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Levinson's Type Generalization of the Edmundson–Lah–Ribarič Inequality

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Abstract. In this paper the authors give a brief historical remark on Edmundson–Madansky and Lah–Ribarič inequalities, which are both special cases of the same inequality, and unify them under the name of Edmundson–Lah–Ribarič inequality. Furthermore, the authors also give a Levinson's type generalization of the Edmundson–Lah–Ribarič inequality, as well as some refinements of the obtained results by constructing certain exponentially convex functions.

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1. Introduction and Some Historical Remarks

The Jensen inequality for convex functions plays a very important role in the Theory of Inequalities due to the fact that it implies a whole series of other classical inequalities. One of the most famous ones amongst them is the so called Lah–Ribarič inequality, which we state in the following theorem.

Theorem 1.1. ([13]) *Let μ be a positive measure on $[0,1]$ and let ϕ be a convex function on $[m,M]$, where $-\infty < m < M < +\infty$. Then for every μ-measurable function f on $[0,1]$ such that $m \leq f(x) \leq M$, $x \in [0,1]$, one has*

$$\frac{\int_0^1 \Phi(f)\mathrm{d}\mu}{\int_0^1 \mathrm{d}\mu} \leq \frac{M-\bar{f}}{M-m}\Phi(m) + \frac{\bar{f}-m}{M-m}\Phi(M), \tag{1.1}$$

where $\bar{f} = \int_0^1 f d\mu / \int_0^1 \mathrm{d}\mu$.

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It was obtained in 1973 by Lah and Ribarič in their paper [13]. Since then, there have been many papers written on the subject of its generalizations and converses. Also, a whole series of monographs in inequalities ([1], [7], [8], [9], [10] and [11]) has been dedicated to classical inequalities, including the Lah–Ribarič inequality.

We also give a probabilistic version of the inequality (1.1), which we will need in this paper:

Theorem 1.2. ([6]) *Let* $X\colon \Omega \to [a,b]$ $(-\infty < a < b < +\infty)$ *be a random variable on probability space* (Ω, p) *and let* $f\colon [a,b] \to \mathbb{R}$ *be a convex function. Then*

$$\mathbb{E}(f(X)) \leq \frac{b-\mathbb{E}(X)}{b-a} f(a) + \frac{\mathbb{E}(X)-a}{b-a} f(b). \tag{1.2}$$

Inequality (1.2) is often referred as Edmundson–Madansky inequality because it was obtained by Edmundson ([6]) in 1956, and Madansky ([15]) in 1959 was the first one to use it in the context of stochastic programming for developing upper bounds on the expectation of convex functions.

For a comprehensive list of recent results on the Edmundson–Madansky inequality, see books [4] and [12].

Next theorem that we state is a generalization of the Lah–Ribarič inequality (1.1) for positive linear functionals which is proved in [3] by Beesack and Pečarić (see also [19, p. 98]):

Theorem 1.3. ([3]) *Let* ϕ *be convex on* $I = [m, M]$ $(-\infty < m < M < \infty)$. *Let* L *be a linear class of real-valued functions on* E *such that* $af + bg \in L$ *for any* $f, g \in L$, $a, b \in \mathbb{R}$ *and* $\mathbf{1} \in L$, *and let* A *be any positive linear functional on* L *with* $A(1) = 1$. *Then for every* $f \in L$ *such that* $\phi(f) \in L$ *(so that* $m \leq f(t) \leq M$ *for all* $t \in E$*), we have*

$$A(\phi(f)) \leq \frac{M-A(f)}{M-m}\phi(m) + \frac{A(f)-m}{M-m}\phi(M). \tag{1.3}$$

One can see that Theorem 1.3 is also a generalization of Theorem 1.2, and that inequalities (1.1) and (1.2) are actually the same inequality, but in different settings. Therefore, from now on we will unify these inequalities under a common name of the Edmundson–Lah–Ribarič inequality.

2. Preliminaries

A well-known Levinson's inequality is stated in the following theorem.

Theorem 2.1. ([14]) *Let* $f\colon \langle 0, 2c\rangle \to \mathbb{R}$ *satisfy* $f''' \geq 0$ *and let* $p_i, x_i, y_i, i = 1, ..., n$ *be such that* $p_i > 0$, $\sum_{i=1}^{n} p_i = 1$, $0 \leq x_i \leq c$ *and*

$$x_1 + y_1 = x_2 + y_2 = \cdots = x_n + y_n. \tag{2.1}$$

Then the following inequality is valid

$$\sum_{i=1}^{n} p_i f(x_i) - f(\bar{x}) \leq \sum_{i=1}^{n} p_i f(y_i) - f(\bar{y}), \tag{2.2}$$

where $\bar{x} = \sum_{i=1}^{n} p_i x_i$ *and* $\bar{y} = \sum_{i=1}^{n} p_i y_i$ *denote the weighted arithmetic means.*

To weaken the assumptions on the differentiability of f, one needs to work with the divided differences. A kth order difference of a function $f\colon I \to \mathbb{R}$ defined on an interval I at distinct points, $x_0, x_1, ..., x_k \in I$, is defined recursively by

$$[x_i]f = f(x_i), \quad \text{for} \quad i = 0, ..., k$$
$$[x_0, ..., x_k]f = \frac{[x_1, ..., x_k]f - [x_0, ..., x_{k-1}]f}{x_k - x_0}.$$

A function $f\colon I \to \mathbb{R}$ is called k-convex if $[x_0, ..., x_k]f \geq 0$ for all choices of $k+1$ distinct points $x_0, x_1, ..., x_k \in I$. If the kth derivative of a convex function exists, then $f^{(k)} \geq 0$, but $f^{(k)}$ may not exist (for properties of divided differences and k-convex functions see [19]).

Remark 2.2. (i) Bullen [5] rescaled Levinson's inequality to a general interval $[a, b]$ and showed that if function f is 3-convex and $p_i, x_i, y_i, i = 1, ..., n$ are such that $p_i > 0$, $\sum_{i=1}^{n} p_i = 1$, $a \leq x_i, y_i \leq b$, (2.1) holds for some $c \in \langle a, b\rangle$ and

$$\max\{x_1, ..., x_n\} \leq \max\{y_1, ..., y_n\}, \tag{2.3}$$

then (2.2) holds.

(ii) Pečarić [17] proved that the inequality (2.2) is valid when one weakens the previous assumption (2.3) to

$$x_i + x_{n-i+1} \leq 2c \quad \text{and} \quad \frac{p_i x_i + p_{n-i+1} x_{n-i+1}}{p_i + p_{n-i+1}} \leq c, \qquad \text{for} \quad i = 1, 2, ..., n$$

(iii) Mercer [16] made a significant improvement by replacing condition (2.1) with a weaker one, i.e. he proved that the inequality (2.2) holds under the following conditions:

$$f''' \geq 0,\ p_i > 0,\ \sum_{i=1}^{n} p_i = 1,\ a \leq x_i, y_i \leq b, \max\{x_1, \ldots, x_n\} \leq \max\{y_1, \ldots, y_n\}$$
$$\sum_{i=1}^{n} p_i (x_i - \bar{x})^2 = \sum_{i=1}^{n} p_i (y_i - \bar{y})^2. \tag{2.4}$$

(iv) Witkowski [20] showed that it is enough to assume that f is 3-convex in Mercer's assumptions. Furthermore, Witkowski weakened the assumption (2.4) and showed that equality can be replaced by inequality in a certain direction.

Furthermore, Baloch, Pečarić and Praljak in their paper [2] introduced a new class of functions $\mathcal{K}_c^1(a, b)$ that extends 3-convex functions and can be interpreted as functions that are "3-convex at point $c \in \langle a, b\rangle$". They showed that $\mathcal{K}_c^1(a, b)$ is the largest class of functions for which Levinson's inequality (2.2) holds under Mercer's assumptions, i.e. that $f \in \mathcal{K}_c^1(a, b)$ if and only if inequality (2.2) holds for arbitrary weights $p_i > 0$, $\sum_{i=1}^{n} p_i = 1$ and sequences x_i and y_i that satisfy $x_i \leq c \leq y_i$ for $i = 1, 2, ..., n$.

We give definition of the class $\mathcal{K}_c^1(a,b)$ extended to an arbitrary interval I in $\mathbb{R}$.

Definition 2.3. Let $f : I \to \mathbb{R}$ and $c \in I^\circ$, where I° is the interior of I. We say that $f \in \mathcal{K}_c^1(I)$ ($f \in \mathcal{K}_c^2(I)$) if there exists a constant D such that the function $F(x) = f(x) - \frac{D}{2}x^2$ is concave (convex) on $\langle -\infty, c] \cap I$ and convex (concave) on $[c, +\infty\rangle \cap I$.

Throughout this paper, $\mathbb{E}(Z)$ and $\mathrm{Var}(Z)$ denote expectation and variance, respectively, of a random variable Z. Pečarić, Praljak and Witkowski in [18] proved the following probabilistic version of Levinson's inequality.

Theorem 2.4. ([18]) *Let $X : \Omega_1 \to I$ and $Y : \Omega_2 \to I$ be two random variables on probability spaces (Ω_1, p) and (Ω_2, q), respectively, such that there exists $c \in I^\circ$ such that*

$$\operatorname{ess\,sup}_{\omega \in \Omega_1} X(\omega) \leq c \leq \operatorname{ess\,sup}_{\omega \in \Omega_2} Y(\omega) \tag{2.5}$$

and

$$\mathrm{Var}(X) = \mathrm{Var}(Y) < \infty.$$

Then for every $f \in \mathcal{K}_c^1(I)$ such that $\mathbb{E}(f(X))$ and $\mathbb{E}(f(Y))$ are finite one has

$$\mathbb{E}(f(X)) - f(\mathbb{E}(X)) \leq \mathbb{E}(f(Y)) - f(\mathbb{E}(Y)).$$

As a simple consequence of the previous theorem, they obtained further generalization of the results obtained in [2].

Corollary 2.5. ([18]) *If $x_i \in I \cap \langle -\infty, c]$, $y_j \in I \cap [c, +\infty\rangle$, $p_i > 0$, $q_j > 0$ for $i = 1, ..., n$ and $j = 1, ..., m$ are such that $\sum_{i=1}^{n} p_i = \sum_{j=1}^{m} q_j = 1$ and $\sum_{i=1}^{n} p_i(x_i - \bar{x})^2 = \sum_{j=1}^{m} q_j(y_j - \bar{y})^2$, then*

$$\sum_{i=1}^{n} p_i f(x_i) - f(\bar{x}) \leq \sum_{j=1}^{m} q_j f(y_j) - f(\bar{y}) \tag{2.6}$$

holds for every $f \in \mathcal{K}_c^1(I)$.

The aim of this paper is to build on the method of Pečarić, Praljak and Witkowski seen in [18] to obtain a Levinson's type generalization of the Edmundson–Lah–Ribarič inequality, as well as to give refinements of the obtained results by constructing certain exponentially convex functions.

3. Results

Our main result is a Levinson's type generalization of the Edmundson–Lah–Ribarič inequality and it is stated in the next theorem.

Theorem 3.1. *Let* $-\infty < a \leq A \leq b \leq B < +\infty$. *Let* $X\colon \Omega_1 \to [a, A]$ *and* $Y\colon \Omega_2 \to [b, B]$ *be two random variables on probability spaces* (Ω_1, p) *and* (Ω_2, q), *respectively, such that* (2.5) *holds and*

$$\frac{A-\mathbb{E}(X)}{A-a}a^2 + \frac{\mathbb{E}(X)-a}{A-a}A^2 - \mathbb{E}(X^2) = \frac{B-\mathbb{E}(Y)}{B-b}b^2 + \frac{\mathbb{E}(Y)-b}{B-b}B^2 - \mathbb{E}(Y^2). \tag{3.1}$$

Then for every $f \in \mathcal{K}_c^1(a, B)$ *such that* $\mathbb{E}(X)$ *and* $\mathbb{E}(Y)$ *are finite one has*

$$\begin{aligned} &\frac{A-\mathbb{E}(X)}{A-a}f(a) + \frac{\mathbb{E}(X)-a}{A-a}f(A) - \mathbb{E}(f(X)) \\ &\quad \leq \frac{B-\mathbb{E}(Y)}{B-b}f(b) + \frac{\mathbb{E}(Y)-b}{B-b}f(B) - \mathbb{E}(f(Y)). \end{aligned} \tag{3.2}$$

Proof. Let $F(x) = f(x) - \frac{D}{2}x^2$, where D is the constant from Definition 2.3. Since $F\colon [a, A] \to \mathbb{R}$ is concave, from the Edmundson–Madansky inequality (1.2) we have

$$\begin{aligned} 0 &\geq \frac{A-\mathbb{E}(X)}{A-a}F(a) + \frac{\mathbb{E}(X)-a}{A-a}F(A) - \mathbb{E}(F(X)) \\ &= \frac{A-\mathbb{E}(X)}{A-a}f(a) + \frac{\mathbb{E}(X)-a}{A-a}f(A) - \mathbb{E}(f(X)) \\ &\quad -\frac{D}{2}\Big(\frac{A-\mathbb{E}(X)}{A-a}a^2 + \frac{\mathbb{E}(X)-a}{A-a}A^2 - \mathbb{E}(X^2)\Big), \end{aligned}$$

and if we rearrange it, we get

$$\begin{aligned} &-\frac{D}{2}\Big(\frac{A-\mathbb{E}(X)}{A-a}a^2 + \frac{\mathbb{E}(X)-a}{A-a}A^2 - \mathbb{E}(X^2)\Big) \\ &\quad \leq -\frac{A-\mathbb{E}(X)}{A-a}f(a) - \frac{\mathbb{E}(X)-a}{A-a}f(A) + \mathbb{E}(f(X)). \end{aligned} \tag{3.3}$$

Similarly, $F\colon [b, B] \to \mathbb{R}$ is convex, so in the same way we obtain

$$\begin{aligned} 0 &\leq \frac{B-\mathbb{E}(Y)}{B-b}F(b) + \frac{\mathbb{E}(Y)-b}{B-b}F(B) - \mathbb{E}(F(Y)) \\ &= \frac{B-\mathbb{E}(Y)}{B-b}f(b) + \frac{\mathbb{E}(Y)-b}{B-b}f(B) - \mathbb{E}(f(Y)) \\ &\quad - \frac{D}{2}\left(\frac{B-\mathbb{E}(Y)}{B-b}b^2 + \frac{\mathbb{E}(Y)-b}{B-b}B^2 - \mathbb{E}(Y^2)\right), \end{aligned}$$

and after rearranging we get

$$\begin{aligned} &\frac{D}{2}\left(\frac{B-\mathbb{E}(Y)}{B-b}b^2 + \frac{\mathbb{E}(Y)-b}{B-b}B^2 - \mathbb{E}(Y^2)\right) \\ &\quad \leq \frac{B-\mathbb{E}(Y)}{B-b}f(b) + \frac{\mathbb{E}(Y)-b}{B-b}f(B) - \mathbb{E}(f(Y)). \end{aligned} \tag{3.4}$$

Now if we add up (3.3) and (3.4), we get

$$
\begin{aligned}
0 &= \frac{D}{2}\Bigg(\frac{B-\mathbb{E}(Y)}{B-b}b^2+\frac{\mathbb{E}(Y)-b}{B-b}B^2-\mathbb{E}(Y^2)\\
&\qquad -\frac{A-\mathbb{E}(X)}{A-a}a^2-\frac{\mathbb{E}(X)-a}{A-a}A^2+\mathbb{E}(X^2)\Bigg)\\
&\le \frac{B-\mathbb{E}(Y)}{B-b}f(b)+\frac{\mathbb{E}(Y)-b}{B-b}f(B)-\mathbb{E}(f(Y))\\
&\quad -\frac{A-\mathbb{E}(X)}{A-a}f(a)-\frac{\mathbb{E}(X)-a}{A-a}f(A)+\mathbb{E}(f(X))
\end{aligned}
$$

which completes the proof. □

Remark 3.2. It is obvious from the proof of the previous theorem that the inequality (3.2) holds if we replace the equality (3.1) by a weaker condition

$$
\begin{aligned}
&D\Bigg(\frac{B-\mathbb{E}(Y)}{B-b}b^2+\frac{\mathbb{E}(Y)-b}{B-b}B^2-\mathbb{E}(Y^2)\\
&\qquad -\frac{A-\mathbb{E}(X)}{A-a}a^2-\frac{\mathbb{E}(X)-a}{A-a}A^2+\mathbb{E}(X^2)\Bigg)\ge 0.
\end{aligned}
$$

Since $f''_-(c)\le D\le f''_+(c)$ (see [2]), if additionally f is convex (resp. concave), this condition can be further weakened to

$$
\begin{aligned}
&\frac{B-\mathbb{E}(Y)}{B-b}b^2+\frac{\mathbb{E}(Y)-b}{B-b}B^2-\mathbb{E}(Y^2)\\
&\quad -\frac{A-\mathbb{E}(X)}{A-a}a^2-\frac{\mathbb{E}(X)-a}{A-a}A^2+\mathbb{E}(X^2)\ge 0 \text{ (resp. } \le 0 \text{).}
\end{aligned}
$$

Remark 3.3. One can easily see from (3.3) and (3.4) that the inequality (3.2) can be rewritten as

$$
\begin{aligned}
&\frac{A-\mathbb{E}(X)}{A-a}f(a)+\frac{\mathbb{E}(X)-a}{A-a}f(A)-\mathbb{E}(f(X))\le 0\\
&\qquad \le \frac{B-\mathbb{E}(Y)}{B-b}f(b)+\frac{\mathbb{E}(Y)-b}{B-b}f(B)-\mathbb{E}(f(Y))
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{A-\mathbb{E}(X)}{A-a}f(a)+\frac{\mathbb{E}(X)-a}{A-a}f(A)-\mathbb{E}(f(X))\le \frac{D}{2}C\\
&\qquad \le \frac{B-\mathbb{E}(Y)}{B-b}f(b)+\frac{\mathbb{E}(Y)-b}{B-b}f(B)-\mathbb{E}(f(Y)),
\end{aligned}
$$

where C is equal to any of the sides in equality (3.1).

The following result gives us a discrete version of Levinson's type generalization of the Edmundson–Lah–Ribarič inequality, and it is obtained as a simple consequence of the previous theorem.

Corollary 3.4. *Let $-\infty < a \leq A \leq c \leq b \leq B < +\infty$. If $x_i \in [a, A]$, $y_j \in [b, B]$, $p_i > 0$, $q_j > 0$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$ are such that $\sum_{i=1}^{n} p_i = \sum_{j=1}^{m} q_j = 1$ and*

$$\frac{A-\bar{x}}{A-a}a^2 + \frac{\bar{x}-a}{A-a}A^2 - \sum_{i=1}^{n} p_i x_i^2 = \frac{B-\bar{y}}{B-b}b^2 + \frac{\bar{y}-b}{B-b}B^2 - \sum_{j=1}^{m} q_j y_j^2, \quad (3.5)$$

where $\bar{x} = \sum_{i=1}^{n} p_i x_i$ and $\bar{y} = \sum_{j=1}^{m} q_j y_j$, then for every $f \in \mathcal{K}_c^1(a, B)$ we have

$$\begin{aligned} &\frac{A-\bar{x}}{A-a}f(a) + \frac{\bar{x}-a}{A-a}f(A) - \sum_{i=1}^{n} p_i f(x_i) \\ &\quad \leq \frac{B-\bar{y}}{B-b}f(b) + \frac{\bar{y}-b}{B-b}f(B) - \sum_{j=1}^{m} q_j f(y_j). \end{aligned} \quad (3.6)$$

Proof. Let X be a discrete random variable that takes value x_i with probability p_i for $i = 1, 2, \ldots, n$ and let Y be a discrete random variable that takes value y_j with probability q_j for $j = 1, 2, \ldots, m$. One can immediately see that X and Y satisfy the conditions from Theorem 3.1, so the inequality (3.6) follows directly from (3.2). □

4. Exponential Convexity

Let $-\infty < a \leq A \leq b \leq B < +\infty$. For fixed random variables $X : \Omega_1 \to [a, A]$ and $Y : \Omega_2 \to [b, B]$ on probability spaces (Ω_1, p) and (Ω_2, q), respectively, such that (2.5) and (3.1) hold.

Motivated by the inequality (3.2), we define the following linear functional, which represents the difference between the right and the left side of the aforementioned inequality with

$$\begin{aligned} \Gamma(f) = &\frac{B-\mathbb{E}(Y)}{B-b}f(b) + \frac{\mathbb{E}(Y)-b}{B-b}f(B) - \mathbb{E}(f(Y)) \\ &-\frac{A-\mathbb{E}(X)}{A-a}f(a) - \frac{\mathbb{E}(X)-a}{A-a}f(A) + \mathbb{E}(f(X)) \end{aligned} \quad (4.1)$$

for functions $f : [a, B] \to \mathbb{R}$ such that $\mathbb{E}(f(X))$ and $\mathbb{E}(f(Y))$ are finite.

From Theorem 3.1 it follows that $\Gamma(f) \geq 0$ for $f \in \mathcal{K}_c^1(a, B)$.

First we will give two mean value results.

Theorem 4.1. *Let $-\infty < a \leq A < c < b \leq B < +\infty$. Let $X : \Omega_1 \to [a, A]$ and $Y : \Omega_2 \to [b, B]$ be two random variables on probability spaces (Ω_1, p) and (Ω_2, q), respectively, such that (2.5) and (3.1) hold and let Γ be the linear functional defined by (4.1). Then for $f \in C^3([a, B])$ there exists $\xi \in [a, B]$ such that*

$$\Gamma(f) = \frac{f'''(\xi)}{6}\left[\frac{B-\mathbb{E}(Y)}{B-b}b^3 + \frac{\mathbb{E}(Y)-b}{B-b}B^3 \right. \quad (4.2)$$

$$\left. - \frac{A-\mathbb{E}(X)}{A-a}a^3 - \frac{\mathbb{E}(X)-a}{A-a}A^3 - \mathbb{E}(Y^3 - X^3)\right] \quad (4.3)$$

Proof. Let $f \in C^3([a,B])$. Function f is bounded, so $\mathbb{E}(f(X))$ and $\mathbb{E}(f(Y))$ are finite and the functional $\Gamma(f)$ is well defined. Furthermore, there exist $m = \min_{x\in[a,B]} f'''(x)$ and $M = \max_{x\in[a,B]} f'''(x)$. The functions

$$f_1(x) = f(x) - \frac{m}{6}x^3 \quad \text{and} \quad f_2(x) = \frac{M}{6}x^3 - f(x)$$

are 3-convex because $f_1'''(x) \geq 0$ and $f_2'''(x) \geq 0$. Consequently, by Theorem 3.1 we have $\Gamma(f_1) \geq 0$ and $\Gamma(f_2) \geq 0$, so we directly get

$$\frac{m}{6}\Gamma(\tilde{f}) \leq \Gamma(f) \leq \frac{M}{6}\Gamma(\tilde{f}) \tag{4.4}$$

where $\tilde{f}(x) = x^3$. Since the function $\tilde{f}$ is 3-convex, by Theorem 3.1 we have

$$\begin{aligned} 0 \leq \Gamma(\tilde{f}) = &\frac{B - \mathbb{E}(Y)}{B-b}b^3 + \frac{\mathbb{E}(Y) - b}{B-b}B^3 \\ &- \frac{A - \mathbb{E}(X)}{A-a}a^3 - \frac{\mathbb{E}(X) - a}{A-a}A^3 - \mathbb{E}(Y^3 - X^3) \end{aligned}$$

If $\Gamma(\tilde{f}) = 0$, then (4.4) implies $\Gamma(f) = 0$, so (4.2) holds for every $\xi \in [a,B]$. Otherwise, dividing (4.4) by $0 < \Gamma(\tilde{f})/6$ we get

$$m \leq \frac{6\Gamma(f)}{\Gamma(\tilde{f})} \leq M,$$

and continuity of f''' insures the existance of $\xi \in [a,B]$ satisfying (4.2). □

Theorem 4.2. *Let a, A, c, b, B, X, Y and Γ be as in Theorem 4.1 and let $f, g \in C^3([a,B])$. If $\Gamma(g) \neq 0$, then there exists $\xi \in [a,B]$ such that*

$$\frac{\Gamma(f)}{\Gamma(g)} = \frac{f'''(\xi)}{g'''(\xi)},$$

or

$$f'''(\xi) = g'''(\xi) = 0.$$

Proof. Let us define function $h \in C^3([a,B])$ by $h(x) = \Gamma(g)f(x) - \Gamma(f)g(x)$. Due to the linearity of Γ we have $\Gamma(h) = 0$. Theorem 4.1 implies that exist $\xi, \xi_1 \in [a,B]$ such that

$$0 = \Gamma(h) = \frac{h'''(\xi)}{6}\Gamma(\tilde{f}),$$

$$0 \neq \Gamma(g) = \frac{g'''(\xi_1)}{6}\Gamma(\tilde{f}),$$

where $\tilde{f}(x) = x^3$. Therefore, $\Gamma(\tilde{f}) \neq 0$, otherwise we would have $\Gamma(g) = 0$, which is contradiction with the assumption $\Gamma(g) \neq 0$, and

$$0 = h'''(\xi) = \Gamma(g)f'''(\xi) - \Gamma(f)g'''(\xi),$$

which gives us the claim of the theorem. □

Next we will give some definitions and basic results regarding the exponential convexity that we will need in the rest of this section. Throughout this section I will denote an interval in $\mathbb{R}$.

Definition 4.3. A function $f\colon I \to \mathbb{R}$ is n-exponentially convex in the Jensen sense on I if

$$\sum_{i,j=1}^{n} \xi_i \xi_j f\left(\frac{x_i + x_j}{2}\right) \geq 0$$

holds for all choices of $\xi_i \in \mathbb{R}$ and every $x_i \in I$, $i = 1, \dots, n$.

A function $f\colon I \to \mathbb{R}$ is n-exponentially convex if it is n-exponentially convex in the Jensen sense and continuous on I.

Remark 4.4. It is clear from the definition that 1-exponentially convex functions in the Jensen sense are in fact nonnegative functions. Also, n-exponentially convex functions in the Jensen sense are k-exponentially convex in the Jensen sense for every $k \in \mathbb{N}$, $k \leq n$.

Definition 4.5. A function $f\colon I \to \mathbb{R}$ is exponentially convex in the Jensen sense on I if it is n-exponentially convex in the Jensen sense for all $n \in \mathbb{N}$.

A function $f\colon I \to \mathbb{R}$ is exponentially convex if it is exponentially convex in the Jensen sense and continuous on I.

Remark 4.6. It is known (and easy to show) that $f\colon I \to \mathbb{R}^+$ is log-convex in the Jensen sense, i.e.

$$f\left(\frac{x_1 + x_2}{2}\right) \leq f(x_1) f(x_2) \quad \text{for all} \quad x_1, x_2 \in I \tag{4.5}$$

if and only if

$$l^2 f(x_1) + 2lm f\left(\frac{x_1 + x_2}{2}\right) + m^2 f(x_2) \geq 0$$

holds for each $l, m \in \mathbb{R}$ and $x_1, x_2 \in I$.

It follows that a positive function is log-convex in the Jensen sense if and only if it is 2-exponentially convex in the Jensen sense. Also, using basic convexity theory it follows that a positive function is log-convex if and only if it is 2-exponentially convex.

The following lemma is equivalent to the definition of convex functions (see [19]).

Lemma 4.7. ([19]) *A function $f\colon I \to \mathbb{R}$ is convex if and only if the inequality*

$$(x_3 - x_2) f(x_1) + (x_1 - x_3) f(x_2) + (x_2 - x_1) f(x_3) \geq 0$$

holds for all $x_1, x_2, x_3 \in I$ such that $x_1 < x_2 < x_3$.

We will also need the following result (see [19]).

Lemma 4.8. ([19]) *If f is a convex function on an interval I and if $x_1 \leq y_1$, $x_2 \leq y_2$, $x_1 \neq x_2$, $y_1 \neq y_2$, then the following inequality is valid*

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(y_2) - f(y_1)}{y_2 - y_1}. \tag{4.6}$$

If the function f is concave then the inequality reverses.

The following result will enable us to construct some new families of exponentially convex functions.

Theorem 4.9. *Let $-\infty < a \leq A \leq b \leq B < +\infty$. Let $X\colon \Omega_1 \to [a, A]$ and $Y\colon \Omega_2 \to [b, B]$ be two random variables on probability spaces (Ω_1, p) and (Ω_2, q), respectively, such that (2.5) and (3.1) hold and let Γ be given by (4.1). Furthermore, let $\Upsilon = \{f_t\colon [a, B] \to \mathbb{R} | t \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions such that for every $t \in J$ $\mathbb{E}(f_t(X))$ and $\mathbb{E}(f_t(Y))$ are finite, and for every four distinct points, $u_0, u_1, u_2, u_3 \in [a, B]$, the mapping $t \longmapsto [u_0, u_1, u_2, u_3]f_t$ is n-exponentially convex in the Jensen sense. Then the mapping $t \longmapsto \Gamma(f_t)$ is n-exponentially convex in the Jensen sense on J. If additionally the mapping $t \longmapsto \Gamma(f_t)$ is continuous on J, then it is is n-exponentially convex on J.*

Proof. For $\xi_i \in \mathbb{R}$ and $t_i \in J$, $i = 1, 2, ldots, n$, we define

$$f(x) = \sum_{i,j=1}^{n} \xi_i \xi_j f_{\frac{t_i + t_j}{2}}(x).$$

Due to linearity of divided differences, from the assumption that the function $t \longmapsto [u_0, u_1, u_2, u_3]f_t$ is n-exponentially convex, we have

$$[u_0, u_1, u_2, u_3]f = \sum_{i,j=1}^{n} \xi_i \xi_j [u_0, u_1, u_2, u_3] f_{\frac{t_i + t_j}{2}} \geq 0.$$

This implies that f is 3-convex, so $f \in \mathcal{K}_1^c$. Due to the linearity of expectation, $\mathbb{E}(f(X))$ and $\mathbb{E}(f(Y))$ are finite, so by Theorem 3.1 we have

$$0 \leq \Gamma(f) = \sum_{i,j=1}^{n} \xi_i \xi_j \Gamma(f_{\frac{t_i + t_j}{2}}).$$

Therefore, the mapping $t \longmapsto \Gamma(f_t)$ is n-exponentially convex in the Jesen sense. If it is also continuous, it is n-exponentially convex by definition. □

If the assumptions of Theorem 4.9 hold for all $n \in \mathbb{N}$, then we immediately get the following corollary.

Corollary 4.10. *Let $-\infty < a \leq A \leq b \leq B < +\infty$. Let $X\colon \Omega_1 \to [a, A]$ and $Y\colon \Omega_2 \to [b, B]$ be two random variables on probability spaces (Ω_1, p) and (Ω_2, q), respectively, such that (2.5) and (3.1) hold and let Γ be given by (4.1). Furthermore, let $\Upsilon = \{f_t\colon [a, B] \to \mathbb{R} | t \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions such that for every $t \in J$ $\mathbb{E}(f_t(X))$ and $\mathbb{E}(f_t(Y))$ are finite, and for every four distinct points, $u_0, u_1, u_2, u_3 \in [a, B]$, the mapping $t \longmapsto [u_0, u_1, u_2, u_3]f_t$ is exponentially convex in the Jensen sense. Then the mapping $t \longmapsto \Gamma(f_t)$ is exponentially convex in the Jensen sense on J. If additionally the mapping $t \longmapsto \Gamma(f_t)$ is continuous on J, then it is is exponentially convex on J.*

Corollary 4.11. *Let $-\infty < a \leq A \leq b \leq B < +\infty$ and let X, Y and Γ be as in Corollary 4.10. Let $\Upsilon = \{f_t\colon [a, B] \to \mathbb{R} | t \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions such that for every $t \in J$ $\mathbb{E}(f_t(X))$ and $\mathbb{E}(f_t(Y))$ are finite, and for every four distinct points, $u_0, u_1, u_2, u_3 \in [a, B]$, the mapping $t \longmapsto [u_0, u_1, u_2, u_3]f_t$ is 2-exponentially convex in the Jensen sense. Then the following statements hold.*

(i) *If the mapping* $t \longmapsto \Gamma(f_t)$ *is continuous on* J*, then for* $r, s, t \in J$ *such that* $r < s < t$ *we have*

$$\Gamma(f_s)^{t-r} \le \Gamma(f_r)^{t-s}\Gamma(f_t)^{s-r} \tag{4.7}$$

(ii) *If the mapping* $t \longmapsto \Gamma(f_t)$ *is strictly positive and differentiable on* J*, then for all* $s, t, u, v \in J$ *such that* $s \le u$ *and* $t \le v$ *we have*

$$\mathfrak{B}_{s,t}(\Upsilon) \le \mathfrak{B}_{u,v}(\Upsilon),$$

where

$$\mathfrak{B}_{s,t}(\Upsilon) = \begin{cases} \left(\frac{\Gamma(f_s)}{\Gamma(f_t)}\right)^{\frac{1}{s-t}}, & s \ne t, \\ \exp\left(\frac{\frac{d}{dt}(\Gamma(f_s))}{\Gamma(f_s)}\right), & s = t. \end{cases} \tag{4.8}$$

Proof. By Theorem 4.9 the mapping $t \longmapsto \Gamma(f_t)$ is 2-exponentially convex in the Jensen sense. From the comment after Remark 4.6 one can see that this mapping is either identically equal to zero (in which case the inequality (4.7) holds with both sides equal to zero) or it is strictly positive and log-convex. Hence if we take $r < s < t$ and $f(t) = \log \Gamma(f_t)$ in Lemma 4.7 we get

$$(t-s)\log\Gamma(f_r) + (r-t)\log\Gamma(f_s) + (s-r)\log\Gamma(f_t) \ge 0,$$

which is equivalent to inequality (4.7).

From (i) it follows that the mapping $t \longmapsto \Gamma(f_t)$ is log-convex on J, which means that the function $t \longmapsto \log\Gamma(f_t)$ is convex on J. Hence, using Lemma 4.8 with $s \le u$, $t \le y$, $s \ne t$, $u \ne v$, we obtain

$$\frac{\log\Gamma(f_s) - \log\Gamma(f_t)}{s-t} \le \frac{\log\Gamma(f_u) - \log\Gamma(f_v)}{u-v},$$

which is

$$\mathfrak{B}_{s,t}(\Upsilon) \le \mathfrak{B}_{u,v}(\Upsilon).$$

Finally, the limiting cases $s = t$ and $u = v$ are obtained by taking the limits $s \to t$ and $u \to v$. □

Now let us consider the following family of functions

$$\Upsilon_1 = \{f_t \colon [a, B] \to \mathbb{R} \mid t \in \mathbb{R}\}, \quad [a, B] \subset \langle 0, +\infty\rangle,$$

defined by

$$f_t(x) = \begin{cases} \dfrac{1}{t(t-1)(t-2)}x^t, & t \ne 0, 1, 2, \\ \dfrac{1}{2}\ln x, & t = 0, \\ -x\ln x, & t = 1, \\ \dfrac{1}{2}x^2\ln x, & t = 2. \end{cases} \tag{4.9}$$

From now on we assume that $\mathbb{E}(f_t(X))$ and $\mathbb{E}(f_t(Y))$ are finite for all the functions f_t given by (4.9).

Since $f_t'''(x) = x^{t-3} \geq 0$, the functions f_t are 3-convex, and the function

$$f(x) = \sum_{i,j=1}^{n} \xi_i \xi_j f_{\frac{t_i+t_j}{2}}(x)$$

satisfies

$$f'''(x) = \sum_{i,j=1}^{n} \xi_i \xi_j f'''_{\frac{t_i+t_j}{2}}(x) = \left(\sum_{i=1}^{n} \xi_i e^{(t_i-3)\ln x} \right)^2 \geq 0,$$

so f is convex. Therefore we have

$$0 \leq [u_0, u_1, u_2, u_3] f = \sum_{i,j=1}^{n} \xi_i \xi_j [u_0, u_1, u_2, u_3] f_{\frac{t_i+t_j}{2}}(x),$$

so the mapping $t \longmapsto [u_0, u_1, u_2, u_3] f_t$ is n-exponentially convex in the Jensen sense. Since this holds for every $n \in \mathbb{N}$, we see that family Υ_1 satisfies the assumptions of Corollary 4.10. Hence, the mapping $t \longmapsto \Gamma(f_t)$ is exponentially convex in the Jensen sense. It is easy to check that that it is also continuous, so the mapping $t \longmapsto \Gamma(f_t)$ is exponentially convex.

If we apply Theorem 4.2 for functions $f = f_t$ and $g = f_s$ given by (4.9), we can conclude that there exists $\xi \in [a, B] \subset \langle 0, +\infty \rangle$ such that

$$\xi = \left(\frac{f_s'''}{f_t'''} \right)^{-1} \left(\frac{\Gamma(f_s)}{\Gamma(f_t)} \right) = \left(\frac{\Gamma(f_s)}{\Gamma(f_t)} \right)^{\frac{1}{s-t}}, \quad s \neq t.$$

Therefore, $\mathfrak{B}_{s,t}(\Upsilon_1)$ given by (4.8) for the family of functions Υ_1 is a mean of the segment $[a, B]$. The limiting cases $s \to t$ can be calculated, and are equal to:

$$\mathfrak{B}_{s,t}(\Upsilon_1) = \begin{cases} \left(\frac{\Gamma(f_s)}{\Gamma(f_t)} \right)^{\frac{1}{s-t}}, & s \neq t, \\ \exp\left(\dfrac{2\Gamma(f_s f_0)}{\Gamma(f_0)} - \dfrac{3s^2 - 6s + 2}{s(s-1)(s-2)} \right), & s = t \neq 0, 1, 2, \\ \exp\left(\dfrac{\Gamma(f_0^2)}{\Gamma(f_0)} + \dfrac{3}{2} \right), & s = t = 0, \\ \exp\left(\dfrac{\Gamma(f_0 f_1)}{\Gamma(f_1)} \right), & s = t = 1, \\ \exp\left(\dfrac{\Gamma(f_0 f_2)}{\Gamma(f_2)} - \dfrac{3}{2} \right), & s = t = 2. \end{cases}$$

From Corollary 4.11(ii) it follows that the means $\mathfrak{B}_{s,t}(\Upsilon_1)$ are monotone in parameters s and t.

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