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Existence of Positive Solutions for Nonlinear Second-Order Impulsive Boundary Value Problems on Time Scales

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Abstract. This paper is concerned with the existence of positive solutions of second-order impulsive boundary value problem with integral boundary conditions on time scales. Existence results of at least three positive solutions are established via a new fixed point theorem in a cone. Also, an example is given to illustrate the effectiveness of our result.

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1. Introduction

The theory of impulsive differential equations is adequate mathematical models for description of evolution processes characterized by the combination of a continuous and jump change of their state. Impulsive differential equations have become an important area of research in recent years of the needs of modern technology, engineering, economics and physics. Moreover, impulsive differential equations are richer in applications compared to the corresponding theory of ordinary differential equations. Many of the mathematical problems encountered in the study of impulsive differential equations cannot be treated with the usual techniques within the standard framework of ordinary differential equations. For the introduction of the basic theory of impulsive equations, see [2,7,9,12,14,18,21,22] and the references therein.

The theory of dynamic equations on time scales goes back to Stefan Hilger [11]. Now, it is still a new area of fairly theoretical exploration in mathematics. The study unifies existing results in differential and finite difference equations, and provides powerful new tools for exploring connections between the traditionally separated fields. We refer the books by Bohner and Peterson [4,5], Lakshmikantham [16] and the papers [1,3,15].

Boundary value problem with integral boundary conditions has been the subject of investigations along the line with impulse differential equations because of their wide applicability in various fields (cf., e.g., [19,23]). Very few works have been done to the existence of positive solutions to boundary value problems for impulsive dynamic equations with integral boundary conditions on time scales (see [8,10,13,17]).

In [6], Boucherif considered the following second-order boundary value problem with integral boundary conditions

$$\begin{cases} y'' = f(t, y(t)), \quad 0 < t < 1, \\ y(0) - ay'(0) = \displaystyle\int_0^1 g_0(s)y(s)ds, \\ y(1) - by'(1) = \displaystyle\int_0^1 g_1(s)y(s)ds. \end{cases}$$

Using Krasnoselskii's fixed point theorem, he obtained the existence criteria of at least one positive solution.

In [12], Hu et al. studied second-order two-point impulsive boundary value problem

$$\begin{cases} -u'' = h(t)f(t, u), \quad t \in J', \\ -\Delta u'|_{t=t_k} = I_k(u(t_k)), \\ \Delta u|_{t=t_k} = \bar{I}_k(u(t_k)), \quad k = 1, 2, \ldots, m, \\ \alpha u(0) - \beta u'(0) = 0, \\ \gamma u(1) + \delta u'(1) = 0. \end{cases}$$

Using the fixed point theorem in cone, they obtained the existence criteria of one or two positive solutions.

In [17], Li and Shu considered following first-order nonlinear impulsive integral boundary value problem on time scales

$$\begin{cases} x^{\triangle}(t) + p(t)x(\sigma(t)) = f\left(t, x\left(\sigma(t)\right)\right), \quad t \in J := [0, T]_{\mathbb{T}} \setminus \left\{t_1, t_2, \ldots, t_m\right\}, \\ \Delta x(t_i) = x(t_i^+) - x(t_i^-) = I_i(u(t_i)), \quad i = 1, 2, \ldots, m, \\ \alpha x(0) - \beta x(\sigma(T)) = \displaystyle\int_0^{\sigma(T)} g(s)x(s)\triangle s. \end{cases}$$

Using the well-known Guo-Krasnoselskii fixed point theorem and Legget–Williams fixed point theorem, some criteria for the existence of at least one, two, and three positive solutions were established for the problem under consideration, respectively.

Motivated by the above results, in this study, we consider the following second-order impulsive boundary value problem (BVP) with integral boundary conditions on time scales

$$\begin{cases} u^{\triangle\triangle}(t)+q(t)f(t,u(t))=0, \quad t\in J:=[0,1]_{\mathbb{T}},\ t\neq t_k,\\ \Delta u|_{t=t_k}=I_k(u(t_k)), \quad k=1,2,\ldots,n,\\ \Delta u^{\triangle}|_{t=t_k}=-J_k(u(t_k)),\\ au(0)-bu^{\triangle}(0)=\displaystyle\int_0^1 g_1(s)u(s)\triangle s,\\ cu(1)+du^{\triangle}(1)=\displaystyle\int_0^1 g_2(s)u(s)\triangle s, \end{cases} \tag{1.1}$$

where $\mathbb{T}$ is a time scale, $0,1\in\mathbb{T}$, $[0,1]_{\mathbb{T}}=[0,1]\cap\mathbb{T}$, $t_k\in(0,1)_{\mathbb{T}}$, $k=1,2,\ldots,n$ with $0<t_1<t_2<\ldots<t_n<1$. $\Delta u|_{t=t_k}$ and $\Delta u^{\triangle}|_{t=t_k}$ denote the jump of $u(t)$ and $u^{\triangle}(t)$ at $t=t_k$, i.e.,

$$\Delta u|_{t=t_k}=u(t_k^+)-u(t_k^-),\ \ \Delta u^{\triangle}|_{t=t_k}=u^{\triangle}(t_k^+)-u^{\triangle}(t_k^-),$$

where $u(t_k^+),u^{\triangle}(t_k^+)$ and $u(t_k^-),u^{\triangle}(t_k^-)$ represent the right-hand limit and left-hand limit of $u(t)$ and $u^{\triangle}(t)$ at $t=t_k$, $k=1,2,\ldots,n$, respectively.

Throughout this paper, we assume that following conditions hold:

- $(C1)$ $a,b,c,d\in[0,\infty)$ with $ac+ad+bc>0$,
- $(C2)$ $f\in\mathcal{C}([0,1]_{\mathbb{T}}\times\mathbb{R}^+,\mathbb{R}^+)$, $q\in\mathcal{C}([0,1]_{\mathbb{T}},\mathbb{R}^+)$,
- $(C3)$ $g_1,\ g_2\in\mathcal{C}([0,1]_{\mathbb{T}},\mathbb{R}^+)$,
- $(C4)$ $I_k\in\mathcal{C}(\mathbb{R}^+,\mathbb{R}^+)$ and $J_k\in\mathcal{C}(\mathbb{R}^+,\mathbb{R}^+)$ are bounded functions such that $(c(1-x)+d)J_k(u(x))>cI_k(u(x))$, for all $x\in\mathbb{R}$.

Using a new fixed point theorem due to Ren et al. [20], we get the existence of at least three positive solutions for the impulsive BVP (1.1). To the authors' knowledge, there is not much work on the existence of positive solutions for the boundary value problem of second-order impulsive boundary value problem with integral boundary conditions on time scales. Our problem is very different from the papers in literature. In fact, our result is also new when $\mathbb{T}=\mathbb{R}$ (the differential case) and $\mathbb{T}=\mathbb{Z}$ (the discrete case). Therefore, the result can be considered as a contribution to this field.

This paper is organized as follows. In Sect. 2, we provide some definitions and preliminary lemmas which are key tools for our main result. We give and prove our main result in Sect. 3. Finally, in Sect. 4, we give an example to demonstrate effective of our result.

2. Preliminaries

In this section, to state the main result of this paper, we need the following lemmas. Throughout the rest of this paper, we assume that the points of impulse t_k are right dense for each $k=1,2,\ldots,n$. Let $J=[0,1]_{\mathbb{T}}$, $J'=J\backslash\{t_1,t_2,\ldots,t_n\}$. We define

$$\begin{aligned}\mathbb{B}=\{u|\ u:[0,1]_{\mathbb{T}}\to\mathbb{R}\text{ is continuous at } t\neq t_k,\ \text{left continuous at } t=t_k,\\ \text{and there exist } u(t_k^-)\text{ and } u(t_k^+)\text{ with } u(t_k^-)=u(t_k)\text{ for } k=1,2,\ldots,n\}.\end{aligned}$$

Then, $\mathbb{B}$ is a real Banach space with the norm $\|u\|=\sup_{t\in[0,1]_{\mathbb{T}}}|u(t)|$. A function $u\in\mathbb{B}\cap C^2(J')$ is called a solution to (1.1) if it satisfies all equations of (1.1).

$$\mathcal{P}=\Big\{u\in\mathbb{B}: u(t) \text{ is nonnegative, nondecreasing on } [0,1]_{\mathbb{T}} \text{ and}$$
$$u^{\triangle}(t) \text{ is nonincreasing on } [0,1]_{\mathbb{T}}\Big\}.$$

Obviously, $\mathcal{P}$ is a cone in $\mathbb{B}$. We note that, for each $u\in\mathcal{P}$,

$$\|u\|=\sup_{t\in[0,1]_{\mathbb{T}}}|u(t)|=u(1).$$

Denote by θ and φ, the solutions of the corresponding homogeneous equation

$$u^{\triangle\triangle}(t)=0,\ \ t\in J, \tag{2.1}$$

under the initial conditions

$$\begin{cases}\theta(0)=b, & \theta^{\Delta}(0)=a,\\ \varphi(1)=d, & \varphi^{\Delta}(1)=-c.\end{cases} \tag{2.2}$$

Using the initial conditions (2.2), we can deduce from Eq. (2.1) for θ and φ the following equations:

$$\theta(t)=b+at,\quad \varphi(t)=d+c(1-t). \tag{2.3}$$

Set

$$D:=\begin{vmatrix}-\displaystyle\int_0^1 g_1(s)\,(b+as)\,\triangle s & \rho-\displaystyle\int_0^1 g_1(s)\,(d+c(1-s))\,\triangle s\\ \rho-\displaystyle\int_0^1 g_2(s)\,(b+as)\,\triangle s & -\displaystyle\int_0^1 g_2(s)\,(d+c(1-s))\,\triangle s\end{vmatrix}, \tag{2.4}$$

and

$$\rho:=ad+ac+bc. \tag{2.5}$$

Lemma 2.1. *Let $(C1)$–$(C4)$ hold. Assume that $D\neq 0$. If $u\in\mathbb{B}\cap C^2(J')$ is a solution of the equation*

$$\begin{aligned}u(t)=&\int_0^1 G\,(t,s)\,q(s)f\,(s,u(s))\,\triangle s+\sum_{k=1}^{n}W_k\,(t,t_k)\\ &+A(f)(b+at)+B(f)(d+c(1-t)),\end{aligned} \tag{2.6}$$

where

$$W_k\,(t,t_k)=\frac{1}{\rho}\begin{cases}(b+at)\,(-cI_k(u(t_k))+(d+c(1-t_k))J_k\,(u(t_k))), & t<t_k,\\ (d+c(1-t))\,(aI_k(u(t_k))+(b+at_k)J_k\,(u(t_k))), & t_k\le t,\end{cases} \tag{2.7}$$

$$G\,(t,s)=\frac{1}{\rho}\begin{cases}(b+a\sigma(s))\,(d+c(1-t)), & \sigma(s)\le t,\\ (b+at)\,(d+c(1-\sigma(s))), & t\le s,\end{cases} \tag{2.8}$$

$$A(f)=\frac{1}{D}\begin{vmatrix}\int_0^1 g_1(s)F(s)\triangle s, & \rho-\int_0^1 g_1(s)(d+c(1-s))\triangle s\\ \int_0^1 g_2(s)F(s)\triangle s & -\int_0^1 g_2(s)(d+c(1-s))\triangle s\end{vmatrix}, \tag{2.9}$$

$$B(f)=\frac{1}{D}\begin{vmatrix}-\int_0^1 g_1(s)(b+as)\triangle s & \int_0^1 g_1(s)F(s)\triangle s\\ \rho-\int_0^1 g_2(s)(b+as)\triangle s & \int_0^1 g_2(s)F(s)\triangle s\end{vmatrix}, \tag{2.10}$$

and

$$F(s):=\int_0^1 G\left(s,r\right)q(r)f\left(r,u(r)\right)\triangle r+\sum_{k=1}^{n}W_k\left(s,t_k\right), \tag{2.11}$$

then, u is a solution of the impulsive boundary value problem (1.1)*.*

Proof. Let u satisfies the integral Eq. (2.6), then we have

$$\begin{aligned}u(t)&=\int_0^t\frac{1}{\rho}(b+a\sigma(s))(d+c(1-t))q(s)f\left(s,u(s)\right)\triangle s\\&\quad+\int_t^1\frac{1}{\rho}(b+at)(d+c(1-\sigma(s)))q(s)f\left(s,u(s)\right)\triangle s\\&\quad+\sum_{0<t_k<t}(d+c(1-t))\left(aI_k(u(t_k))+(b+at_k)J_k\left(u(t_k)\right)\right)\\&\quad+\sum_{t<t_k<1}(b+at)\left(-cI_k(u(t_k))+(d+c(1-t_k))J_k\left(u(t_k)\right)\right)\\&\quad+A(f)(b+at)+B(f)(d+c(1-t)),\end{aligned}$$

$$\begin{aligned}u^{\triangle}(t)&=-\int_0^t\frac{c}{\rho}(b+a\sigma(s))q(s)f\left(s,u(s)\right)\triangle s\\&\quad+\int_t^1\frac{a}{\rho}(d+c(1-\sigma(s)))q(s)f\left(s,u(s)\right)\triangle s\\&\quad-\sum_{0<t_k<t}c\left(aI_k(u(t_k))+(b+at_k)J_k\left(u(t_k)\right)\right)\\&\quad+\sum_{t<t_k<1}a\left(-cI_k(u(t_k))+(d+c(1-t_k))J_k\left(u(t_k)\right)\right)\\&\quad+A(f)a-B(f)c,\\u^{\triangle\triangle}(t)&=\frac{1}{\rho}\left(-c\left(b+a\sigma(t)\right)-a\left(d+c(1-\sigma(t))\right)\right)q(t)f\left(t,u(t)\right)\\&=-q(t)f\left(t,u(t)\right).\end{aligned}$$

So, we get $u^{\triangle\triangle}(t)+q(t)f(t,u(t))=0$. Since

$$\begin{aligned}u(0)=&\int_0^1\frac{b}{\rho}\left(d+c(1-\sigma(s))\right)q(s)f\left(s,u(s)\right)\triangle s\\&+\sum_{k=1}^{n}b\left(-cI_k(u(t_k))+(d+c(1-t_k))J_k\left(u(t_k)\right)\right)\\&+A(f)b+B(f)(d+c),\\u^{\triangle}(0)=&\int_0^1\frac{a}{\rho}(d+c(1-\sigma(s)))q(s)f\left(s,u(s)\right)\triangle s\\&+\sum_{k=1}^{n}a\left(-cI_k(u(t_k))+(d+c(1-t_k))J_k\left(u(t_k)\right)\right)\\&+A(f)a-B(f)c,\end{aligned}$$

we have

$$\begin{aligned}au(0)-bu^{\triangle}(0)&=B(f)\left(ad+ac+bc\right)\\&=\int_0^1 g_1(s)\bigg(\int_0^1 G\left(s,r\right)q(r)f\left(r,u(r)\right)\triangle r+\sum_{k=1}^{n}W_k\left(s,t_k\right)\\&\qquad+A(f)(b+as)+B(f)(d+c(1-s))\bigg)\triangle s.\end{aligned}\tag{2.12}$$

Since,

$$\begin{aligned}u(1)=&\int_0^1\frac{d}{\rho}(b+a\sigma(s))q(s)f\left(s,u(s)\right)\triangle s\\&+\sum_{k=1}^{n}d\left(aI_k(u(t_k))+(b+at_k)J_k\left(u(t_k)\right)\right)+A(f)(b+a)+B(f)d,\\u^{\triangle}(1)=&-\int_0^1\frac{c}{\rho}(b+a\sigma(s))q(s)f\left(s,u(s)\right)\triangle s\\&+\sum_{k=1}^{n}-c\left(aI_k(u(t_k))+(b+at_k)J_k\left(u(t_k)\right)\right)+A(f)a-B(f)c,\end{aligned}$$

we have

$$\begin{aligned}cu(1)+du^{\triangle}(1)&=A(f)\left(ad+ac+bc\right)\\&=\int_0^1 g_2(s)\bigg(\int_0^1 G\left(s,r\right)q(r)f\left(r,u(r)\right)\triangle r+\sum_{k=1}^{n}W_k\left(s,t_k\right)\\&\qquad+A(f)(b+as)+B(f)(d+c(1-s))\bigg)\triangle s.\end{aligned}\tag{2.13}$$

From (2.5), (2.12) and (2.13), we get

$$\begin{cases} \left[-\int_0^1 g_1(s)(b+as)\triangle s\right] A(f) + \left[\rho - \int_0^1 g_1(s)(d+c(1-s))\triangle s\right] B(f) \\ \quad = \int_0^1 g_1(s)F(s)\triangle s \\ \left[\rho - \int_0^1 g_2(s)(b+as)\triangle s\right] A(f) + \left[-\int_0^1 g_2(s)(d+c(1-s))\triangle s\right] B(f) \\ \quad = \int_0^1 g_2(s)F(s)\triangle s \end{cases},$$

which implies that $A(f)$ and $B(f)$ satisfy (2.9) and (2.10), respectively. □

Lemma 2.2. *Let* $(C1)$–$(C4)$ *hold. Assume*

$(C5)$ $D < 0,\ \rho - \int_0^1 g_2(s)(b+as)\triangle s > 0,\ a - \int_0^1 g_1(s)\triangle s > 0.$

Then for $u \in \mathbb{B} \cap C^2(J')$ *with* $f, q \geq 0$, *the solution* u *of the problem* (1.1) *satisfies* $u(t) \geq 0$ *for* $t \in [0,1]_{\mathbb{T}}$.

Proof. It is an immediate subsequence of the facts that $G \geq 0$ on $[0,1]_{\mathbb{T}} \times [0,1]_{\mathbb{T}}$ and $A(f) \geq 0,\ B(f) \geq 0$. □

Lemma 2.3. *Let* $(C1)$–$(C5)$ *hold. Assume*

$(C6)$ $c - \int_0^1 g_2(s)\triangle s < 0.$

Then the solution $u \in \mathbb{B} \cap C^2(J')$ *of the problem* (1.1) *satisfies* $u^{\triangle}(t) \geq 0$ *for* $t \in [0,1]_{\mathbb{T}}$.

Proof. Assume that the inequality $u^{\triangle}(t) < 0$ holds. Since $u^{\triangle}(t)$ is nonincreasing on $[0,1]_{\mathbb{T}}$, one can verify that

$$u^{\triangle}(1) \leq u^{\triangle}(t),\ t \in [0,1]_{\mathbb{T}}.$$

From the boundary conditions of the problem (1.1), we have

$$-\frac{c}{d}u(1) + \frac{1}{d}\int_0^1 g_2(s)u(s)\triangle s \leq u^{\triangle}(t) < 0.$$

The last inequality yields

$$-cu(1) + \int_0^1 g_2(s)u(s)\triangle s < 0.$$

Therefore, we obtain that

$$\int_0^1 g_2(s)u(1)\triangle s < \int_0^1 g_2(s)u(s)\triangle s < cu(1),$$

i.e.,

$$\left(c - \int_0^1 g_2(s)\triangle s\right) u(1) > 0.$$

According to Lemma 2.2, we have that $u(1) \geq 0$. So, $c - \int_0^1 g_2(s)\triangle s > 0$. However, this contradicts to condition $(C6)$. Consequently, $u^{\triangle}(t) \geq 0$ for $t \in [0,1]_{\mathbb{T}}$. □

Define $T : \mathcal{P} \longrightarrow \mathbb{B}$ by

$$Tu(t) = \int_0^1 G(t,s)\, q(s) f(s,u(s)) \triangle s + \sum_{k=1}^{n} W_k(t,t_k) \tag{2.14}$$

$$+A(f)(b+at) + B(f)(d + c(1-t)),$$

where $W_k,\ G,\ A(f)$ and $B(f)$ are defined as in (2.7), (2.8), (2.9) and (2.10), respectively.

Lemma 2.4. *Let $(C1)-(C6)$ hold. Then $T : \mathcal{P} \to \mathcal{P}$ is completely continuous.*

Proof. By Arzela–Ascoli theorem, we can easily prove that operator T is completely continuous. □

3. Main Result

The following fixed point theorem is fundamental and important to the proof of our main result.

Definition 3.1. Let $\mathbb{B}$ be a Banach space. Given a nonnegative continuous function γ on a cone $\mathcal{P} \subset \mathbb{B}$, for each $c > 0$ we define the set $\mathcal{P}(\gamma, c) = \{x \in \mathcal{P} : \gamma(x) < c\}$.

Lemma 3.2. [20] *Let $\mathcal{P}$ be a cone in a real Banach space $\mathbb{B}$. Let α, β and γ be three increasing, nonnegative and continuous functionals on $\mathcal{P}$, satisfying for some $c > 0$ and $M > 0$ such that*

$$\gamma(x) \leq \beta(x) \leq \alpha(x),\ \ \|x\| \leq M\gamma(x)$$

for all $x \in \overline{\mathcal{P}(\gamma, c)}$. Suppose there exists a completely continuous operator $T : \overline{\mathcal{P}(\gamma, c)} \to \mathcal{P}$ and $0 < a < b < c$ such that

(i) $\gamma(Tx) < c$, *for all* $x \in \partial\mathcal{P}(\gamma, c)$;
(ii) $\beta(Tx) > b$, *for all* $x \in \partial\mathcal{P}(\beta, b)$;
(iii) $\mathcal{P}(\alpha, a) \neq \emptyset$, *and* $\alpha(Tx) < a$, *for all* $x \in \partial\mathcal{P}(\alpha, a)$.

Then, T has at least three fixed points, $x_1,\ x_2$ and $x_3 \in \overline{\mathcal{P}(\gamma, c)}$ such that

$$0 \leq \alpha(x_1) < a < \alpha(x_2), \quad \beta(x_2) < b < \beta(x_3), \quad \gamma(x_3) < c.$$

Now, we consider the existence of at least three positive solutions for the impulsive boundary value problem (1.1) by the fixed point theorem in [20].

Suppose $\eta,\ \zeta \in \mathbb{T}$ with $0 < \eta < \zeta < 1$. We define the increasing, nonnegative, continuous functionals $\gamma,\ \beta$, and α on $\mathcal{P}$ by

$$\gamma(u) = \max_{t\in[0,\eta]_{\mathbb{T}}} u(t) = u(\eta),$$
$$\beta(u) = \min_{t\in[\eta,\zeta]_{\mathbb{T}}} u(t) = u(\eta),$$
$$\alpha(u) = \max_{t\in[0,\zeta]_{\mathbb{T}}} u(t) = u(\zeta).$$

It is obvious that for each $u \in \mathcal{P},\ \gamma(u) \le \beta(u) \le \alpha(u)$. In addition, for each $u \in \mathcal{P}$, since $u^{\triangle}$ is nonincreasing on $[0,1]_{\mathbb{T}}$ we have $\gamma(u) = u(\eta) \ge \eta u(1)$. Thus, $\|u\| \le \frac{1}{\eta}\gamma(u),\quad \forall u \in \mathcal{P}$.

For convenience, we denote

$$H(s) := \int_0^1 G\left(s,r\right) q(r)\triangle r + \frac{n}{\rho}(c+d)(2a+b),$$

$$A := \frac{1}{D}\begin{vmatrix} \int_0^1 g_1(s)H(s)\triangle s & \rho - \int_0^1 g_1(s)\left(d+c(1-s)\right)\triangle s \\ \int_0^1 g_2(s)H(s)\triangle s & -\int_0^1 g_2(s)\left(d+c(1-s)\right)\triangle s \end{vmatrix},$$

$$B := \frac{1}{D}\begin{vmatrix} -\int_0^1 g_1(s)\left(b+as\right)\triangle s & \int_0^1 g_1(s)H(s)\triangle s \\ \rho - \int_0^1 g_2(s)\left(b+as\right)\triangle s & \int_0^1 g_2(s)H(s)\triangle s \end{vmatrix},$$

$$\Lambda := \int_0^1 G(\zeta,s)q(s)\triangle s + \frac{n}{\rho}(c+d)(2a+b) + \left(b+a\zeta\right)A + (d+c(1-\zeta))B,$$
$$\Omega := \int_\eta^1 G(\eta,s)q(s)\triangle s.$$

Theorem 3.3. *Suppose the assumptions of* $(C1)$–$(C6)$ *are satisfied. Let there exist positive numbers* $l < m < r$ *such that*

$$l < \eta m < \frac{\eta\Lambda}{\Omega} m < r,$$

and assume that $f,\ I_k$ *and* J_k *satisfies the following conditions:*

$(C7)$ $f(t,u) < \frac{r}{\Lambda},\ I_k(u(t_k)) \le \frac{r}{\Lambda},\ J_k(u(t_k)) \le \frac{r}{\Lambda}$ *for all* $(t,u) \in [0,1]_{\mathbb{T}} \times \left[0, \frac{r}{\eta}\right],\quad k = 1,2,\ldots,n,$

$(C8)$ $f(t,u) > \frac{m}{\Omega}$, *for all* $(t,u) \in [\eta,1]_{\mathbb{T}} \times \left[m, \frac{m}{\eta}\right]$,

$(C9)$ $f(t,u) < \frac{l}{\Lambda},\ I_k(u(t_k)) \le \frac{l}{\Lambda},\ J_k(u(t_k)) \le \frac{l}{\Lambda}$ *for all* $(t,u) \in [0,1]_{\mathbb{T}} \times \left[0, \frac{l}{\eta}\right],\quad k = 1,2,\ldots,n.$

Then the boundary value problem (1.1) *has at least three positive solutions* u_1, u_2 *and* u_3 *belong to* $\overline{\mathcal{P}(\gamma,r)}$ *such that*

$$0 \leq \alpha(u_1) < l < \alpha(u_2),\ \beta(u_2) < m < \beta(u_3),\ \gamma(u_3) < r.$$

Proof. We define the completely continuous operator T by (2.14). So, it is easy to check that $T : \overline{\mathcal{P}(\gamma,r)} \to \mathcal{P}$.

We now show that all the conditions of Lemma 3.2 are satisfied. To show that condition (i) of Lemma 3.2, we choose $u \in \partial\mathcal{P}(\gamma,r)$. Then $\gamma(u) = \max_{t\in[0,\eta]_{\mathbb{T}}} u(t) = u(\eta) = r$, this implies that $0 \leq u(t) \leq r$ for $t \in [0,\eta]_{\mathbb{T}}$. If we recall that $\|u\| \leq \frac{1}{\eta}\gamma(u) = \frac{1}{\eta}r$. So, we have

$$0 \leq u(t) \leq \frac{r}{\eta}, \quad t \in [0,1]_{\mathbb{T}}.$$

Then assumption $(C7)$ implies for all $(t,u) \in [0,1]_{\mathbb{T}} \times \left[0, \frac{r}{\eta}\right], \quad k = 1,2,\ldots,n,$

$$f(t,u) < \frac{r}{\Lambda},\ I_k(u(t_k)) \leq \frac{r}{\Lambda},\ J_k(u(t_k)) \leq \frac{r}{\Lambda}.$$

Therefore,

$$\begin{aligned}
\gamma(Tu) &= \max_{t\in[0,\eta]_{\mathbb{T}}} (Tu)(t) = (Tu)(\eta) \\
&= \int_0^1 G(\eta,s)\,q(s)f(s,u(s))\,\triangle s + \sum_{k=1}^{n} W_k(\eta,t_k) \\
&\quad + A(f)(b+a\eta) + B(f)(d+c(\zeta)) \\
&< \left(\int_0^1 G(\zeta,s)\,q(s)\triangle s + \frac{n}{\rho}(c+d)(2a+b) + (b+a\zeta)A \right. \\
&\quad \left. + (d+c(1-\zeta))B\right)\frac{r}{\Lambda} = r.
\end{aligned}$$

Hence, condition (i) is satisfied.

Second, we show that (ii) of Lemma 3.2 is satisfied. For this, we take $u \in \partial\mathcal{P}(\beta,m)$. Then, $\beta(u) = \min_{t\in[\eta,\zeta]_{\mathbb{T}}} u(t) = u(\eta) = m$, this means $u(t) \geq m$, for all $t \in [\eta,1]_{\mathbb{T}}$. Noticing that $\|u\| \leq \frac{1}{\eta}\gamma(u) \leq \frac{1}{\eta}\beta(u) = \frac{m}{\eta}$, we get

$$m \leq u(t) \leq \frac{m}{\eta}, \quad \text{for } t \in [\eta,1]_{\mathbb{T}}.$$

Then, assumption $(C8)$ implies $f(t,u) > \frac{m}{\Omega}$. Therefore,

$$\begin{aligned}
\beta(Tu) &= \min_{t\in[\eta,\zeta]_{\mathbb{T}}} (Tu)(t) = (Tu)(\eta) \\
&\geq \int_\eta^1 G(\eta,s)\,q(s)f(s,u(s))\,\triangle s > \frac{m}{\Omega}\int_\eta^1 G(\eta,s)\,q(s)\triangle s = m.
\end{aligned}$$

So, we get $\beta(Tu) > m$. Hence, condition (ii) is satisfied.

Finally, we show that the condition (iii) of Lemma 3.2 is satisfied. We note that $u(t) = \dfrac{l}{3},\ t \in [0,1]_{\mathbb{T}}$ is a member of $\mathcal{P}(\alpha, l)$, and so $\mathcal{P}(\alpha, l) \neq \emptyset$.

Now, let $u \in \partial\mathcal{P}(\alpha, l)$. Then $\alpha(u) = \max_{t\in[0,\zeta]_{\mathbb{T}}} u(t) = u(\zeta) = l$. This implies $0 \le u(t) \le l,\ t \in [0,\zeta]_{\mathbb{T}}$. Noticing that $\|u\| \le \dfrac{1}{\eta}\gamma(u) \le \dfrac{1}{\eta}\alpha(u) = \dfrac{l}{\eta}$, we get

$$0 \le u(t) \le \frac{l}{\eta}, \quad \text{for } t \in [0,1]_{\mathbb{T}}.$$

By the assumption $(C9)$, we have for all $(t,u) \in [0,1]_{\mathbb{T}} \times \left[0, \dfrac{l}{\eta}\right]$,

$$f(t,u) < \frac{l}{\Lambda},\ I_k(u(t_k)) \le \frac{l}{\Lambda},\ J_k(u(t_k)) \le \frac{l}{\Lambda}, \quad k = 1,2,\ldots,n.$$

Therefore, we get

$$\begin{aligned}
\alpha(Tu) &= \max_{t\in[0,\zeta]_{\mathbb{T}}} (Tu)(t) = (Tu)(\zeta) \\
&< \left(\int_0^1 G(\zeta,s)\,q(s)\triangle s + \frac{n}{\rho}(c+d)(2a+b) \right. \\
&\qquad \left. +(b+a\zeta)A + (d+c(1-\zeta))B \right) \frac{l}{\Lambda} = l.
\end{aligned}$$

So, we get $\alpha(Tu) < l$. Thus, (iii) of Lemma 3.2 is satisfied.

Therefore, by Lemma 3.2, the impulsive boundary value problem (1.1) has at least three positive solutions u_1, u_2 and u_3 belong to $\overline{\mathcal{P}(\gamma, r)}$ such that

$$0 < \alpha(u_1) < l < \alpha(u_2),\ \beta(u_2) < m < \beta(u_3),\ \gamma(u_3) < r.$$

The proof of Theorem 3.3 is complete. □

4. An Example

Example. In BVP (1.1), suppose that $\mathbb{T} = [0,1],\ q(t) = g_1(t) = g_2(t) = 1,\ a = 3,\ b = 1,\ c = \dfrac{1}{4}$ and $d = 4$, i.e.,

$$\begin{cases}
u''(t) + f(t,u(t)) = 0, \quad t \in [0,1], \quad t \neq \dfrac{1}{4}, \\
\Delta u|_{t=\frac{1}{4}} = I_1\left(u\left(\dfrac{1}{4}\right)\right), \\
\Delta u'|_{t=\frac{1}{4}} = -J_1\left(u\left(\dfrac{1}{4}\right)\right), \\
3u(0) - u'(0) = \displaystyle\int_0^1 u(s)ds, \\
\dfrac{1}{4}u(1) + 4u'(1) = \displaystyle\int_0^1 u(s)ds,
\end{cases} \tag{4.1}$$

where

$$f(t,u)=\begin{cases}0.6, & u\in[0,25],\\ \frac{1}{25}(372u-9285), & u\in(25,30],\\ \frac{1}{8}(5u+450), & u\in(30,150],\\ 150, & u>150,\end{cases}$$

$$I_1(u)=\frac{3}{100}u,\quad u\geq 0,\quad J_1(u)=\frac{1}{500}u,\quad u\geq 0.$$

By simple calculation, we get $\rho=13,\ \theta(t)=1+3t,\ \varphi(t)=\frac{17}{4}-\frac{1}{4}t,\ D=-\frac{663}{8},\ A=B=\frac{2027}{3978}$ and

$$G(t,s)=\frac{1}{13}\begin{cases}(1+3s)\left(\frac{17}{4}-\frac{1}{4}t\right), & s\leq t,\\ (1+3t)\left(\frac{17}{4}-\frac{1}{4}s\right), & t\leq s.\end{cases}$$

Set $\eta=\frac{1}{5},\ \zeta=\frac{1}{2}$, then we get $\Omega=\frac{656}{1625},\ \Lambda=\frac{7751}{1224}$. Taking $l=5,\ m=30$ and $r=1000$, it is easy to check that

$$5<6<\frac{2519075}{802944}30<1000,$$

and the conditions $(C1)$–$(C6)$ are satisfied. Now, we show that conditions $(C7)$–$(C9)$ are satisfied:

$$\begin{aligned}
&f(t,u(t))\leq 150<\frac{r}{\Lambda}=\frac{1224000}{7751},\ I_1\left(u\left(\frac{1}{4}\right)\right)\leq 150<\frac{r}{\Lambda}=\frac{1224000}{7751},\\
&J_1\left(u\left(\frac{1}{4}\right)\right)\leq 10<\frac{r}{\Lambda}=\frac{1224000}{7751}\ \text{for } (t,u(t))\in[0,1]\times[0,5000];\\
&f(t,u(t))\geq 75>\frac{m}{\Omega}=\frac{24375}{328}\ \text{for } (t,u(t))\in\left[\frac{1}{5},1\right]\times[30,150];\\
&f(t,u(t))=0.6<\frac{l}{\Lambda}=\frac{6120}{7751},\ I_1\left(u\left(\frac{1}{4}\right)\right)\leq\frac{3}{4}<\frac{l}{\Lambda}=\frac{6120}{7751},\\
&J_1\left(u\left(\frac{1}{4}\right)\right)\leq\frac{1}{20}<\frac{l}{\Lambda}=\frac{6120}{7751}\ \text{for } (t,u(t))\in[0,1]\times[0,25].
\end{aligned}$$

So, all conditions of Theorem 3.3 hold. Thus by Theorem 3.3, the BVP (4.1) has at least three positive solutions $u_1,\ u_2$ and u_3 belong to $\overline{\mathcal{P}(\gamma,1000)}$ such that

$$0 \leq \max_{t\in[0,\frac{1}{2}]} u_1(t) < 5 < \max_{t\in[0,\frac{1}{2}]} u_2(t),$$
$$\min_{t\in[\frac{1}{5},\frac{1}{2}]} u_2(t) < 30 < \min_{t\in[\frac{1}{5},\frac{1}{2}]} u_3(t),$$
$$\max_{t\in\left[0,\frac{1}{5}\right]} u_3(t) < 1000.$$

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