_1,\varphi_2}$ with Gaussian windows

$$\varphi_1(t) = \varphi_2(t) = \pi^{-d/4}\exp(-t^2/2) \quad \text{and with a radial symbol} \quad a \in L^1(\mathbb{R}^{2d}).$$

Such operators are named Daubechies operators afterward. The eigenfunctions of Daubechies operators are Hermite functions:

$$h_n(t) = (-1)^n \pi^{-1/4}(2^n n!)^{-1/2}\exp(t^2/2)(\exp(-t^2))^{(n)}, \quad n = 0,1,\ldots,$$

and eigenvalues can be explicitly calculated. This is an important issue in applications, cf. [27].

Note that Hermite functions belong not only to $\mathcal{S}^{(1)}(\mathbb{R}^d)$, but also to test function spaces for quasi-analytic ultradistributions. In that context, Hermite functions give rise to important representation theorems [24].

Localization operators of the form $\langle L_{\chi_\Omega} f, g\rangle = \iint_\Omega W(f,g)$, where

$$W(f,g)(x,\omega) = \int f\left(x+\frac{t}{2}\right)\overline{g\left(x-\frac{t}{2}\right)}e^{-2\pi i\omega t}\,dt, \quad f,g\in L^2(\mathbb{R}), \qquad (1.5)$$

is the Wigner transform (see [16], it is also known as the cross-Wigner distribution), were studied in [27] in the context of signal analysis. There it is proved that if $\Omega \subset [-B,B]\times[-T,T]$ is an open set such that all its cross sections in both ω and x directions consist of at most M intervals, then the eigenfunctions of L_{χ_Ω} belong to $\mathcal{S}^{(1)}(\mathbb{R}^d)$.

Inverse problem for a simply connected localization domain Ω has been recently studied in [1]. It is proved that if one of the eigenfunctions of Daubechies operator is a Hermite function, then Ω is a disc centered at the origin.

In abstract harmonic analysis, localization operators on a locally compact group G and $L^p(G)$, $1\leq p\leq\infty$, were studied in [40] where one can find a product formula and Schatten–von Neumann properties of localization operators; see also [4].

Since the beginning of the twenty-first century, localization operators in the context of modulation spaces were studied by many authors, cf. [6,7,12,15,34,35]. See also the references given there.

For example, different choices of windows and symbols of localization operators give rise to different continuity, compactness and Schatten–von Neumann properties [6,9,34,35], composition formulas and Fredholm property [7,12], multilinear versions [8], eigenvalue and eigenfunctions estimates [1,13,27].

We will use the standard tools in the study of localization operators such as

(a) STFT, cross-Wigner distribution, and the Weyl transform representation of localization operators,
(b) continuity properties of pseudo-differential operators, and
(c) convolution and multiplication in Lebesgue and modulation spaces.

In particular, we prove the Weyl connection (see Lemma 3.3) in a different manner from that in [3,16].

Notation. The Schwartz class is denoted by $\mathcal{S}(\mathbb{R}^d)$, and the space of tempered distributions by $\mathcal{S}'(\mathbb{R}^d)$. We use brackets $\langle f,g\rangle$ to denote the extension of the inner product $\langle f,g\rangle = \int f(t)\overline{g(t)}dt$ on $L^2(\mathbb{R}^d)$ to any pair of dual spaces. The Fourier transform is normalized to be $\hat f(\omega) = \mathcal{F}f(\omega) = \int f(t)e^{-2\pi i t\omega}dt$. The involution f^* is $f^*(\cdot) = \overline{f(-\cdot)}$, and the convolution of f and g is given by $f*g(x) = \int f(x-y)g(y)dy$, when the integral exists.

We denote by $\langle\cdot\rangle^s$ the polynomial weights

$$\langle(x,\omega)\rangle^s = (1+|x|^2+|\omega|^2)^{s/2}, \quad (x,\omega)\in\mathbb{R}^{2d}, \quad s\in\mathbb{R},$$

and $\langle x\rangle = \langle 1+|x|^2\rangle$, when $x\in\mathbb{R}^d$.

We use the notation $A\lesssim B$ to indicate $A\le cB$ for a suitable constant $c>0$, whereas $A\asymp B$ means that $c^{-1}A\le B\le cA$ for some $c\ge 1$.

Recall, the Schatten class S_p, $1\le p<\infty$, consists of all compact operators with singular values in l^p, and S_∞ is the space of bounded operators on $L^2(\mathbb{R}^d)$. The singular values of a compact operator $L\in S_\infty$ are the eigenvalues of $\sqrt{L^*L}$.

2. Modulation Spaces

Modulation spaces are defined through decay and integrability conditions on STFT, which makes them suitable for time–frequency analysis and for the study of localization operators in particular. The definition is given in terms of weighted mixed-norm Lebesgue spaces.

In general, a weight $w(\cdot)$ on $\mathbb{R}^d$ is a non-negative and continuous function. By $L^p_w(\mathbb{R}^d)$, $p\in[1,\infty]$ we denote the weighted Lebesgue space defined by the norm

$$\|f\|_{L^p_w} = \|fw\|_{L^p} = \left(\int |f(x)|^p w(x)^p dx\right)^{1/p},$$

with the usual modification when $p=\infty$. When $w(x)=\langle x\rangle^t$ $t\in\mathbb{R}$, we use the notation $L^p_t(\mathbb{R}^d)$ instead.

Similarly, the weighted mixed-norm space $L^{p,q}_w(\mathbb{R}^{2d})$, $p,q\in[1,\infty]$ consists of (Lebesgue) measurable functions on $\mathbb{R}^{2d}$ such that

$$\|F\|_{L^{p,q}_w} = \left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|F(x,\omega)|^p w(x,\omega)^p dx\right)^{q/p} d\omega\right)^{1/q} < \infty,$$

where $w(x,\omega)$ is a weight on $\mathbb{R}^{2d}$.

In particular, when $w(x,\omega)=\langle x\rangle^t\langle\omega\rangle^s$, $s,t\in\mathbb{R}$, we will use the notation $L^{p,q}_w(\mathbb{R}^{2d}) = L^{p,q}_{t,s}(\mathbb{R}^{2d})$.

Now, modulation space $M^{p,q}_{s,t}(\mathbb{R}^d)$ consists of distributions whose STFT belong to $L^{p,q}_{t,s}(\mathbb{R}^{2d})$:

Definition 2.1. *Let $\phi\in\mathcal{S}(\mathbb{R}^d)\backslash 0$, $s,t\in\mathbb{R}$ and $p,q\in[1,\infty]$. The* modulation space $M^{p,q}_{s,t}(\mathbb{R}^d)$ *consists of all $f\in\mathcal{S}'(\mathbb{R}^d)$ s. t.*

$$\|f\|_{M^{p,q}_{s,t}} \equiv \left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|V_\phi f(x,\omega)\langle x\rangle^t\langle\omega\rangle^s|^p\, dx\right)^{q/p} d\omega\right)^{1/q} < \infty \tag{2.1}$$

(*with obvious interpretation of the integrals when $p=\infty$ or $q=\infty$*).

In special cases, we use the usual abbreviations: $M^{p,p}_{0,0}=M^p$, $M^{p,p}_{t,t}=M^p_t$, etc.

Remark 2.2. Notice that the original definition given in [14] contains more general, *submultiplicative* weights. We restrict ourselves to $w(x,\omega) = \langle x\rangle^t \langle \omega\rangle^s$, $s,t \in \mathbb{R}$, since the convolution and multiplication estimates which will be used later on involve weighted spaces with such polynomial weights. The weights of (almost) exponential growth are used in the study of Gelfand–Shilov spaces and their duals in cf. [9,22,31,36]. We refer to [19] for a survey on the most important types of weights commonly used in time–frequency analysis.

The following theorem lists some of the basic properties of modulation spaces. We refer to [14,18] for its proof.

Theorem 2.3. *Let $p, q, p_j, q_j \in [1,\infty]$ and $s, t, s_j, t_j \in \mathbb{R}$, $j = 1, 2$. Then:*

(1) $M^{p,q}_{s,t}(\mathbb{R}^d)$ *are Banach spaces, independent of the choice of* $\phi \in \mathcal{S}(\mathbb{R}^d)\backslash 0$*;*
(2) *if* $p_1 \le p_2$, $q_1 \le q_2$, $s_2 \le s_1$ *and* $t_2 \le t_1$, *then*

$$\mathcal{S}(\mathbb{R}^d) \subseteq M^{p_1,q_1}_{s_1,t_1}(\mathbb{R}^d) \subseteq M^{p_2,q_2}_{s_2,t_2}(\mathbb{R}^d) \subseteq \mathcal{S}'(\mathbb{R}^d);$$

(3) $\cap_{s,t} M^{p,q}_{s,t}(\mathbb{R}^d) = \mathcal{S}(\mathbb{R}^d), \quad \cup_{s,t} M^{p,q}_{s,t}(\mathbb{R}^d) = \mathcal{S}'(\mathbb{R}^d)$.

Modulation spaces include the following well-known function spaces:

(a) $M^2(\mathbb{R}^d) = L^2(\mathbb{R}^d)$.
(b) $M^2_{t,0}(\mathbb{R}^d) = L^2_t(\mathbb{R}^d)$.
(c) *Sobolev spaces* $M^2_{0,s}(\mathbb{R}^d) = H^2_s(\mathbb{R}^d) = \{f \mid \hat{f}(\omega)\langle\omega\rangle^s \in L^2(\mathbb{R}^d)\}$.
(d) *Shubin spaces* $M^2_s(\mathbb{R}^d) = L^2_s(\mathbb{R}^d) \cap H^2_s(\mathbb{R}^d) = Q_s(\mathbb{R}^d)$, cf. [28].

3. Main Results

We introduce the *Young functional*:

$$\mathsf{R}(\mathrm{p}) \equiv 2 - \frac{1}{p_0} - \frac{1}{p_1} - \frac{1}{p_2}, \qquad p = (p_0, p_1, p_2) \in [1,\infty]^3. \tag{3.1}$$

When $\mathsf{R}(\mathrm{p}) = 0$, the Young inequality for convolution reads as

$$\|f_1 * f_2\|_{L^{p_0'}} \le \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}}, \quad f_j \in L^{p_j}(\mathbb{R}^d), \;\; j = 1, 2.$$

We give a version of this inequality for weighted Lebesgue spaces when $0 \le \mathsf{R}(\mathrm{p}) \le 1/2$.

The following theorem is an extension of the Young inequality to the case of weighted Lebesgue spaces and modulation spaces.

Theorem 3.1. *Let $s_j, t_j \in \mathbb{R}$, $p_j, q_j \in [1,\infty]$, $j = 0, 1, 2$. Assume that $0 \le \mathsf{R}(\mathrm{p}) \le 1/2$, $\mathsf{R}(\mathrm{q}) \le 1$,*

$$0 \le t_j + t_k, j, k = 0, 1, 2, \quad j \ne k, \tag{3.2}$$

$$0 \le t_0 + t_1 + t_2 - d \cdot \mathsf{R}(\mathrm{p}), \quad \text{and} \tag{3.3}$$

$$0 \le s_0 + s_1 + s_2, \tag{3.4}$$

with strict inequality in (3.3) when $\mathsf{R}(\mathrm{p}) > 0$ and $t_j = d \cdot \mathsf{R}(\mathrm{p})$ for some $j = 0, 1, 2$.

Then, $(f_1, f_2) \mapsto f_1 * f_2$ on $C_0^\infty(\mathbb{R}^d)$ extends uniquely to a continuous map from

(1) $L^{p_1}_{t_1}(\mathbb{R}^d) \times L^{p_2}_{t_2}(\mathbb{R}^d)$ *to* $L^{p_0'}_{-t_0}(\mathbb{R}^d)$;
(2) $M^{p_1,q_1}_{s_1,t_1}(\mathbb{R}^d) \times M^{p_2,q_2}_{s_2,t_2}(\mathbb{R}^d)$ *to* $M^{p_0',q_0'}_{-s_0,-t_0}(\mathbb{R}^d)$.

For the proof we refer to [37]. It is based on a detailed study of an auxiliary three-linear map over carefully chosen regions in $\mathbb{R}^d$; see Subsections 3.1 and 3.2 in [37]. This result extends multiplication and convolution properties obtained in [26]. Moreover, the result is sharp in the following sense.

Proposition 3.2. *Let* $p_j, q_j \in [1,\infty]$ *and* $s_j, t_j \in \mathbb{R}$, $j = 0,1,2$. *Assume that at least one of the following statements hold true:*

(1) *The map* $(f_1, f_2) \mapsto f_1 * f_2$ *on* $C_0^\infty(\mathbb{R}^d)$ *is continuously extendable to a map from* $L^{p_1}_{t_1}(\mathbb{R}^d) \times L^{p_2}_{t_2}(\mathbb{R}^d)$ *to* $L^{p_0'}_{-t_0}(\mathbb{R}^d)$;
(2) *The map* $(f_1, f_2) \mapsto f_1 * f_2$ *on* $C_0^\infty(\mathbb{R}^d)$ *is continuously extendable to a map from* $M^{p_1,q_1}_{s_1,t_1}(\mathbb{R}^d) \times M^{p_2,q_2}_{s_2,t_2}(\mathbb{R}^d)$ *to* $M^{p_0',q_0'}_{-s_0,-t_0}(\mathbb{R}^d)$;

Then, (3.2) *and* (3.3) *hold true.*

Again, we refer to [37] for the proof.

Next, we give a Weyl transform representation of localization operators.

Let L_σ be the Weyl pseudo-differential operator with the Weyl symbol $\sigma \in \mathcal{S}^{(1)'}(\mathbb{R}^{2d})$:

$$\langle L_\sigma f, g\rangle = \langle \sigma, W(g,f)\rangle, \quad f,g \in \mathcal{S}^{(1)}(\mathbb{R}^d). \tag{3.5}$$

In fact, if $\sigma \in \mathcal{S}^{(1)}(\mathbb{R}^{2d})$, then the Weyl pseudo-differential operator L_σ is defined as the oscillatory integral:

$$L_\sigma f(x) = \iint \sigma(\frac{x+y}{2}, \omega) f(y) e^{2\pi(x-y)\cdot\omega} dy d\omega, \quad f \in \mathcal{S}^{(1)}(\mathbb{R}^{2d}).$$

It extends to each $\sigma \in \mathcal{S}^{(1)'}(\mathbb{R}^{2d})$, and then L_σ is continuous from $\mathcal{S}^{(1)}(\mathbb{R}^{2d})$ to $\mathcal{S}^{(1)'}(\mathbb{R}^{2d})$. With this definition, (3.5) is proved in, e.g., [16,18,39].

Next, we establish the so-called Weyl connection, which shows that the set of localization operators is a subclass of the set of Weyl operators. Although the same result (in the context of the Schwartz class) can be found elsewhere ([3,16]), it is given here to be self-contained. The proof is based on kernel theorem for Gelfand–Shilov spaces, and direct calculation.

Lemma 3.3. *If* $a \in \mathcal{S}^{(1)'}(\mathbb{R}^{2d})$ *and* $\varphi_1, \varphi_2 \in \mathcal{S}^{(1)}(\mathbb{R}^d)$, *then the localization operator* $A_a^{\varphi_1,\varphi_2}$ *is Weyl pseudo-differential operator with the Weyl symbol* $\sigma = a * W(\varphi_2, \varphi_1)$; *in other words,*

$$A_a^{\varphi_1,\varphi_2} = L_{a*W(\varphi_2,\varphi_1)}. \tag{3.6}$$

The proof of Lemma 3.3 is given in Sect. 3.3. Note that Lemma 3.3 can be proved in the quasi-analytic case by the same arguments. However, in this paper we do not need such an extension. Notice also that in the literature the symbol a in Lemma 3.3 is called the *anti-Wick symbol* of the Weyl pseudo-differential operator L_σ.

Lemma 3.3 describes the localization operators in terms of the convolution. The smoothing effect of convolution gives boundedness of localization operators over different spaces even if a is a distribution. It is shown in [6] that polynomial weights give rise to boundedness result for operators with symbols which are compactly supported tempered distributions. We refer to [9] for related results in the context of non-quasi-analytic classes. Moreover, representation theorems based on the heat kernel and parametrix techniques lead to trace-class result for certain quasi-analytic distributions; see [10].

We start with an estimate of modulation space norm of the cross-Wigner distribution.

Proposition 3.4. *Let $t_j \in \mathbb{R}$, $p_j \in [1,\infty]$, $j = 0,1,2$, $0 \le \mathsf{R}(\mathrm{p}) \le 1/2$, $0 \le t_j + t_k$, $j,k = 0,1,2$, $j \ne k$, and $0 \le t_0 + t_1 + t_2 - d \cdot \mathsf{R}(\mathrm{p})$, with strict inequality when $\mathsf{R}(\mathrm{p}) > 0$ and $t_j = d \cdot \mathsf{R}(\mathrm{p})$ for some $j = 0,1,2$.*

If $\varphi_j \in M^{p_j}_{t_j}(\mathbb{R}^d)$, $j = 1,2$, then the map $(\varphi_1,\varphi_2) \mapsto W(\varphi_2,\varphi_1)$ where W is the cross-Wigner distribution given by (1.5) is a sesquilinear continuous map from $M^{p_2}_{t_2}(\mathbb{R}^d) \times M^{p_1}_{t_1}(\mathbb{R}^d)$ to $M^{1,p_0'}_{-t_0,0}(\mathbb{R}^{2d})$.

Proof. Note that in the $2d-$case,

$$\|F\|_{M^{p,q}_{s,t}(\mathbb{R}^{2d})} = \left(\int_{\mathbb{R}^{2d}} \left(\int_{\mathbb{R}^{2d}} |V_\Phi F(z,\zeta)\langle z\rangle^t \langle \zeta\rangle^s|^p \, dz\right)^{q/p} d\zeta\right)^{1/q}$$

for some $\Phi \in \mathcal{S}(\mathbb{R}^{2d})\backslash 0$.

Let $\psi_1,\psi_2 \in \mathcal{S}(\mathbb{R}^d)\backslash 0$. Then,

$$\|W(\varphi_2,\varphi_1)\|_{M^{1,p_0'}_{-t_0,0}} \asymp \|(V_{W(\psi_1,\psi_2)} W(\varphi_2,\varphi_1))(z,\zeta)\langle\zeta\rangle^{-t_0}\|_{L^{1,p_0'}},$$

where $z,\zeta \in \mathbb{R}^{2d}$.

By the relation between the Wigner transform and the STFT:

$$W(f,g)(x,\omega) = 2^d e^{4\pi i x\cdot\omega} V_{\overline{g^*}} f(2x,2\omega), \quad f,g \in \mathcal{S}^{(1)}(\mathbb{R}^d)$$

(see [18, Lemma 4.3.1] and the proof of [18, Lemma 14.5.1 (b)]), it follows that

$$\begin{aligned}
&(V_{W(\psi_1,\psi_2)} W(\varphi_2,\varphi_1))(z,\zeta)\\
&\quad = e^{-2\pi i z_2\zeta_2} \overline{V_{\psi_1}\varphi_1}(z_1 + \frac{\zeta_2}{2}, z_2 - \frac{\zeta_1}{2}) V_{\psi_2}\varphi_2(z_1 - \frac{\zeta_2}{2}, z_2 + \frac{\zeta_1}{2}),
\end{aligned}$$

with $z = (z_1,z_2) \in \mathbb{R}^d \times \mathbb{R}^d$ and $\zeta = (\zeta_1,\zeta_2) \in \mathbb{R}^d \times \mathbb{R}^d$. Hence

$$\begin{aligned}
&\|W(\varphi_2,\varphi_1)\|_{M^{1,p_0'}_{-t_0,0}}\\
&\asymp \left(\int_{\mathbb{R}^{2d}} \left(\int_{\mathbb{R}^{2d}} |V_{\psi_1}\varphi_1(z_1 + \frac{\zeta_2}{2}, z_2 - \frac{\zeta_1}{2})||V_{\psi_2}\varphi_2(z_1 - \frac{\zeta_2}{2}, z_2 + \frac{\zeta_1}{2})|dz\right) \langle\zeta\rangle^{-t_0 p_0'} d\zeta\right)^{1/p_0'}\\
&= \left(\int_{\mathbb{R}^{2d}} (|V_{\psi_1}\varphi_1| * |\overline{V_{\psi_2}\varphi_2}^*|)(\zeta_2,-\zeta_1)\langle(\zeta_2,-\zeta_1)\rangle^{-t_0 p_0'} d\zeta\right)^{1/p_0'},
\end{aligned}$$

where the convolution is obtained from the integration over z after the the change of variables $(z_1,z_2) \mapsto (z_1 - \frac{\zeta_2}{2}, z_2 + \frac{\zeta_1}{2})$; see also [6, Proposition 2.5]. Therefore,

$$\|W(\varphi_2,\varphi_1)\|_{M^{1,p_0'}_{-t_0,0}} \lesssim \|\, |V_{\psi_1}\varphi_1| * |V_{\psi_2}\varphi_2|^* \,\|_{L^{p_0'}_{-t_0}} \lesssim \|V_{\psi_1}\varphi_1\|_{L^{p_1}_{t_1}} \|V_{\psi_2}\varphi_2\|_{L^{p_2}_{t_2}},$$

where the last estimate follows from Theorem 3.1 (1). □

Remark 3.5. Our result extends some known estimates. In particular, when $s = -t_0 = t_1 = t_2 \geq 0$, $p = p_0' = p_2 \in [1, \infty]$ and $p_1 = 1$, we obtain

$$\|W(\varphi_2, \varphi_1)\|_{M^{1,p}_{s,0}} \lesssim \|\varphi_1\|_{M^1_s} \|\varphi_2\|_{M^p_s}, \tag{3.7}$$

that is [6, Proposition 2.5] (with a slightly different notation).

In certain situations (in particular, when $p_0' \neq p_2$), we obtain sharper estimates for the modulation space norm of the cross-Wigner distribution with respect to (3.7). For example, if $p_1 = 1$, $p_2 = \infty$ and $p \geq 2$, we obtain

$$\|W(\varphi_2, \varphi_1)\|_{M^{1,p}_{-t_0,0}} \lesssim \|\varphi_1\|_{M^1_{t_1}} \|\varphi_2\|_{M^\infty_{t_2}},$$

with $0 < t_0 + t_1 + t_2 - d \cdot \mathsf{R}(\mathrm{p})$ when $p = \infty$ and $t_j = d \cdot \mathsf{R}(\mathrm{p})$ for some $j = 0, 1, 2$.

3.1. Continuity Properties

In this subsection, we use the relation between the Weyl pseudo-differential operators and localization operators, from Lemma 3.3, and convolution results for modulation spaces from Theorem 3.1 to obtain continuity results for $A_a^{\varphi_1,\varphi_2}$ for different choices of windows and symbol.

Let σ be the Weyl symbol of L_σ. By [18, Theorem 14.5.2], if $\sigma \in M^{\infty,1}(\mathbb{R}^{2d})$, then L_σ is bounded on $M^{p,q}(\mathbb{R}^d)$, $1 \leq p, q \leq \infty$. This result has a long history starting with the Calderon–Vaillancourt theorem on boundedness of pseudo-differential operators with smooth and bounded symbols on $L^2(\mathbb{R}^d)$ [5]. It is extended by Sjöstrand in [29] where $M^{\infty,1}$ is used as appropriate symbol class. Sjöstrand's results were thereafter further extended in [18,20,21,32–34].

Theorem 3.6. *Let the assumptions of Theorem 3.1 hold. If $\varphi_j \in M^{p_j}_{t_j}(\mathbb{R}^d)$, $j = 1, 2$, and $a \in M^{\infty,r}_{u,v}(\mathbb{R}^{2d})$ where $1 \leq r \leq p_0$, $u \geq t_0$ and $v \geq d\mathsf{R}(\mathrm{p})$ with $v > d\mathsf{R}(\mathrm{p})$ when $\mathsf{R}(\mathrm{p}) > 0$, then $A_a^{\varphi_1,\varphi_2}$ is bounded on $M^{p,q}(\mathbb{R}^d)$, for all $1 \leq p, q \leq \infty$ and the operator norm satisfies the uniform estimate*

$$\|A_a^{\varphi_1,\varphi_2}\|_{op} \lesssim \|a\|_{M^{\infty,r}_{u,v}} \|\varphi_1\|_{M^{p_1}_{t_1}} \|\varphi_2\|_{M^{p_2}_{t_2}}.$$

Proof. Let $\varphi_j \in M^{p_j}_{t_j}(\mathbb{R}^d)$, $j = 1, 2$. Then, by Proposition 3.4 it follows that $W(\varphi_2, \varphi_1) \in M^{1,p_0'}_{-t_0,0}(\mathbb{R}^{2d})$. This fact, together with Theorem 3.1 (2), implies that

$$a * W(\varphi_2, \varphi_1) \in M^{\tilde{p},1}(\mathbb{R}^{2d}), \quad \tilde{p} \geq 2,$$

if the involved parameters fulfill the conditions of the theorem. Concerning the Lebesgue parameters, it is easy to see that $\tilde{p} \geq 2$ is equivalent to $\mathsf{R}(\mathrm{p}) = \mathsf{R}(p, \infty, 1) \in [0, 1/2]$, and that $1 \leq r \leq p_0$ is equivalent to $\mathsf{R}(\mathrm{q}) = \mathsf{R}(\infty, r, p_0') \leq 1$. It is also straightforward to check that the choice of the weight parameters u and v implies that $a * W(\varphi_2, \varphi_1) \in M^{\tilde{p},1}(\mathbb{R}^{2d})$, $\tilde{p} \geq 2$.

In particular, if $\tilde{p} = \infty$ then $a * W(\varphi_2, \varphi_1) \in M^{\infty,1}(\mathbb{R}^{2d})$. From [18, Theorem 14.5.2] (and Lemma 3.3), it follows that $A_a^{\varphi_1,\varphi_2}$ is bounded on $M^{p,q}(\mathbb{R}^d)$, $1 \leq p, q \leq \infty$.

The operator norm estimate also follows from [18, Theorem 14.5.2]. □

Remark 3.7. When $p_1 = p_2 = 1$, $r = p_0 = \infty$ and $t_1 = t_2 = -t_0 = s \geq 0$, $u = -s$, $v = 0$ we recover the celebrated Cordero–Gröchenig Theorem, [6, Theorem 3.2], in the case of polynomial weights, with the uniform estimate

$$\|A_a^{\varphi_1,\varphi_2}\|_{op} \lesssim \|a\|_{M^\infty_{-s,0}} \|\varphi_1\|_{M^1_s} \|\varphi_2\|_{M^1_s}$$

in our notation.

We remark that a modification of Theorem 3.6 can be obtained by using [18, Theorem 14.5.6] instead. This result allows symbols from weighted modulation spaces. We leave to the reader to check how to change the conditions on weight parameters in Theorem 3.6 in such case.

3.2. Schatten–von Neumann Properties

In this subsection, we use known results on Weyl pseudo-differential operators with symbol σ, their connection to localization operators from Lemma 3.3, and convolution properties of modulation spaces. References to the proof of the following well-known theorem can be found in [6].

Theorem 3.8. *Let σ be the Weyl symbol of L_σ.*

(1) *If $\sigma \in M^1(\mathbb{R}^{2d})$, then $\|L_\sigma\|_{S_1} \lesssim \|\sigma\|_{M^1}$.*
(2) *If $\sigma \in M^p(\mathbb{R}^{2d})$, $1 \leq p \leq 2$, then $\|L_\sigma\|_{S_p} \lesssim \|\sigma\|_{M^p}$.*
(3) *If $\sigma \in M^{p,p'}(\mathbb{R}^{2d})$, $2 \leq p \leq \infty$, then $\|L_\sigma\|_{S_p} \lesssim \|\sigma\|_{M^{p,p'}}$.*

The Schatten–von Neumann properties in the following Theorem are formulated in the spirit of [6]; see also [32,33]. Note that more general weights are considered in [34,35], leading to different types of results.

Theorem 3.9. *Let the assumptions of Theorem 3.1 hold, $1 \leq q \leq \infty$, and let $v \geq d\mathsf{R}(\mathrm{p})$ with $v > d\mathsf{R}(\mathrm{p})$ when $\mathsf{R}(\mathrm{p}) > 0$.*

(1) *If $1 \leq p \leq 2$ and $p \leq r \leq 2p/(2-p)$, then the mapping $(a, \varphi_1, \varphi_2) \mapsto A_a^{\varphi_1,\varphi_2}$ is bounded from $M^{r,q}_{-s,v} \times M^1_s \times M^p_s$ into S_p, that is*

$$\|A_a^{\varphi_1,\varphi_2}\|_{S_p} \lesssim \|a\|_{M^{r,q}_{0,t}} \|\varphi_1\|_{M^1_s} \|\varphi_2\|_{M^p_s}.$$

(2) *If $2 \leq p \leq \infty$ and $p \leq r$, then the mapping $(a, \varphi_1, \varphi_2) \mapsto A_a^{\varphi_1,\varphi_2}$ is bounded from $M^{r_1,r_2}_{u,v} \times M^1_s \times M^{p'}_s$ into S_p, that is*

$$\|A_a^{\varphi_1,\varphi_2}\|_{S_p} \lesssim \|a\|_{M^{r,q}_{0,t}} \|\varphi_1\|_{M^1_s} \|\varphi_2\|_{M^{p'}_s}.$$

Proof. (1) By Proposition 3.4, it follows that $W(\varphi_2, \varphi_1) \in M^{1,p_w}_{-t_0,0}(\mathbb{R}^{2d})$, with $t_0 \geq -s$ and $p_w \in [2p/(p+2), p]$. Therefore, $W(\varphi_2, \varphi_1) \in M^{1,p}_{s,0}(\mathbb{R}^{2d})$.

This and Theorem 3.1 (2) imply $a * W(\varphi_2, \varphi_1) \in M^p(\mathbb{R}^{2d})$. The result now follows from Theorem 3.8 (2).

(2) By Proposition 3.4, it follows that $W(\varphi_2, \varphi_1) \in M^{1,p_w}_{-t_0,0}(\mathbb{R}^{2d})$, with $t_0 \geq -s$ and $p_w \in [p', 2p'/(p'+2), p]$. Therefore, $W(\varphi_2, \varphi_1) \in M^{1,p'}_{s,0}(\mathbb{R}^{2d})$.

The statement follows from Theorem 3.1 (2) and Theorem 3.8 (3), similarly to the previous case. □

Remark 3.10. A particular choice: $r = p$, $q = \infty$ and $v = 0$ gives [6, Theorem 3.4].

We finish with necessary conditions whose proofs follow from the proofs of Theorems 4.3 and 4.4 in [6] and are therefore omitted.

Theorem 3.11. *Let the assumptions of Theorem 3.1 hold and let $a \in \mathcal{S}'(\mathbb{R}^{2d})$.*

(1) *If there exists a constant $C = C(a) > 0$ depending only on the symbol a such that*

$$\|A_a^{\varphi_1,\varphi_2}\|_{S_\infty} \leq C\|\varphi_1\|_{M^{p_1}_{t_1}}\|\varphi_2\|_{M^{p_2}_{t_2}},$$

for all $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$, then $a \in M^{\infty,r}_{u,v}(\mathbb{R}^{2d})$ where $1 \leq r \leq p_0$, $u \geq t_0$ and $v \geq d\mathsf{R}(\mathrm{p})$ with $v > d\mathsf{R}(\mathrm{p})$ when $\mathsf{R}(\mathrm{p}) > 0$.

(2) *If there exists a constant $C = C(a) > 0$ depending only on the symbol a such that*

$$\|A_a^{\varphi_1,\varphi_2}\|_{S_2} \leq C\|\varphi_1\|_{M^1}\|\varphi_2\|_{M^1}$$

for all $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$, then $a \in M^{2,\infty}(\mathbb{R}^{2d})$.

3.3. Proof of Lemma 3.3

We first note that the integrals below are absolutely convergent and that the change of order of integration is allowed. Moreover, when suitably interpreted, certain oscillatory integrals are meaningful in $\mathcal{S}^{(1)'}(\mathbb{R}^d)$. In particular, if δ denotes the Dirac distribution, then the Fourier inversion formula in the sense of distributions gives $\int e^{2\pi i x\omega} d\omega = \delta(x)$, wherefrom $\iint \phi(x) e^{2\pi i(x-y)\omega} dx d\omega = \phi(y)$, when $\phi \in \mathcal{S}^{(1)}(\mathbb{R}^d)$.

Next, we will use the following version of the Schwartz kernel theorem [23]. If T is any linear and continuous operator from $\mathcal{S}^{(1)}(\mathbb{R}^{2d})$ to $\mathcal{S}^{(1)'}(\mathbb{R}^{2d})$, then there exists a uniquely determined $k \in \mathcal{S}^{(1)'}(\mathbb{R}^{2d})$ such that

$$\langle Tf, g\rangle = \langle k, g \otimes \overline{f}\rangle, \quad f, g \in \mathcal{S}^{(1)}(\mathbb{R}^{2d}).$$

We refer to [25] for the proof; see also [38].

Let us show that the kernel of $A_a^{\varphi_1,\varphi_2}$ coincides with the kernel of L_σ when $\sigma = a * W(\varphi_2, \varphi_1)$.

From (1.4), it follows that:

$$\begin{aligned}\langle A_a^{\varphi_1,\varphi_2} f, g\rangle &= \iint_{\mathbb{R}^{2d}} a(x,\omega)\left(\int_{\mathbb{R}^d} f(y)\overline{M_\omega T_x \varphi_1}(y)dy\right)\left(\int_{\mathbb{R}^d} \overline{g}(t) M_\omega T_x \varphi_2(t)dt\right) dx d\omega \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(y)\overline{g}(t)\left(\iint_{\mathbb{R}^{2d}} a(x,\omega)\overline{M_\omega T_x \varphi_1}(y) M_\omega T_x \varphi_2(t) dx d\omega\right) dt dy \\ &= \langle k, g \otimes \overline{f}\rangle,\end{aligned}$$

where

$$k(t,y) = \int_{\mathbb{R}^{2d}} a(x,\omega)\overline{M_\omega T_x \varphi_1}(y) M_\omega T_x \varphi_2(t) dx d\omega. \tag{3.8}$$

Next, we calculate the kernel of $L_{a*W(\varphi_2,\varphi_1)}$. We will use the covariance property of the Wigner transform:

$$W(T_xM_\omega f, T_xM_\omega g)(p,q) = \int_{\mathbb{R}^d} T_xM_\omega f\left(p+\frac{t}{2}\right)\overline{T_xM_\omega g\left(p-\frac{t}{2}\right)}e^{-2\pi iqt}dt$$
$$= \int_{\mathbb{R}^d} e^{2\pi i\left(p+\frac{t}{2}\right)\omega} f\left(p+\frac{t}{2}-x\right)\overline{e^{-2\pi i\left(p-\frac{t}{2}\right)\omega} g\left(p-\frac{t}{2}-x\right)}e^{-2\pi iqt}dt$$
$$= \int_{\mathbb{R}^d} f\left(p-x+\frac{t}{2}\right)\overline{g\left(p-x-\frac{t}{2}\right)}e^{-2\pi i(q-\omega)t}dt = W(f,g)(p-x,q-\omega).$$

Note also that $W(f,g) = \overline{W(g,f)}$.

We have:

$$a * W(\varphi_2,\varphi_1)(p,q) = \iint_{\mathbb{R}^{2d}} a(x,\omega)W(\varphi_2,\varphi_1)(p-x,q-\omega)dxd\omega$$
$$= \iint_{\mathbb{R}^{2d}} a(x,\omega)W(T_xM_\omega\varphi_2, T_xM_\omega\varphi_1)(p,q)dxd\omega$$
$$= \iint_{\mathbb{R}^{2d}} a(x,\omega)\left(\int_{\mathbb{R}^d} T_xM_\omega\varphi_2\left(p+\frac{s}{2}\right)\overline{T_xM_\omega\varphi_1}\left(p-\frac{s}{2}\right)e^{-2\pi iqs}ds\right)dxd\omega$$
$$= \iint_{\mathbb{R}^{2d}} a(x,\omega)\left(\int_{\mathbb{R}^d} M_\omega T_x\varphi_2\left(p+\frac{s}{2}\right)\overline{M_\omega T_x\varphi_1}\left(p-\frac{s}{2}\right)e^{-2\pi iqs}ds\right)dxd\omega,$$

where we have used the commutation relation $T_xM_\omega = e^{-2\pi ix\cdot\omega}M_\omega T_x$.

Now,

$$\langle L_{a*W(\varphi_2,\varphi_1)}f,g\rangle = \langle a*W(\varphi_2,\varphi_1), W(g,f)\rangle$$
$$= \iint_{\mathbb{R}^{2d}} a(x,\omega)\iint_{\mathbb{R}^{2d}}\left(\iint_{\mathbb{R}^{2d}} M_\omega T_x\varphi_2\left(p+\frac{s}{2}\right)\overline{M_\omega T_x\varphi_1}\left(p-\frac{s}{2}\right)e^{-2\pi iq(s-r)}\right.$$
$$\left.\times\,\overline{g}\left(p+\frac{r}{2}\right)f\left(p-\frac{r}{2}\right)dsdr\right)dpdqdxd\omega$$
$$= \iint_{\mathbb{R}^{2d}} a(x,\omega)\int_{\mathbb{R}^d}\left(\iint_{\mathbb{R}^{2d}} M_\omega T_x\varphi_2\left(p+\frac{s}{2}\right)\overline{M_\omega T_x\varphi_1}\left(p-\frac{s}{2}\right)\right.$$
$$\left.\times\,\overline{g}\left(p+\frac{r}{2}\right)f\left(p-\frac{r}{2}\right)\delta(s-r)dsdr\right)dpdxd\omega$$
$$= \iint_{\mathbb{R}^{2d}} a(x,\omega)\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} M_\omega T_x\varphi_2\left(p+\frac{s}{2}\right)\overline{M_\omega T_x\varphi_1}\left(p-\frac{s}{2}\right)\right.$$
$$\left.\times\,\overline{g}\left(p+\frac{s}{2}\right)f\left(p-\frac{s}{2}\right)ds\right)dpdxd\omega,$$

where we use the above-mentioned interpretation of the oscillatory integral which appears above. Finally, the change of variable $p+\frac{s}{2}=t$ and $p-\frac{s}{2}=y$ gives

$$\langle L_{a*W(\varphi_2,\varphi_1)}f,g\rangle = \iint_{\mathbb{R}^{2d}}\iint_{\mathbb{R}^{2d}} a(x,\omega)M_\omega T_x\varphi_2(t)\overline{M_\omega T_x\varphi_1}(y)dxd\omega$$
$$\times\,\overline{g}(t)f(y)dtdy = \langle k, g\otimes\overline{f}\rangle,$$

where k is given by (3.8). By the uniqueness of the kernel we conclude that $A_a^{\varphi_1,\varphi_2} = L_{a*W(\varphi_2,\varphi_1)}$, and the proof is complete.

4. Concluding Remarks

Continuity of localization operators on modulation spaces with general sub-multiplicative weights is used in the study of ultradistributional symbols in [9]. One can find a trace-class result when the symbol is a compactly supported non-quasi-analytic ultradistribution. Notice that such weights may have at most exponential growth at infinity.

By the Hardy theorem for modulation spaces, cf. [22] if the weight grows at infinity faster than $C\exp(a|z|^2)$, $z \in \mathbb{R}^{2d}$, for some $a \geq \pi/2$, the corresponding modulation space is trivial.

When the growth at infinity of the involved weight is between $C_1\exp(b|z|)$, and $C_2\exp(c|z|^2)$, $z \in \mathbb{R}^{2d}$, for some $b > 0$ and some small positive c, the corresponding test function spaces are quasi-analytic.

Boundedness results in the framework of quasi-analytic ultradistributions require different techniques. For example, a trace-class result for quasi-analytic ultradistributions given in [10] is based on representations of quasi-analytic ultradistributions based on the heat kernel method, and corresponding growth properties of the STFT. Notice that representations via Hermite functions given in [24,25] cannot be used in that context, since quasi-analytic spaces of test functions are not dense in test function spaces for compactly supported ultradistributions.

An extension of the results from the present paper is possible if the convolution and multiplication results from [37] can be extended to general sub-multiplicative weights. This requires new nontrivial estimates and will be done in a separate paper.

References

[1] Abreu, L.D., Dörfler, M.: An inverse problem for localization operators, *Inverse Problems* 28(11), pp. 115001, 16 (2012)

[2] Berezin, F.A.: Wick and anti-wick symbols of operators. Mat. Sb. N. S. **86**(128), 578–610 (1971)

[3] Boggiatto, P., Cordero, E., Gröchenig, K.: Generalized anti-wick operators with symbols in distributional Sobolev spaces. Integral Equ. Oper. Theory **48**, 427–442 (2004)

[4] Boggiatto, P., Oliaro, A., Wong, M.W.: L^p boundedness and compactness of localization operators. J. Math. Anal. Appl. **322**(1), 193–206 (2006)

[5] Calderón, A.-P., Vaillancourt, R.: On the boundedness of pseudo-differential operators. J. Math. Soc. Japan **23**, 374–378 (1971)

[6] Cordero, E., Gröchenig, K.: Time-frequency analysis of localization operators. J. Funct. Anal **205**(1), 107–131 (2003)

[7] Cordero, E., Gröchenig, K.: Symbolic calculus and Fredholm property for localization operators. J. Fourier Anal. Appl **12**(4), 371–392 (2006)

[8] Cordero, E., Okoudjou, K.A.: Multilinear localization operators J. Math. Anal. Appl. **325**(2), 1103–1116 (2007)

[9] Cordero, E., Pilipović, S., Rodino, L., Teofanov, N.: Localization operators and exponential weights for modulation spaces. Mediterr. J. Math **2**(4), 381–394 (2005)

[10] Cordero, E., Pilipović, S., Rodino, L., Teofanov, N.: Quasianalytic Gelfand-Shilov spaces with application to localization operators. Rocky Mt. J. Math **40**(4), 1123–1147 (2010)

[11] Cordero, E., Rodino, L.: Wick calculus: a time-frequency approach. Osaka J. Math **42**(1), 43–63 (2005)

[12] Cordero, E., Rodino, L., Gröchenig, K.: Localization operators and time-frequency analysis, in Harmonic, wavelet and p-adic analysis, World Sci. Publ., pp. 83–110 (2007)

[13] Daubechies, I.: Time-frequency localization operators: a geometric phase space approach. IEEE Trans. Inform. Theory **34**(4), 605–612 (1988)

[14] Feichtinger, H.G.: Modulation spaces on locally compact abelian groups, Technical Report, University Vienna, 1983. and also in Wavelets and Their Applications, Krishna, M., Radha, R., Thangavelu, S. (eds.) Allied Publishers, pp. 99–140 (2003)

[15] Feichtinger, H.G., Nowak, K.: A first survey of Gabor multipliers. In: Advances in Gabor Analysis, Birkhäuser, 99–128 (2003)

[16] Folland, G.B.: Harmonic analysis in phase space. Princeton Univ. Press, Princeton (1989)

[17] Gelfand, I.M., Shilov, G.E.: Generalized functions II. Academic Press, New York (1968)

[18] Gröchenig, K.: Foundations of time-frequency analysis. Birkhäuser, Boston (2001)

[19] Gröchenig, K.H.: Weight functions in time-frequency analysis. In: Rodino, L., Wong M.W. (eds.) Pseudodifferential operators: partial differential equations and time-frequency analysis. Fields Inst. Comm. **52**, 343–366 (2007)

[20] Gröchenig, K.H., Heil, C.: Modulation spaces and pseudo-differential operators. Integr. Equ. Oper. Th. **34**, 439–457 (1999)

[21] Gröchenig, K.H., Heil, C.: Modulation spaces as symbol classes for pseudodifferential operators, In: Krishna, M., Radha, R., Thangavelu, S. (eds.) Wavelets and their applications, Allied Publishers Private Limited, New Dehli, pp. 151–170 (2003)

[22] Gröchenig, K., Zimmermann, G.: Hardy's theorem and the short-time Fourier transform of Schwartz functions. J. Lond. Math. Soc. **63**, 205–214 (2001)

[23] Hörmander, L.: The analysis of linear partial differential Operators, vol. I, Springer, Berlin (1983)

[24] Langenbruch, M.: Hermite functions and weighted spaces of generalized functions. Manuscr. Math **119**, 269–285 (2006)

[25] Lozanov–Crvenkovic, Z., Perisic, D., Taskovic, M.: Gelfand-Shilov spaces, Structural and Kernel theorems, Preprint. arXiv:0706.2268v2

[26] Pilipović, S., Teofanov, N., Toft, J.: Micro-local analysis in Fourier Lebesgue and modulation spaces, II. J. Pseudo Differ. Oper. Appl. **1**, 341–376 (2010)

[27] Ramanathan, J., Topiwala, P.: Time-frequency localization via the Weyl correspondence. SIAM J. Math. Anal. **24**(5), 1378–1393 (1993)

[28] Shubin, M.A.: Pseudodifferential Operators and Spectral Theory. Springer, Berlin, second edition (2001)

[29] Sjöstrand, J.: An algebra of pseudodifferential operators. Math. Res. Lett. **1**, 185–192 (1994)

[30] Slepian, D.: Some comments on Fourier analysis, uncertainty and modeling. SIAM Rev. **25**(3), 379–393 (1983)

[31] Teofanov, N.: Ultradistributions and time-frequency analysis. In: Pseudodifferential operators and related topics, operator theory: advances and applications, Boggiatto, P., Rodino, L., Toft, J., Wong, M.W. (eds) Birkhäuser, 164,173–191 (2006)

[32] Toft, J.: Continuity properties for modulation spaces with applications to pseudo-differential calculus, I. J. Funct. Anal **207**, 399–429 (2004)

[33] Toft, J.: Continuity properties for modulation spaces with applications to pseudo-differential calculus, II. Ann. Global Anal. Geom. **26**, 73–106 (2004)

[34] Toft, J.: Continuity and Schatten properties for pseudo-differential operators on modulation spaces. In: Oper. Theory Adv. Appl., 172, 173–206, Birkhäuser (2007)

[35] Toft, J.: Continuity and Schatten properties for Toeplitz operators on modulation spaces. In: Oper. Theory Adv. Appl., 172, 313–328, Birkhäuser (2007)

[36] Toft, J.: The Bargmann transform on modulation and Gelfand-Shilov spaces, with applications to Toeplitz and pseudo-differential operators. J. Pseudo Differ. Oper. Appl. **3**(2), 145–227 (2012)

[37] Toft, J., Johansson, K., Pilipović, S., Teofanov, N.: Sharp convolution and multiplication estimates in weighted spaces, analysis and applications, accepted for publication (2014)

[38] Toft, J., Khrennikov, A., Nilsson, B., Nordebo, S.: Decompositions of Gelfand-Shilov kernels into kernels of similar class. J. Math. Anal. Appl. **396**(1), 315–322 (2012)

[39] Wong, M.W.: Weyl transforms, Springer (1998)

[40] Wong, M.W.: Wavelet transforms and localization operators, Birkhäuser Basel (2002)

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