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A Blow-Up Result in a Nonlinear Wave Equation with Delay

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Abstract. In this paper, we consider a nonlinear wave equation with delay. We show that under suitable conditions on the initial data, the weights of the damping, the delay term and the nonlinear source, the energy of solutions blows up in a finite time.

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Keywords. Nonlinear source, delay time, wave system, blowup.

1. Introduction

In this paper, we consider the following nonlinear damped wave equation with delay

$$\begin{cases} u_{tt}(x,t) - \operatorname{div}\left(|\nabla u(x,t)|^{m-2}\nabla u(x,t)\right) & \\ \quad +\mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = b\,|u(x,t)|^{p-2}\,u(x,t), & \text{in } \Omega\times(0,\infty), \\ u_t\,(x,t-\tau) = f_0\,(x,t-\tau)\,, & \text{in } (0,\tau)\,, \\ u(x,t) = 0, & \text{on } \partial\Omega\times[0,+\infty) \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $p > m \geq 2, b, \mu_1$ are positive constants, μ_2 is a real number, and $\tau > 0$ represents the time delay. Time delays arise in many applications because, in most instances, physical, chemical, biological, thermal, and economic phenomena naturally not only depend on the present state but also on some past occurrences. In recent years, the control of PDEs with time delay effects has become an active area of research. In many cases, it has been shown that delay is a source of instability and even an arbitrarily small delay may destabilize a system which is uniformly asymptotically stable in the absence of delay unless additional conditions or control terms have been used. For instance, it well known that the system

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$$\begin{cases} u_{tt}(x,t) - \triangle u(x,t) + a_0 u_t(x,t) + a u_t(x,t-\tau) = 0 & \text{in } \Omega \times (0,\infty), \\ u(x,t) = 0 & \text{in } \Gamma_0 \times [0,\infty), \\ \frac{\partial u}{\partial v}(x,t) = 0 & \text{in } \Gamma_1 \times (0,\infty), \end{cases} \tag{1.2}$$

in the absence of delay ($a = 0, a_0 > 0$) is exponentially stable, see [13,32]. In the presence of delay ($a > 0$), Nicaise and Pignotti [20] examined system (1.2) and proved, under the assumption that the weight of the feedback with delay is smaller than that without delay ($a < a_0$), the energy is exponentially stable. When $a \geq a_0$, they produced a sequence of delays for which the corresponding solution is instable. The same results were obtained for the case of boundary delay. We refer the reader to [1] for an abstract treatment to this problem and to [19,22,23] for analogous results in the case of time-varying delay. When the delay term in (1.2) is replaced by the distributed delay of the form

$$\int_{\tau_1}^{\tau_2} a(s) u_t(x,t-s) \mathrm{d}s,$$

exponential stability results were obtained in [21] under the condition

$$\int_{\tau_1}^{\tau_2} a(s) \mathrm{d}s < a_0.$$

In the absence of the delay term ($a = 0$), problem (1.2) has been extensively studied and many results concerning global existence and nonexistence have been proved. For instance, for the equation

$$u_{tt} - \Delta u + a u_t |u_t|^m = b|u|^\gamma u, \quad \text{in } \Omega \times (0,\infty), \tag{1.3}$$

$m, \gamma \geq 0$, it is well known that, for $a = 0$, the source term $bu|u|^\gamma$, ($\gamma > 0$) causes finite-time blowup of solutions with negative initial energy (see [4]). The interaction between the damping and the source terms was first considered by Levine [10,11] in the linear damping case ($m = 0$). He showed that solutions with negative initial energy blow up in finite time. Georgiev and Todorova [7] extended Levine's result to the nonlinear damping case ($m > 0$). In their work, the authors introduced a different method and showed that solutions with negative energy continue to exist globally 'in time' if $m \geq \gamma$ and blow up in finite time if $\gamma > m$ and the initial energy is sufficiently negative. This last blow-up result has been extended to solutions with negative initial energy only by Messaoudi [15] and others. For results of same nature, we refer the reader to Levine and Serrin [9], and Vitillaro [27], Messaoudi and Said-Houari [17].

For problem (1.3) in $\mathbb{R}^n$, we mention, among others, the work of Levine, Serrin and Park [12], Todorova [25,26], Messaoudi [16], and Zhou [31]. Recently, Autuori et al. [3] investigated the global nonexistence for nonlinear Kirchhoff system. They established their result using the classical potential well and the concavity method when the initial energy is controlled above by a critical value.

In [14], Messaoudi considered the following initial-boundary value problem in the presence of the viscoelastic term

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)\mathrm{d}\tau + u_t|u_t|^{m-2} = u|u|^{p-2}, & \text{in } \Omega \times (0,\infty) \\ u(x,t) = 0, \quad x \in \partial\Omega,\ t \geq 0 \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \quad x \in \Omega, \end{cases} \tag{1.4}$$

where Ω is a bounded domain of $\mathbb{R}^n$ $(n \geq 1)$ with a smooth boundary $\partial\Omega, p > 2, m \geq 1$, and $g : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a positive nonincreasing function. He showed, under suitable conditions on g, that solutions with initial negative energy blow up in finite time if $p > m$ and continue to exist if $m \geq p$. This result has been later pushed, by the same author [18], to certain solutions with positive initial energy. A similar result was also obtained by Wu [29] using a different method. Motivated by the above works, Wu and Lin [30] showed that the global nonexistence results for

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(x,s)\mathrm{d}s + Q(x,t,u_t) = f(x,u), \quad \text{in } \Omega \times (0,\infty),$$

can be extended to a bigger region.

More blow-up results can also be found in many interesting works. For example, Autuori et al. [2] discussed a strongly damped Kirchhoff–Love model, containing an intrinsic dissipative term of Kelvin–Voigt type, and proved a global nonexistence and a priori estimates for the life span of maximal solutions. In [6], Feng et al. studied a semilinear wave equation with a nonlinear boundary dissipation. They looked into the interaction between the boundary damping and the interior source. Under appropriate assumptions on the initial data, two blow-up results with positive initial energy were established. In [8], Guo and Rammaha focused on the life span of solutions of systems of nonlinear wave equations with a supercritical source and proved, under some restrictions on the parameters and for negative initial energy, that every weak solution of the system blows up in finite time. Also, in [24], Ouchenane et al. proved that solutions of a system of wave equations with viscoelastic term, degenerate damping, and strong nonlinear sources acting in both equations cannot exist globally provided that the initial data are sufficiently large. In [28], Wang discussed a class of fourth-order wave equations with a linear damping term and a superlinear source term. He showed the uniqueness and the existence of a local solution and gave necessary and sufficient conditions for global existence and finite-time blow up of these solutions. Moreover, the potential well depth was estimated.

In the present work, we are concerned with problem (1.1). We prove, under suitable conditions on the initial data that the energy blows up in finite time. To the best of our knowledge, this is the first work that deals with blow up of solutions to problems involving delay. The contents of this paper is organized as follows. In Sect. 2, we prepare some material needed in our proof and state the energy functional. In Sect. 3, we state and prove our main result.

2. Preliminaries

In this section, we transform the equation in (1.1) to a system, using the idea of [17] and introduce the associated energy. We also refer the reader to [5] for existence of solutions to nonlinear problems with delay.

So, we introduce the new variable

$$z(x,\rho,t) = u_t(x,t-\tau\rho), \quad x\in\Omega,\ \rho\in(0,1),\ t>0.$$

Thus, we have

$$\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \quad x\in\Omega,\ \rho\in(0,1),\ t>0.$$

Then, problem (1.1) takes the form

$$\begin{cases} u_{tt}(x,t) - \operatorname{div}\left(|\nabla u(x,t)|^{m-2}\nabla u(x,t)\right) + \mu_1 u_t(x,t) & \\ \quad +\mu_2 z(x,1,t) = b\,|u(x,t)|^{p-2}u(x,t), & \text{in } \Omega\times(0,\infty)\\ \tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, & \text{in } \Omega\times(0,1)\times(0,\infty)\\ z(x,\rho,0) = f_0(x,-\rho\tau), & \text{in } \Omega\times(0,1)\\ u(x,t) = 0, & \text{on } \partial\Omega\times[0,1)\\ u(x,0) = u_0(x), & \text{in } \Omega. \end{cases} \tag{2.1}$$

Next, we introduce the energy functional

$$E(t) := \frac{1}{2}\|u_t\|_2^2 + \frac{1}{m}\|\nabla u\|_m^m - \frac{b}{p}\|u\|_p^p + \frac{\xi}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)\mathrm{d}\rho\mathrm{d}x, \quad t\ge 0, \tag{2.2}$$

where

$$\tau|\mu_2| < \xi < \tau\left(2\mu_1 - |\mu_2|\right), \quad \mu_1 > |\mu_2|. \tag{2.3}$$

We also set

$$H(t) = -E(t) = \frac{b}{p}\|u\|_p^p - \frac{1}{2}\|u_t\|_2^2 - \frac{1}{m}\|\nabla u\|_m^m - \frac{\xi}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)\mathrm{d}\rho\mathrm{d}x. \tag{2.4}$$

The following lemma shows that the associated energy of the problem under the condition $\mu_1 > |\mu_2|$ is decreasing.

Lemma 2.1. *Let u be the solution of (2.1). Then, for some $C_0 \ge 0$,*

$$E'(t) \le -C_0\int_0^1 \left(u_t^2 + z^2(x,1,t)\right)\mathrm{d}x \le 0. \tag{2.5}$$

Proof. Multiplying Eq. $(2.1)_1$ by u_t and integrating over $(0,1)$ and $(2.1)_2$ by $(\xi/\tau)z$ and integrating over $(0,1)\times\Omega$ with respect to ρ and x summing up we get

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left(\frac{1}{2}\|u_t\|_2^2 + \frac{1}{m}\|\nabla u\|_m^m - \frac{b}{p}\|u\|_p^p\right)\mathrm{d}x + \frac{\xi}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\int_0^1 z^2(x,\rho,t)\mathrm{d}\rho\mathrm{d}x\\ &\quad = -\mu_1\int_\Omega u_t^2\mathrm{d}x - \frac{\xi}{\tau}\int_\Omega\int_0^1 zz_\rho(x,\rho,t)\mathrm{d}\rho\mathrm{d}x - \mu_2\int_\Omega u_t z(x,1,t)\mathrm{d}x. \end{aligned} \tag{2.6}$$

We, now, estimate the last two terms of the right-hand side of (2.6) as follows

$$
\begin{aligned}
-\frac{\xi}{\tau}\int_\Omega\int_0^1 zz_\rho(x,\rho,t)\mathrm{d}\rho\mathrm{d}x &= -\frac{\xi}{2\tau}\int_\Omega\int_0^1 \frac{\partial}{\partial\rho}z^2(x,\rho,t)\mathrm{d}\rho\mathrm{d}x\\
&= \frac{\xi}{2\tau}\int_\Omega \left(z^2(x,0,t)-z^2(x,1,t)\right)\mathrm{d}x = \frac{\xi}{2\tau}\left(\int_\Omega u_t^2\mathrm{d}x - \int_\Omega z^2(x,1,t)\mathrm{d}x\right)
\end{aligned}
$$

and

$$
-\mu_2\int_\Omega u_t z(x,1,t)\mathrm{d}x \le \frac{|\mu_2|}{2}\left(\int_\Omega u_t^2\mathrm{d}x + \int_\Omega z^2(x,1,t)\mathrm{d}x\right).
$$

Hence, we obtain

$$
\frac{\mathrm{d}E(t)}{\mathrm{d}t} \le -\left(\mu_1 - \frac{\xi}{2\tau} - \frac{|\mu_2|}{2}\right)\int_\Omega u_t^2\mathrm{d}x - \left(\frac{\xi}{2\tau} - \frac{|\mu_2|}{2}\right)\int_\Omega z^2(x,1,t)\mathrm{d}x.
$$

Using (2.3), we have, for some $C_0 > 0$,

$$
E'(t) \le -C_0\int_\Omega \left(u_t^2 + z^2(x,1,t)\right)\mathrm{d}x \le 0.
$$

□

Lemma 2.2. *Suppose that* $m < p \le \frac{mn}{n-m}$, *if* $n > m$ *and* $p > m$ *if* $n \le m$. *Then there exists a positive constant* $C > 1$ *depending on* Ω *only such that*

$$
\|u\|_p^s \le C\left[\|u\|_p^p + \|\nabla u\|_m^m\right],
$$

for any $u \in W_0^{1,m}(\Omega)$ *and* $m \le s \le p$.

Proof. If $\|u\|_p \ge 1$ then

$$
\|u\|_p^s \le \|u\|_p^p.
$$

If $\|u\|_p \le 1$ then, $\|u\|_p^s \le \|u\|_p^m$. Using Sobolev embedding theorems, we have

$$
\|u\|_p^s \le \|u\|_p^m \le C\|\nabla u\|_m^m.
$$

As a result, we have

Corollary 2.3. *Let the assumptions of Lemma 2.2 hold. Then we have the following*

$$
\|u\|_p^s \le C\left[\frac{b+p}{p}\|u\|_p^p - H(t) - \frac{1}{2}\|u_t\|_2^2 - \frac{\xi}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)\mathrm{d}\rho\mathrm{d}x\right],
$$

for all $t \in [0,T)$.

3. Blowup

In this section, we state and prove our main result.

Theorem 3.1. *Suppose that* $m < p \leq \frac{mn}{n-m}$, *if* $n > m$ *and* $p > m$ *if* $n \leq m$. *Assume further that*

$$E(0) := \frac{1}{2}\left\|u_1\right\|_2^2 + \frac{1}{m}\left\|\nabla u_0\right\|_m^m - \frac{b}{p}\left\|u_0\right\|_p^p + \frac{\xi}{2}\int_\Omega\int_0^1 f_0(x,-\rho\tau)\mathrm{d}\rho\mathrm{d}x < 0. \tag{3.1}$$

Then the solution (1.4) *blows up in finite time.*

The result in (2.6) implies that

$$E(t) \leq E(0) \leq 0. \tag{3.2}$$

Hence,

$$H'(t) = -E'(t) = C_0\int_0^1\left(u_t^2 + z^2(x,1,t)\right)\mathrm{d}x \geq C_0\int_0^1 z^2(x,1,t)\mathrm{d}x \geq 0, \tag{3.3}$$

and

$$0 < H(0) \leq H(t) \leq \frac{b}{p}\left\|u\right\|_p^p. \tag{3.4}$$

We set

$$L(t) = H^{1-\alpha}(t) + \varepsilon\int_\Omega uu_t\mathrm{d}x + \frac{\mu_1\varepsilon}{2}\int_\Omega u^2\mathrm{d}x, \quad t \geq 0, \tag{3.5}$$

where $\varepsilon > 0$ to be specified later and

$$0 < \alpha < \frac{p-m}{2p}. \tag{3.6}$$

A direct differentiation of $L(t)$ gives

$$\begin{aligned} L'(t) &= (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon\int_\Omega u_t^2\mathrm{d}x + \varepsilon\int_\Omega uu_{tt}\mathrm{d}x + \mu_1\varepsilon\int_\Omega uu_t\mathrm{d}x \\ &= (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon\left\|u_t\right\|_2^2 - \varepsilon\left\|\nabla u\right\|_m^m - \varepsilon\mu_2\int_\Omega uz(x,1,t)\mathrm{d}x \\ &\quad + \varepsilon b\int_\Omega |u|^p\mathrm{d}x. \end{aligned} \tag{3.7}$$

Using

$$\varepsilon\mu_2\int_\Omega uz(x,1,t)\mathrm{d}x \leq \varepsilon|\mu_2|\left(\delta\int_\Omega u^2\mathrm{d}x + \frac{1}{4\delta}\int_\Omega z^2(x,1,t)\mathrm{d}x\right), \quad \text{for any } \delta > 0,$$

and (3.3), we obtain

$$\begin{aligned} L'(t) \geq &\left[(1-\alpha)H^{-\alpha}(t) - \frac{\varepsilon|\mu_2|}{4\delta C_0}\right]H'(t) + \varepsilon\left\|u_t\right\|_2^2 - \varepsilon\left\|\nabla u\right\|_m^m + \varepsilon b\left\|u\right\|_p^p \\ &- \varepsilon\delta|\mu_2|\left\|u\right\|_2^2 \end{aligned} \tag{3.8}$$

Of course (3.8) remains valid even if δ is time dependant since the integral is taken over the x-variable. Therefore by taking δ so that $\frac{|\mu_2|}{4\delta C_0} = kH^{-\alpha}(t)$, for large k to be specified later, and substituting in (3.8) we arrive at

$$L'(t) \geq [(1-\alpha) - \varepsilon k] H^{-\alpha}(t)H'(t) + \varepsilon \|u_t\|_2^2 - \varepsilon \|\nabla u\|_m^m - \frac{\varepsilon |\mu_2|^2}{4kC_0} H^\alpha \|u\|_2^2$$
$$+\varepsilon b \|u\|_p^p . \tag{3.9}$$

Using (3.4), we find

$$H^\alpha(t) \leq \left(\frac{b}{p}\right)^\alpha \|u\|_p^{\alpha p} ,$$

thus

$$H^\alpha \|u\|_2^2 \leq c \|u\|_p^{2+\alpha p} \quad \text{for some } c > 0. \tag{3.10}$$

Inserting (3.10) in (3.9), for $0 < a < 1$, we arrive at

$$\begin{aligned} L'(t) &\geq [(1-\alpha) - \varepsilon k] H^{-\alpha}(t)H'(t) + \varepsilon \|u_t\|_2^2 - \varepsilon \|\nabla u\|_m^m \\ &\quad -\frac{\varepsilon c |\mu_2|^2}{4kC_0} \|u\|_p^{2+2\alpha} + \varepsilon ab \|u\|_p^p \\ &\quad +\varepsilon (1-a) \left[pH + \frac{p}{2} \|u_t\|_2^2 + \frac{p}{m} \|\nabla u\|_m^m + \frac{p\xi}{2} \int_\Omega \int_0^1 z^2(x,\rho,t) \mathrm{d}\rho \mathrm{d}x\right] \\ &\geq [(1-\alpha) - \varepsilon k] H^{-\alpha}(t)H'(t) + \varepsilon \left(1 + \frac{p(1-a)}{2}\right) \|u_t\|_2^2 \\ &\quad +\varepsilon \left(\frac{p(1-a)}{m} - 1\right) \|\nabla u\|_m^m + \varepsilon ab \|u\|_p^p + \varepsilon (1-a) pH \\ &\quad +\varepsilon (1-a) \frac{p\xi}{2} \int_\Omega \int_0^1 z^2(x,\rho,t) \mathrm{d}\rho \mathrm{d}x - \frac{\varepsilon c |\mu_2|^2}{4kC_0} \|u\|_p^{2+\alpha p} . \end{aligned}$$

Then we use Corollary 2.3, for $s = 2 + \alpha p \leq p$, to deduce that

$$\|u\|_p^{2+\alpha p} \leq C \left[\frac{b+p}{p} \|u\|_p^p - H(t) - \frac{1}{2} \|u_t\|_2^2 - \frac{\xi}{2} \int_\Omega \int_0^1 z^2(x,\rho,t) \mathrm{d}\rho \mathrm{d}x\right].$$

Thus

$$\begin{aligned} L'(t) &\geq [(1-\alpha) - \varepsilon k] H^{-\alpha}(t)H'(t) + \varepsilon \left(1 + \frac{p(1-a)}{2} + \frac{C |\mu_2|^2}{8k}\right) \|u_t\|_2^2 \\ &\quad + \varepsilon \left[(1-a) \frac{p\xi}{2} + \frac{C\xi |\mu_2|^2}{8kC_0}\right] \int_\Omega \int_0^1 z^2(x,\rho,t) \mathrm{d}\rho \mathrm{d}x \\ &\quad + \varepsilon \left[(1-a) p + \frac{C |\mu_2|^2}{4kC_0}\right] H(t) + \varepsilon \left(\frac{p(1-a)}{m} - 1\right) \|\nabla u\|_m^m \\ &\quad + \varepsilon \left[ab - C \left(\frac{b+p}{p}\right) \frac{|\mu_2|^2}{4kC_0}\right] \|u\|_p^p . \end{aligned} \tag{3.11}$$

At this point, we choose $a > 0$ so small that makes

$$\frac{p(1-a)}{2} - 1 > 0,$$

the constant k large so that

$$ab - C\left(\frac{2b+p}{p}\right)\frac{|\mu_2|^2}{4kC_0} > 0$$

and ε so small so that

$$(1-\alpha) - \varepsilon k > 0,$$
$$H(0) + \varepsilon\int_\Omega u_0 u_1 \mathrm{d}x > 0.$$

Thus, for $\lambda > 0$, estimation (3.11) becomes

$$L'(t) \geq \lambda\left[\|u_t\|_2^2 + \|\nabla u\|_m^m + H(t) + \|u\|_p^p + \int_\Omega\int_0^1 z^2(x,\rho,t)\mathrm{d}\rho\mathrm{d}x\right]. \quad (3.12)$$

Consequently, we have

$$L(t) \geq L(0) > 0, \quad t \geq 0.$$

Next, using Hölder's inequality and the embedding $\|u\|_2 \leq C\|u\|_p$, we have

$$\int_\Omega u u_t \mathrm{d}x \leq \|u\|_2\|u_t\|_2 \leq C\|u\|_p\|u_t\|_2.$$

Young's inequality yields

$$\left|\int_\Omega u u_t \mathrm{d}x\right|^{1/(1-\alpha)} \leq C\left(\|u\|_p^{\mu/(1-\alpha)} + \|u_t\|_2^{\theta/(1-\alpha)}\right), \quad (3.13)$$

for $1/\mu + 1/\theta = 1$. To be able to use Lemma 2.2, we take $\theta = 2(1-\alpha)$ which gives $\mu/(1-\alpha) = 2/(1-2\alpha) \leq p$.

Therefore, for $s = 2/(1-2\alpha)$, (3.13) becomes

$$\left|\int_\Omega u u_t \mathrm{d}x\right|^{1/(1-\alpha)} \leq C\left(\|u\|_p^s + \|u_t\|_2^2\right).$$

Again Lemma 2.2, gives

$$\left|\int_\Omega u u_t \mathrm{d}x\right|^{1/(1-\alpha)} \leq C\left[H(t) + \|u_t\|_2^2 + \|u\|_p^p\right].$$

Therefore,

$$\begin{aligned} L^{1/(1-\alpha)}(t) &= \left(H^{1-\alpha}(t) + \varepsilon\int_\Omega u u_t \mathrm{d}x + \frac{\mu_1\varepsilon}{2}\int_\Omega u^2 \mathrm{d}x\right)^{1/(1-\alpha)} \\ &\leq C\left[H(t) + \left|\int_\Omega u u_t \mathrm{d}x\right|^{1/(1-\alpha)} + \|u\|_2^{2/(1-\alpha)}\right] \\ &\leq C\left[H(t) + \left|\int_\Omega u u_t \mathrm{d}x\right|^{1/(1-\alpha)} + \|u\|_p^{2/(1-\alpha)}\right] \\ &\leq C\left[H(t) + \|u_t\|_2^2 + \|u\|_p^p\right], \quad t \geq 0. \end{aligned} \quad (3.14)$$

Combining (3.12) and (3.14), we arrive at

$$L'(t) \geq \Lambda L^{1/(1-\alpha)}(t), \quad t \geq 0. \tag{3.15}$$

where Λ is a positive constant depending only on λ and C.

A simple integration of (3.15) over $(0, t)$ yields

$$L^{\alpha/(1-\alpha)}(t) \geq \frac{1}{L^{-\alpha/(1-\alpha)}(0) - \Lambda\alpha t/(1-\alpha)}.$$

Therefore, $L(t)$ blows up in time

$$T \leq T^* = \frac{1-\alpha}{\Lambda\alpha L^{-\alpha/(1-\alpha)}(0)}.$$

This completes the proof.

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