Mediterr. J. Math. 13 (2016), 53–64
DOI 10.1007/s00009-014-0493-z
0378-620X/16/010053-12
*published online* November 26, 2014


**Mediterranean Journal of Mathematics**

# Power-Commuting Generalized Skew Derivations in Prime Rings

Luisa Carini, Vincenzo De Filippis and Giovanni Scudo

**Abstract.** Let $R$ be a non-commutative prime ring of characteristic different from 2 with extended centroid $C$, $F \neq 0$ a generalized skew derivation of $R$, and $n \geq 1$ such that $[F(x), x]^n = 0$, for all $x \in R$. Then there exists an element $\lambda \in C$ such that $F(x) = \lambda x$, for all $x \in R$.

**Mathematical Subject Classification.** 16N60, 16W25, 47B47, 47B48.

**Keywords.** Prime rings, differential identities, generalized skew derivations.

## 1. Introduction

Let $R$ be a prime ring of characteristic different from 2 with center $Z(R)$, extended centroid $C$, right Martindale quotient ring $Q_r$ and symmetric Martindale quotient ring $Q$.

An additive mapping $d : R \to R$ is a *derivation* on $R$ if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. Let $a \in R$ be a fixed element. A map $d : R \to R$ defined by $d(x) = [a, x] = ax - xa$, $x \in R$, is a derivation on $R$, which is called *inner derivation* defined by $a$. Many results in literature indicate how the global structure of a ring $R$ is often tightly connected to the behaviour of additive mappings defined on $R$. A well-known result of Posner [20] states that if $d$ is a derivation of $R$ such that $[d(x), x] \in Z(R)$, for any $x \in R$, then either $d = 0$ or $R$ is commutative. In [15] Lanski generalizes the result of Posner to a Lie ideal. Later in [2] the authors prove the following:

**Theorem 1.** *Let $R$ be a prime ring of characteristic different from 2, $L$ a non-central Lie ideal of $R$, $d$ a non-zero derivation of $R$ such that $[d(u), u]^n \in Z(R)$, for any $u \in L$. Then $R$ satisfies $s_4$.*

*In particular, if $d$ satisfies $[d(u), u]^n = 0$, for any $u \in L$, then $R$ is commutative.*

More recently in [9] the author considers a similar situation in the case the derivation $d$ is replaced by a generalized derivation. More specifically an additive map $G : R \to R$ is said to be a generalized derivation if there exists a derivation $d$ of $R$ such that, for all $x, y \in R$, $G(xy) = G(x)y + xd(y)$. Basic examples of generalized derivations are the usual derivations on $R$ and left or right $R$-module mappings from $R$ into itself. An important example is a map of the form $G(x) = ax + xb$, for some $a, b \in R$; such generalized derivations are called *inner*.

Generalized derivations have been primarily studied on operator algebras. Therefore, any investigation from the algebraic point of view might be interesting (see for example [12,16]).

In [9] the author proves the following:

**Theorem 2.** *Let $R$ be a prime ring of characteristic different from 2 with right quotient ring $U$ and extended centroid $C$, $G \neq 0$ a generalized derivation of $R$, $L$ a non-central Lie ideal of $R$ and $n \geq 1$ such that $[G(u), u]^n = 0$, for all $u \in L$. Then there exists an element $a \in C$ such that $G(x) = ax$, for all $x \in R$, unless when $R$ satisfies $s_4$ and there exist $b \in U$, $\beta \in C$ such that $G(x) = bx + xb + \beta x$, for all $x \in R$.*

*In particular, if $[G(x), x]^n = 0$, for all $x \in R$, then there exists an element $a \in C$ such that $G(x) = ax$, for all $x \in R$.*

In [21], Wang considers a similar situation in the case the derivation $d$ is replaced by a non-trivial automorphism $\sigma$ of $R$ and proves the following:

**Theorem 3.** *Let $R$ be a prime ring with center $Z$, $L$ a noncentral Lie ideal of $R$, and $\sigma$ a nontrivial automorphism of $R$ such that $[u^\sigma, u]^n \in Z$ for all $u \in L$. If either $char(R) > n$ or $char(R) = 0$, then $R$ satisfies $s_4$.*

Here, we continue this line of investigation and we examine what happens in case $F \neq 0$ is a generalized skew derivation of $R$ such that $[F(x), x]^n = 0$ for all $x \in R$, and $n \geq 1$. More specifically, let $\alpha$ be an automorphism of a ring $R$. An additive map $D : R \to R$ is called an *$\alpha$-derivation* (or a *skew derivation*) on $R$ if $D(xy) = D(x)y + \alpha(x)D(y)$ for all $x, y \in R$. In this case $\alpha$ is called an *associated automorphism* of $D$. Basic examples of $\alpha$-derivations are the usual derivations and the map $\alpha - id$, where $id$ denotes the identity map. Let $b \in Q$ be a fixed element. Then a map $D : R \to R$ defined by $D(x) = bx - \alpha(x)b$, $x \in R$, is an $\alpha$-derivation on $R$ and it is called an *inner $\alpha$-derivation* (an *inner skew derivation*) defined by $b$. If a skew derivation $D$ is not inner, then it is called *outer*.

An additive mapping $F : R \to R$ is called a *generalized $\alpha$-derivation* (or a *generalized skew derivation*) on $R$ if there exists an additive mapping $D$ on $R$ such that $F(xy) = F(x)y + \alpha(x)D(y)$ for all $x, y \in R$. The map $D$ is uniquely determined by $F$ and it is called an *associated additive map* of $F$. Moreover, it turns out that $D$ is always an $\alpha$-derivation (see [17,18] for more details).

Let us also mention that an automorphism $\alpha : R \to R$ is *inner* if there exists an invertible $q \in Q$ such that $\alpha(x) = qxq^{-1}$ for all $x \in R$. If an automorphism $\alpha \in \mathrm{Aut}(R)$ is not inner, then it is called *outer*.

The result we obtain is the following:

**Theorem 4.** *Let $R$ be a non-commutative prime ring of characteristic $\neq 2$ with extended centroid $C$, $F \neq 0$ a generalized skew derivation of $R$, and*

*$n \geq 1$ such that $[F(x), x]^n = 0$, for all $x \in R$. Then there exists an element $\lambda \in C$ such that $F(x) = \lambda x$, for all $x \in R$.*

## 2. Preliminaries

We denote the set of all skew-derivations on $Q$ by $\mathrm{SDer}(Q)$. By a skew-derivation word we mean an additive map $\Delta$ of the form $\Delta = d_1 d_2 \dots d_m$, with each $d_i \in \mathrm{SDer}(Q)$. Then a skew-differential polynomial is a generalized polynomial, with coefficients in $Q$, of the form $\Phi(\Delta_j(x_i))$ involving noncommutative indeterminates $x_i$ on which the derivations words $\Delta_j$ act as unary operations. The skew-differential polynomial $\Phi(\Delta_j(x_i))$ is said a skew-differential identity on a subset $T$ of $Q$ if it vanishes for any assignment of values from $T$ to its indeterminates $x_i$.

To prove our result, we need to recall the following known facts:

**Fact 1.** In [8] Chuang and Lee investigate polynomial identities with skew derivations. More precisely in [8, Theorem 1] they prove that if $D$ is an outer skew derivation of $R$ which satisfies the generalized polynomial identity $\Phi(x_i, D^k(x_j))$, then:

1. If $D$ is not left-algebraic modulo inner skew derivations, then $R$ satisfies the generalized polynomial identity $\Phi(x_i, y_{kj})$, where $x_i$ and $y_{kj}$ are distinct indeterminates.
2. If $D$ is algebraic modulo inner skew derivations such that the minimal order $m$ of such algebraic dependence is strictly bigger than $k$, then $R$ satisfies the generalized polynomial identity $\Phi(x_i, y_{kj})$, where $x_i$ and $y_{kj}$ are distinct indeterminates.

As a consequence of this result, we would like to point out that, if $k = 1$, that is $\Phi(x_i, D(x_j))$ is a generalized polynomial identity for $R$, then, in any case, $\Phi(x_i, y_j)$ is also a generalized polynomial identity for $R$, where $x_i$ and $y_j$ are distinct indeterminates.

**Fact 2.** Let $R$ be a prime ring and $I$ a two-sided ideal of $R$. Then $I$, $R$ and $Q$ satisfy the same generalized polynomial identities with coefficients in $Q$ (see [4]). Furthermore, $I$, $R$ and $Q$ satisfy the same generalized polynomial identities with automorphisms (Theorem 1 in [6]).

**Fact 3.** Recall that, in case $\mathrm{char}(R) = 0$, an automorphism $\alpha$ of $Q$ is called *Frobenius* if $\alpha(x) = x$ for all $x \in C$. Moreover, in case $\mathrm{char}(R) = p \geq 2$, an automorphism $\alpha$ is *Frobenius* if there exists a fixed integer $t$ such that $\alpha(x) = x^{p^t}$ for all $x \in C$. In [6, Theorem 2] Chuang proves that if $\Phi(x_i, \alpha(x_i))$ is a generalized polynomial identity for $R$, where $R$ is a prime ring and $\alpha \in \mathrm{Aut}(R)$ an automorphism of $R$ which is not Frobenius, then $R$ also satisfies the non-trivial generalized polynomial identity $\Phi(x_i, y_i)$, where $x_i$ and $y_i$ are distinct indeterminates.

**Fact 4.** Let $R$ be a domain and $\alpha \in \mathrm{Aut}(R)$ an automorphism of $R$ which is outer. In [14] Kharchenko proves that if $\Phi(x_i, \alpha(x_i))$ is a generalized polynomial identity for $R$, then $R$ also satisfies the non-trivial generalized polynomial identity $\Phi(x_i, y_i)$, where $x_i$ and $y_i$ are distinct indeterminates.

Finally, let us mention that if $R$ is a prime ring satisfying a non-trivial generalized polynomial identity and $\alpha$ an automorphism of $R$ such that $\alpha(x) = x$ for all $x \in C$, then $\alpha$ is an inner automorphism of $R$ [1, Theorem 4.7.4].

## 3. The Inner Case

In this section, we assume there exist $a, b \in Q$ and $F : R \to R$, such that $F(x) = ax + \alpha(x)b$, for all $x \in R$. In particular, we would like to consider the case $[ar + \alpha(r)b, r]^n = 0$ for all $r \in R$. We first prove the case when there exists an invertible element $q \in Q$ such that $\alpha(x) = qxq^{-1}$, for all $x \in R$. To do this, we also need the following lemma (the result is contained in [10]):

**Lemma 1.** *Let $F$ be a infinite field and $n \geq 2$. If $A_1, \ldots, A_k$ are not scalar matrices in $M_n(F)$ then there exists some invertible matrix $P \in M_n(F)$ such that each matrix $PA_1P^{-1}, \ldots, PA_kP^{-1}$ has all non-zero entries.*

**Proposition 1.** *Let $R$ be a prime ring of characteristic different from 2, and $a, b, q \in Q$. If $q$ is an invertible element of $Q$ and there exists a fixed integer $n \geq 1$ such that*

$$[ar + qrq^{-1}b, r]^n = 0 \tag{1}$$

*for all $r \in R$, then one of the following holds:*

1. *$a, b, q \in C$;*
2. *both $q^{-1}b \in C$ and $a + b \in C$.*

*Proof.* First, we notice that in case $q^{-1}b \in C$, then, by (1), we get $[(a + b)r, r]^n = 0$, for all $r \in R$, and by [9], it follows $a + b \in C$. Moreover, in case $q \in C$, we also have $[ar + rb, r]^n = 0$, for all $r \in R$, and the conclusion follows again by [9] (see Theorem 2).

Hence, in the following we may assume that both $q^{-1}b \notin C$ and $q \notin C$, so (1) is a non-trivial generalized polynomial identity for $R$. By Fact 2 and if we denote $p = q^{-1}b$ in (1), it follows that $Q$ satisfies

$$[ax + qxp, x]^n. \tag{2}$$

Hence, by [19] $Q$ is a primitive ring, which is isomorphic to a dense subring of the ring of linear transformations of a vector space $V$ over $C$, containing nonzero linear transformations of finite rank. Suppose first that $\dim_C V \geq 3$. Since $p \notin C$, there exists $v \in V$ such that $\{v, pv\}$ are linearly $C$-independent. Moreover, since $\dim_C V \geq 3$, there is $w \in V$ such that $\{v, pv, w\}$ are linearly $C$-independent vectors. By the density of $Q$, there exists $r \in Q$ such that

$$rv = 0, \quad rw = v, \quad rpv = q^{-1}w.$$

Hence

$$0 = [ar + qrp, r]^n v = (-1)^n v \neq 0$$

which is a contradiction.

Finally, we consider the case: $\dim_C V \leq 2$. Of course if $\dim_C V = 1$, then $Q$ is commutative, a contradiction. Thus, we may assume $\dim_C V = 2$, that is $Q = M_2(C)$, the ring al $2 \times 2$ matrices over the field $C$. Of course in this case, we may assume that $M_2(C)$ satisfies

$$[ax + qxp, x]^2. \tag{3}$$

Assume first that $C$ is infinite. Since both $q \notin C$ and $p \notin C$, by Lemma 1 there exists some invertible matrix $A \in M_2(C)$ such that each matrix $AqA^{-1}, ApA^{-1}$ has all non-zero entries. Moreover, it is easy to prove that

$$\left[(AaA^{-1})x + (AqA^{-1})x(ApA^{-1}), x\right]^2$$

is still a generalized identity for $M_2(C)$.

Denote $e_{ij}$ the usual matrix unit, with 1 in the $(i, j)$-entry and zero elsewhere and $AaA^{-1} = a'$, $AqA^{-1} = q' = \sum q_{lm}e_{lm}$, $ApA^{-1} = p' = \sum p_{lm}e_{lm}$, for suitable $q_{lm}, p_{lm} \in C$. Let $r = e_{ij}$, for any $j \neq i$, therefore, by (3), we have that $(-e_{ij}qe_{ij}p)^2 e_{ij} = 0$, which implies the contradiction $q_{ji}p_{ji} = 0$.

Assume now that $C$ is finite. Let $K$ be an infinite field which is an extension of the field $C$ and let $\overline{R} = M_2(K) \cong R \otimes_C K$. The generalized polynomial identity $[ax + qxp, x]^2$ is homogeneous of degree 4 in the indeterminate $x$.

Hence its complete linearization is a multilinear generalized polynomial identity $\Theta(x_1, y_1, z_1, t_1)$ in 4 indeterminates, moreover,

$$\Theta(x_1, x_1, x_1, x_1) = 4[ax_1 + qx_1p, x_1]^2.$$

Clearly, the multilinear polynomial $\Theta(x_1, y_1, z_1, t_1)$ is a generalized polynomial identity for $R$ and $\overline{R}$ too. Since $\mathrm{char}(C) \neq 2$, we obtain $[ar+qrp, r]^2 = 0$, for all $r \in \overline{R}$, and the conclusion follows from the above argument. □

**Lemma 2.** *Let $R$ be a non-commutative prime ring of characteristic different from 2, $\alpha : R \to R$ an automorphism of $R$, such that $[\alpha(x), x]^n = 0$ for all $x \in R$. Then, $\alpha$ is the identity map on $R$.*

*Proof.* By Main Theorem in [5], the ring $R$ satisfies a generalized polynomial identity. We notice that if there exists an element $q \in Q$ such that $\alpha(x) = qxq^{-1}$, for all $x \in R$, then the conclusion follows from Proposition 1. Therefore, we assume that $\alpha$ is an outer automorphism of $R$ and prove that a number of contradictions occurs.

By Theorem 1 in [6] $R$ and $Q$ satisfy the same generalized polynomial identities with automorphisms and hence $[\alpha(x), x]^n$ is also an identity for $Q$. Since $R$ is a GPI-ring, by [19] $Q$ is a primitive ring, which is isomorphic to a dense subring of the ring of linear transformations of a vector space $V$ over a division ring $D$.

In case $Q$ is a domain, by Fact 4, we have that $Q$ satisfies $[y, x]^n$, which leads to the contradiction that $Q$ is commutative. Thus, we may assume that $\dim_D V \geq 2$.

By [13, p. 79], there exists a semi-linear automorphism $T \in \mathrm{End}(V)$ such that $\alpha(x) = TxT^{-1}$ for all $x \in Q$. Hence, $Q$ satisfies $[TxT^{-1}, x]^n$.

We notice that, if for any $v \in V$ there exists $\lambda_v \in D$ such that $T^{-1}v = v\lambda_v$, then, by a standard argument, it follows that there exists a unique $\lambda \in D$

such that $T^{-1}v = v\lambda$, for all $v \in V$ (see for example Lemma 1 in [7]). In this case

$$\alpha(x)v = (TxT^{-1})v = Txv\lambda$$

and

$$(\alpha(x) - x)v = T(xv\lambda) - xv = T(T^{-1}xv) - xv = 0$$

which implies the contradiction that $\alpha$ is the identity map, since $V$ is faithful.

Therefore, there exists $v \in V$ such that $\{v, T^{-1}v\}$ are linearly $D$-independent.

Consider first the case $\dim_D V \geq 3$. Thus, there exists $w \in V$ such that $\{w, v, T^{-1}v\}$ are linearly $D$-independent. Moreover, by the density of $Q$, there exists $r \in Q$ such that

$$rv = 0, \quad rT^{-1}v = T^{-1}w, \quad rw = -v.$$

Hence, by the main assumption we get again the contradiction

$$0 = (TrT^{-1}r - rTrT^{-1})^n v = v \neq 0.$$

Therefore, we have just to consider the case when $\dim_D V = 2$.

Note that there exists $w \in V$ such that $w \notin vD$ and $Tw \notin vD$: In fact, on the contrary, for all $w \in V$ we have that either $w \in vD$ or $w \in (T^{-1}v)D$. Then it follows that $V = (vD) \cup (T^{-1}v)D$ is union of two proper subspaces, but this is a contradiction since $Q$ is not a domain and $\dim_D V \neq 1$.

Thus, there exist $w \in V$, $\lambda, \mu, \eta, \theta \in D$ (where $\mu \neq 0$ and $\theta \neq 0$) such that

$$w = v\lambda + (T^{-1}v)\mu \tag{4}$$

$$Tw = v\eta + (T^{-1}v)\theta \tag{5}$$

moreover, by applying the semi-linear automorphism $T$ on (4), we also get

$$Tw = (Tv)\tau(\lambda) + v\tau(\mu) \tag{6}$$

where $\tau : D \to D$ is the automorphism of $D$ associated to $T$. Notice that $\tau(\mu) \neq 0$, since $\mu \neq 0$.

Comparing (6) with (5) we have

$$v(\eta - \tau(\mu)) + (T^{-1}v)\theta - (Tv)\tau(\lambda) = 0$$

where $\tau(\lambda) \neq 0$, since $\theta \neq 0$ and $\{v, T^{-1}v\}$ are $D$-independent. Denote $\tau(\mu) = \mu'$ and $\tau(\lambda) = \lambda'$, so that

$$Tv = \big(v(\eta - \tau(\mu)) + (T^{-1}v)\theta\big)\lambda'^{-1}.$$

By the main assumption, we also know that $Q$ satisfies $[T(x+y)T^{-1}, x+y]^n$, that is $Q$ satisfies

$$\begin{aligned}(TxT^{-1}x - xTxT^{-1} + TyT^{-1}x - xTyT^{-1}\\ + TxT^{-1}y - yTxT^{-1} + TyT^{-1}y - yTyT^{-1})^n\end{aligned} \tag{7}$$

By the density of $Q$, there exist $r_1, r_2 \in Q$ such that

$$r_1v = 0, \quad r_2v = 0, \quad r_1T^{-1}v = -v, \quad r_2T^{-1}v = 0.$$

It follows that

$$r_1(Tv) = v\theta\lambda'^{-1}$$

$$r_2(Tv) = 0$$

$$(Tr_1T^{-1}r_1 - r_1Tr_1T^{-1} + Tr_2T^{-1}r_1 - r_1Tr_2T^{-1} + Tr_1T^{-1}r_2 - r_2Tr_1T^{-1} + Tr_2T^{-1}r_2 - r_2Tr_2T^{-1})v = v\theta\lambda'^{-1}$$

and by (7) we have the following contradiction

$$0 = (Tr_1T^{-1}r_1 - r_1Tr_1T^{-1} + Tr_2T^{-1}r_1 - r_1Tr_2T^{-1} + Tr_1T^{-1}r_2 - r_2Tr_1T^{-1} + Tr_2T^{-1}r_2 - r_2Tr_2T^{-1})^n v = v(\theta\lambda'^{-1})^n \neq 0.$$

□

**Lemma 3.** *Let $R$ be a non-commutative prime ring of characteristic different from 2, $b, c \in Q$, $\alpha : R \to R$ an outer automorphism of $R$, such that $[bx + \alpha(x)c, x]^n = 0$ for all $x \in R$. Then $b \in C$ and $c = 0$.*

*Proof.* In the following, we assume that either $b \notin C$ or $c \neq 0$.

Hence, by [5] $R$ is a GPI-ring and $Q$ is also GPI-ring by [4]. By Martindale's theorem in [19], $Q$ is a primitive ring having non-zero socle and its associated division ring $D$ is finite dimensional over $C$. Hence $Q$ is isomorphic to a dense subring of the ring of linear tranformations of a vector space $V$ over $D$, containing non-zero linear transformations of finite rank.

As remarked in Lemma 2, there exists a semi-linear automorphism $T \in \mathrm{End}(V)$ such that $\alpha(x) = TxT^{-1}$ for all $x \in Q$. Hence, $Q$ satisfies $[bx + TxT^{-1}c, x]^n$.

We notice that, if for any $v \in V$ there exists $\lambda_v \in D$ such that $T^{-1}cv = v\lambda_v$, then, by a standard argument it follows that there exists a unique $\lambda \in D$ such that $T^{-1}cv = v\lambda$, for all $v \in V$. In this case,

$$(bx + \alpha(x)c)v = (bx + TxT^{-1}c)v = bxv + T(xv\lambda)$$

$$bxv + T((xv)\lambda) = bxv + T(T^{-1}cxv)$$

$$bxv + cxv = (b + c)xv.$$

Hence, for all $v \in V$,

$$(bx + \alpha(x)c - (b + c)x)v = 0$$

which implies $bx+\alpha(x)c = (b+c)x$, for all $x \in Q$, since $V$ is faithful. Therefore, we have both $[(b+c)x, x]^n = 0$ and $\alpha(x)c = cx$, for all $x \in Q$. Thus, $b+c \in C$ follows from Theorem 2. Moreover, since $Q$ satisfies $\alpha(x)c = cx$ and $\alpha(x)$-word degree is 1 then, by Theorem 3 in [6], $yc - cx$ is an identity for $Q$. This implies $c \in C$. Therefore, $b \in C$ and either $c = 0$ or $\alpha(x) = x$ for all $x \in Q$, that is $\alpha$ is the identity map on $Q$. In any case, we get a contradiction.

In light of the previous argument, we may suppose there exists $v \in V$ such that $\{v, T^{-1}cv\}$ are linearly $D$-independent.

Consider first the case $\dim_D V \geq 3$. Thus, there exists $w \in V$ such that $\{w, v, T^{-1}cv\}$ are linearly $D$-independent. Moreover, by the density of $Q$, there exists $r \in Q$ such that

$$rv = 0, \quad rT^{-1}cv = T^{-1}w, \quad rw = -v.$$

Hence, by the main assumption, we get the contradiction

$$0 = (br^2 + TrT^{-1}cr - rbr - rTrT^{-1}c)^n v = v \neq 0.$$

Therefore, we have just to consider the case when $\dim_D V \leq 2$.

If $C$ is finite, then $D$ is also finite. Thus $D$ is a field by Wedderburn's Theorem. Note that, if $\dim_D V = 1$ then $Q \cong D$ and so $Q$ is commutative, a contradiction.

On the other hand, if $\dim_D V = 2$, then $Q \cong M_2(D)$, for $D$ a field. Of course in this case $Q$ satisfies $[bx + \alpha(x)c, x]^2$. Therefore, the $\alpha(x)$-word degree is strictly less than $\mathrm{char}(R)$, when $\mathrm{char}(R) \neq 0$. By Theorem 3 in [6], $Q$ satisfies $[bx + yc, x]^2$ and again by Theorem 2, we get $b, c \in C$. Moreover, if $c \neq 0$, we also have $[\alpha(x), x]^2 = 0$ for all $x \in Q$ and by Lemma 2, $\alpha$ is the identity map on $Q$, which is again a contradiction.

In light of the previous argument, in all that follows we may consider $C$ infinite.

If $\alpha$ is not Frobenius, then, by Fact 3, one has that $R$ satisfies $[bx + yc, x]^n = 0$. In particular, for $x = y$, we get $b, c \in C$ by Theorem 2. From this we also have $c[y, x]^n = 0$, which implies the contradiction that $R$ is commutative, since $c \neq 0$.

Let now $\alpha$ be Frobenius. Note that if $\mathrm{char}(R) = 0$, we have $\alpha(x) = x$ for all $x \in R$ since $\alpha$ is Frobenius. By [1, Theorem 4.7.4] this implies that $\alpha$ is inner, a contradiction. Thus, we may assume that $\mathrm{char}(R) = p > 2$ and $\alpha(\gamma) = \gamma^{p^t}$, for all $\gamma \in C$ and some nonzero fixed integer $t$. Moreover, there exists $\lambda \in C$ such that $\lambda^{p^t} \neq \lambda$, that is $\lambda^{p^t - 1} \neq 0$.

In particular, we choose $\gamma \in C$ such that $\gamma = \lambda^{p^t - 1} \neq 0$. In the main relation we replace $x$ by $\lambda x$ and obtain that $R$ satisfies

$$[b(\lambda x) + \lambda^{p^t} \alpha(x)c, \lambda x]^n$$

that is

$$\lambda^{2n}\big([bx, x] + \gamma[\alpha(x)c, x]\big)^n.$$

If denote $\Phi(x) = [bx, x]$ and $\Omega(x) = [\alpha(x)c, x]$, it follows that $(\Phi(r) + \gamma\Omega(r))^n = 0$ for all $r \in R$. Expanding the last one, we get

$$\sum_{i=0}^{n} \gamma^i \left( \sum_{(i, n-i)} \varphi_1 \cdot \varphi_2 \cdots \varphi_n \right) = 0$$

where the inside summations are taken over all permutations of $n - i$ terms of the form $\Phi(x)$ and $i$ terms of the form $\Omega(x)$. This means that each summation inside has exactly $n - i$ terms of the form $\Phi(x)$ and $i$ terms of the form $\Omega(x)$ but in some different order. For any $j = 0, \ldots, n$, denote $y_j = \sum_{(j, n-j)} \varphi_1 \cdot$

$\varphi_2 \ldots \varphi_n$, then we can write

$$y_0 + \gamma y_1 + \gamma^2 y_2 + \cdots + \gamma^n y_n = 0. \tag{8}$$

Replacing in the previous argument $\lambda$ successively by $1, \lambda, \lambda^2, \ldots, \lambda^n$, the Eq. (8) gives the system of equations

$$\begin{aligned} &y_0 + y_1 + y_2 + \cdots + y_n = 0 \\ &y_0 + \gamma y_1 + \gamma^2 y_2 + \cdots + \gamma^n y_n = 0 \\ &y_0 + \gamma^2 y_1 + \gamma^4 y_2 + \cdots + \gamma^{2n} y_n = 0 \\ &y_0 + \gamma^3 y_1 + \gamma^6 y_2 + \cdots + \gamma^{3n} y_n = 0 \\ &\cdots\cdots\cdots\cdots \\ &y_0 + \gamma^n y_1 + \gamma^{2n} y_2 + \cdots + \gamma^{n^2} y_n = 0. \end{aligned} \tag{9}$$

Moreover, since $C$ is infinite, there exist infinitely many $\lambda \in C$ such that $\lambda^{i(p^t-1)} \neq 1$ for $i = 1, \ldots, n$, that is there exist infinitely many $\gamma = \lambda^{p^t-1} \in C$ such that $\gamma^i \neq 1$ for $i = 1, \ldots, n$. Hence, the Vandermonde determinant (associated with the system (9))

$$\begin{vmatrix} 1 & 1 & \cdots & \cdots & 1 \\ 1 & \gamma & \gamma^2 & \cdots & \gamma^n \\ 1 & \gamma^2 & \gamma^4 & \cdots & \gamma^{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \gamma^n & \gamma^{2n} & \cdots & \gamma^{n^2} \end{vmatrix} = \prod_{0 \leq i < j \leq n} (\gamma^i - \gamma^j)$$

is not zero. Thus, we can solve the above system (9) and obtain $y_i = 0$ $(i = 0, \ldots, n)$. In particular, $y_0 = 0$, that is $[br, r]^n = 0$ for all $r \in R$. Applying again the result in Theorem 2, we have $b \in C$. In light of this, to prove our result, in what follows we can assume $c \neq 0$. Therefore, by the main assumption, it follows that $R$ satisfies $[\alpha(x)c, x]^n$. Now, replace $x$ by $x + \lambda$, so that $R$ satisfies $[\alpha(x)c + \lambda^{p^t} c, x]^n$, that is for all $r \in R$

$$([\alpha(r)c, r] + \lambda^{p^t}[c, r])^n = 0.$$

Here, denote $\mu = \lambda^{p^t}$, $\Delta(r) = [\alpha(r)c, r]$, $\Psi(r) = [c, r]$, thus, $(\Delta(r) + \mu\Psi(r))^n = 0$, for all $r \in R$. Expanding this last, we get

$$\sum_{i=0}^{n} \mu^i \left( \sum_{(i,n-i)} \chi_1 \cdot \chi_2 \cdots \chi_n \right) = 0$$

where, as in the above argument, the summations inside are taken over all permutations of $n - i$ terms of the form $\Psi(x)$ and $i$ terms of the form $\Delta(x)$. For any $j = 0, \ldots, n$, denote $z_j = \sum_{(j,n-j)} \chi_1 \cdot \chi_2 \ldots \chi_n$, then we can write

$$z_0 + \mu z_1 + \mu^2 z_2 + \cdots + \mu^n z_n = 0. \tag{10}$$

Replacing in the previous argument $\lambda$ successively by $1, \lambda, \lambda^2, \ldots, \lambda^n$, the Eq. (10) gives the system of equations

$$\begin{aligned} &z_0 + z_1 + z_2 + \cdots + z_n = 0 \\ &z_0 + \mu z_1 + \mu^2 z_2 + \cdots + \mu^n z_n = 0 \\ &z_0 + \mu^2 z_1 + \mu^4 z_2 + \cdots + \mu^{2n} z_n = 0 \\ &z_0 + \mu^3 z_1 + \mu^6 z_2 + \cdots + \mu^{3n} z_n = 0 \\ &\cdots\cdots\cdots\cdots \\ &z_0 + \mu^n z_1 + \mu^{2n} z_2 + \cdots + \mu^{n^2} z_n = 0. \end{aligned} \tag{11}$$

By repeating the same process above, there exist infinitely many $\lambda \in C$ such that $\lambda^{ip^t} \neq 1$ for $i = 1, \ldots, n$, that is there exist infinitely many $\mu = \lambda^{p^t} \in C$ such that $\mu^i \neq 1$ for $i = 1, \ldots, n$. Hence, the Vandermonde determinant (associated with the system (11))

$$\begin{vmatrix} 1 & 1 & \cdots & \cdots & 1 \\ 1 & \mu & \mu^2 & \cdots & \mu^n \\ 1 & \mu^2 & \mu^4 & \cdots & \mu^{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \mu^n & \mu^{2n} & \cdots & \mu^{n^2} \end{vmatrix} = \prod_{0 \leq i < j \leq n} (\mu^i - \mu^j)$$

is not zero. Thus, we can solve the above system (11) and obtain $z_i = 0$ $(i = 0, \ldots, n)$. In particular, $z_0 = 0$, that is $[c, r]^n = 0$ for all $r \in R$. By Theorem 2, in [11], $c \in C$. Since $c \neq 0$, it follows that $R$ satisfies $[\alpha(x), x]^n$ and by Lemma 2, we conclude that $\alpha$ is the identity map on $R$ which contradicts the hypothesis that $\alpha$ is an outer automorphism of $R$. □

## 4. The Proof of Main Result

Here, we can finally prove the main Theorem of this paper. We remark that Chang, in [3] shows that any (right) generalized skew derivation of $R$ can be uniquely extended to the right Martindale quotient ring $Q_r$ of $R$ as follows: a (right) generalized skew derivation is an additive mapping $F : Q_r \longrightarrow Q_r$ such that $F(xy) = F(x)y + \alpha(x)\delta(y)$ for all $x, y \in Q_r$, where $\delta$ is a skew derivation of $R$ and $\alpha$ is an automorphism of $R$. Notice that there exists $F(1) = a \in Q_r$ such that $F(x) = ax + \delta(x)$ for all $x \in R$.

### 4.1. Proof of Theorem 4

As we said above and by our main assumption, $R$ satisfies $[ax + \delta(x), x]^n$.

Assume first that $\delta$ is an outer skew derivation. By Fact 1, $R$ also satisfies $[ax+y, x]^n$ and in particular the component $[ax, x]^n$. By Theorem 2, it follows that $a \in C$, therefore, $[y, x]^n$ is an identity for $R$, that is $R$ is commutative, a contradiction.

Let now $\delta$ be an inner skew derivation, that is there exists $b \in Q$ such that $\delta(x) = bx - \alpha(x)b$, for all $x \in R$. Hence $R$ satisfies $[(a+b)x - \alpha(x)b, x]^n$.

In case there exists an invertible element $q \in Q$ such that $\alpha(x) = qxq^{-1}$, then by Proposition 1 we get:

1. either $a, b, q \in C$, that is $F(x) = ax$.
2. or $q^{-1}b \in C$ and $F(x) = ax$, with $a \in C$.

On the other hand, if $\alpha$ is an outer automorphism of $R$, then, by applying Lemma 3 we get:

1. either $a \in C$ and $b = 0$, that is $F(x) = ax$;
2. or $a, b \in C$, $\alpha$ is the identity map on $R$ and $F(x) = ax$.

## References


[1] Beidar, K.I., Martindale, W.S., III, Mikhalev, A.V.: Rings with Generalized Identities, Pure and Applied Mathematics, Dekker, New York (1996)

[2] Carini, L., De Filippis, V.: Commutators with power central values on a Lie ideal. Pac. J. Math. **193/2**, 296–278 (2000)

[3] Chang, J.-C.: On the identity $h(x) = af(x)+g(x)b$. Taiwan. J. Math. **7**, 103–113 (2003)

[4] Chuang, C.-L.: GPIs having coefficients in Utumi quotient rings. Proc. Am. Math. Soc. **103**, 723–728 (1988)

[5] Chuang, C.-L.: Differential identities with automorphisms and antiautomorphisms I. J. Algebra **149**, 371–404 (1992)

[6] Chuang, C.-L.: Differential identities with automorphisms and antiautomorphisms II. J. Algebra **160**, 130–171 (1993)

[7] Chuang, C.-L, Chou, M.-C, Liu, C.-K: Skew derivations with annihilating Engel conditions. J. Publ. Math. Debrecen **68/1–2**, 161–170 (2006)

[8] Chuang, C.-L, Lee, T.-K: Identities with a single skew derivation. J. Algebra **288**, 59–77 (2005)

[9] De Filippis, V.: Generalized derivations and commutators with nilpotent values on Lie ideals. Tamsui Oxford J. Math. Sci. **22/2**, 167–175 (2006)

[10] De Filippis, V., Di Vincenzo, O.M.: Vanishing derivations and centralizers of generalized derivations on multilinear polynomials. Comm. Algebra **40/6**, 1918–1932 (2012)

[11] Herstein, I.N.: Center-like elements in prime rings. J. Algebra **60/2**, 567–574 (1979)

[12] Hvala, B.: Generalized derivations in rings. Comm. Algebra **26**, 1147–1166 (1998)

[13] Jacobson, N.: Structure of Rings. American Mathematical Society, Providence (1964)

[14] Kharchenko, V.K.: Generalized identities with automorphisms. Algebra Log. **14**, 132–148 (1975)

[15] Lanski, C.: Differential identities, Lie ideals and Posner's theorems. Pac. J. Math. **134/2**, 275–297 (1988)

[16] Lee, T.-K.: Generalized derivations of left faithful rings. Comm. Algebra **27**, 4057–4073 (1999)

[17] Lee, T.-K.: Generalized skew derivations characterized by acting on zero products. Pac. J. Algebra **216**, 293–301 (2004)

[18] Lee, T.-K, Liu, K.-S: Generalized skew derivations with algebraic values of bounded degree. Houston J. Math. **39/3**, 733–740 (2013)

[19] Martindale, W.S. III.: Prime rings satisfying a generalized polynomial identity. J. Algebra **12**, 576–584 (1969)

[20] Posner, E.C.: Derivations in prime rings. Proc. Am. Math. Soc. **8**, 1093–1100 (1957)

[21] Wang, Y.: Power-centralizing automorphisms of Lie ideals in prime rings. Comm. Algebra **34**, 609–615 (2006)

Luisa Carini, Vincenzo De Filippis and Giovanni Scudo
Department of Mathematics
University of Messina,
98166, Messina, Italy
e-mail: lcarini@unime.it

Vincenzo De Filippis
e-mail: defilippis@unime.it

Giovanni Scudo
e-mail: gscudo@unime.it

Received: February 21, 2014.
Revised: October 17, 2014.
Accepted: November 13, 2014.