Mediterr. J. Math. 12 (2015), 1245–1264
DOI 10.1007/s00009-014-0473-3
0378-620X/15/041245-20
published online September 18, 2014

Mediterranean Journal of Mathematics

Existence and Ulam Stability for Partial Impulsive Discontinuous Fractional Differential Inclusions in Banach Algebras

Saïd Abbas and Mouffak Benchohra

Abstract. In this paper, we investigate some existence and Ulam's type stability concepts of fixed point inclusions for a class of partial discontinuous fractional-order differential inclusions with impulses in Banach Algebras. Our results are obtained using weakly Picard operators theory.

Mathematics Subject Classification. 26A33, 34G20, 34A40, 45N05, 47H10.

Keywords. Partial fractional differential inclusion, left-sided mixed Riemann–Liouville integral, Caputo fractional-order derivative, fixed point inclusion, Banach algebra, impulse, Ulam–Hyers stability.

1. Introduction

The fractional calculus represents a powerful tool in applied mathematics to study a myriad of problems from different fields of science and engineering, with many break-through results found in mathematical physics, finance, hydrology, biophysics, thermodynamics, control theory, statistical mechanics, astrophysics, cosmology and bioengineering [12,26]. There has been a significant development in fractional differential equations and in the impulse theory in recent years; see the monographs of Abbas et al. [7], Kilbas et al. [19], Lakshmikantham et al. [20], Miller and Ross [23], Samoilenko and Perestyuk [30], the papers of Abbas et al. [1–6,8], Vityuk et al. [32,33], and the references therein.

The stability of functional equations was originally raised by Ulam in 1940 in a talk given at Wisconsin University (for more details see [31]). The first answer to Ulam's question was given by Hyers in 1941 in the case of Banach spaces in [13]. Thereafter, this type of stability is called the Ulam–Hyers stability. In 1978, Rassias [27] provided a remarkable generalization of the Ulam–Hyers stability of mappings by considering variables. The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Thus, the stability question of functional equations is how do the solutions of the

Birkhäuser

inequality differ from those of the given functional equation? Considerable attention has been given to the study of the Ulam–Hyers and Ulam–Hyers–Rassias stability of all kinds of functional equations; one can see the monographs of [14,15]. Bota-Boriceanu and Petrusel [9], Petru et al. [24,25], and Rus [28,29] discussed the Ulam–Hyers stability for operatorial equations and inclusions. Castro and Ramos [10], and Jung [17] considered the Hyers–Ulam–Rassias stability for a class of Volterra integral equations. Ulam stability for fractional differential equations with Caputo derivative is proposed by Wang et al. [35,36]. Some stability results for fractional integral equation are obtained by Wei et al. [38]. More details from historical point of view, and recent developments of such stabilities are reported in [16,22,28,34,37,38].

In this paper, we discuss the existence and the Ulam–Hyers–Rassias stability for the following fractional partial impulsive discontinuous differential inclusions of the form

$$\begin{cases} {}^cD^r_{\theta_k}\left(\frac{u(x,y)}{f(x,y,u(x,y))}\right) \in G(x,y,u(x,y)); & (x,y)\in J_k;\ k=0,\dots,m, \\ u(x_k^+,y)=u(x_k^-,y)+I_k(u(x_k^-,y)); & y\in[0,b],\ k=1,\dots,m, \\ u(x,0)=\varphi(x); \quad x\in[0,a], \quad u(0,y)=\psi(y); & y\in[0,b], \end{cases} \tag{1}$$

where $a,b>0,\ J_0=[0,x_1]\times[0,b],\ J_k:=(x_k,x_{k+1}]\times[0,b];\ k=1,\dots,m,\ \theta_k=(x_k,0);\ k=0,\dots,m,\ 0=x_0<x_1<\cdots<x_m<x_{m+1}=a,\ {}^cD^r_{\theta_k}$ is the fractional Caputo derivative of order $r=(r_1,r_2)\in(0,1]\times(0,1]$, $f:J\times\mathbb{R}\to\mathbb{R}^*$ is a given continuous function, $J=[0,a]\times[0,b],\ \mathbb{R}^*=\mathbb{R}\backslash\{0\}$, $G:J\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$, $\mathcal{P}(\mathbb{R})$ is the class of all nonempty subsets of $\mathbb{R}$, $I_k:\mathbb{R}\to\mathbb{R};\ k=1,\dots,m$ are given functions satisfying suitable conditions and $\varphi:[0,a]\to\mathbb{R},\ \psi:[0,b]\to\mathbb{R}$ are given absolutely continuous functions with $\varphi(0)=\psi(0)$. Here, $u(x_k^+,y)$ and $u(x_k^-,y)$ denote the right and left limits of $u(x,y)$ at $x=x_k$, respectively. We introduce some concepts about Ulam stability of impulsive partial fractional differential inclusions.

2. Preliminaries

Denote $L^1(J)$ the space of Lebesgue-integrable functions $u:J\to\mathbb{R}$ with the norm

$$\|u\|_{L^1}=\int_0^a\int_0^b|u(x,y)|\,\mathrm{d}y\,\mathrm{d}x.$$

As usual, by $AC(J)$ we denote the space of absolutely continuous functions from J into $\mathbb{R}$, and $\mathcal{C}:=C(J)$ is the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm $\|.\|_\infty$ defined by

$$\|u\|_\infty=\sup_{(x,y)\in J}|u(x,y)|.$$

In all what follows consider the Banach space

$$\begin{aligned}\mathcal{PC}:=\big\{u:J\to\mathbb{R}:\ & u\in C(J_k);\ k=0,1,\dots,m,\\ & \text{and there exist } u(x_k^-,y) \text{ and } u(x_k^+,y);\ k=1,\dots,m,\\ & \text{with } u(x_k^-,y)=u(x_k,y) \text{ for each } y\in[0,b]\big\},\end{aligned}$$

with the norm

$$\|u\|_{\mathcal{PC}} = \sup_{(x,y)\in J} |u(x,y)|.$$

Define a multiplication "$\cdot$" by

$$(u \cdot v)(x,y) = u(x,y)v(x,y) \text{ for each } (x,y) \in J.$$

Then, $\mathcal{PC}$ is a Banach algebra with above norm and multiplication.

Let (X,d) be a metric space induced from the normed space $(X, \|.\|)$. Denote $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ closed}\}$, $\mathcal{P}_{bd}(X) = \{Y \in \mathcal{P}(X) : Y \text{ bounded}\}$, $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact}\}$ and $\mathcal{P}_{cp,cv}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact and convex}\}$.

Definition 2.1. A multivalued map $T : X \to \mathcal{P}(X)$ is convex (closed) valued if $T(x)$ is convex (closed) for all $x \in X$, T is called upper semicontinuous (u.s.c.) on X if for each $x_0 \in X$, the set $T(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $T(x_0)$, there exists an open neighborhood N_0 of x_0 such that $T(N_0) \subseteq N$. T is lower semicontinuous (l.s.c.) if the set $\{t \in X : T(t) \cap B \neq \emptyset\}$ is open for any open set B in X. T is said to be completely continuous if $T(B)$ is relatively compact for every $B \in \mathcal{P}_{bd}(X)$. T has a fixed point if there is $x \in X$ such that $x \in T(x)$. The fixed point set of the multivalued operator T will be denoted by $Fix(T)$. The graph of T will be denoted by $Graph(F) := \{(u,v) \in X \times \mathcal{P}(X) : v \in T(u)\}$.

Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to [0,\infty) \cup \{\infty\}$ given by

$$H_d(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\right\},$$

where $d(A,b) = \inf_{a\in A} d(a,b)$, $d(a,B) = \inf_{b\in B} d(a,b)$. Then, $(\mathcal{P}_{bd,cl}(X), H_d)$ is a Hausdorff metric space.

Definition 2.2. For each $u \in \mathcal{C}$, define the set of selections of the multivalued $F : J \times \mathcal{C} \to \mathcal{P}(\mathcal{C})$ by

$$S_{F,u} = \{v :\in L^1(J) : v(x,y) \in F(x,y,u(x,y));\ (x,y) \in J\}.$$

Definition 2.3. A multivalued map $G : J \to \mathcal{P}_{cl}(X)$ is said to be measurable if for every $v \in E$ the function $(x,y) \to d(v, G(x,y)) = \inf\{d(v,z) : z \in G(x,y)\}$ is measurable.

Definition 2.4. A multivalued map $G : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if

(i) $(x, y) \longmapsto G(x, y, u)$ is measurable for each $u \in \mathbb{R}$;
(ii) $u \longmapsto G(x, y, u)$ is upper semicontinuous for almost all $(x, y) \in J$.
G is said to be L^1-Carathéodory if $(i), (ii)$ and the following condition hold;
(iii) For each $c > 0$, there exists a positive function $\sigma_c \in L^1(J)$ such that

$$\begin{aligned}\|G(x,y,u)\|_{\mathcal{P}} &= \sup\{\|g\| : g \in G(x,y,u)\}\\ &\le \sigma_c(x,y) \ \text{ for all } |u| \le c \text{ and for } a.e.\ (x,y) \in J.\end{aligned}$$

Lemma 2.5. [18] *Let G be a completely continuous multivalued map with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e., $u_n \to u,\ w_n \to w,\ w_n \in G(u_n)$ imply $w \in G(u)$).*

Lemma 2.6. [21] *Let X be a Banach space. Let $G : J \times X \to \mathcal{P}(X)$ be an L^1-Carathéodory multivalued mapping with $S_{G,u} \neq \emptyset$, and let $\mathcal{L}$ be a linear continuous mapping from $L^1(J, X)$ into $C(J, X)$, then the operator*

$$\mathcal{L} \circ S_{G,u} : C(J,X) \to \mathcal{P}_{cp,cv}(C(J,X)),$$

$$u \mapsto \mathcal{L}(S_{G,u})(u),$$

is a closed graph operator in $C(J, X) \times C(J, X)$.

Now, we introduce notations and definitions concerning to partial fractional calculus theory.

Definition 2.7. [32] Let $\theta = (0, 0)$, $r_1, r_2 \in (0, \infty)$ and $r = (r_1, r_2)$. For $f \in L^1(J)$, the expression

$$(I_\theta^r f)(x,y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t)\,\mathrm{d}t\,\mathrm{d}s,$$

is called the left-sided mixed Riemann–Liouville integral of order r, where $\Gamma(.)$ is the (Euler's) Gamma function defined by $\Gamma(\xi) = \int_0^\infty t^{\xi-1}e^{-t}\,\mathrm{d}t;\ \xi > 0$.

In particular,

$$\begin{aligned}(I_\theta^\theta f)(x,y) &= f(x,y),\ (I_\theta^\sigma f)(x,y)\\ &= \int_0^x \int_0^y f(s,t)\,\mathrm{d}t\,\mathrm{d}s;\ \text{for almost all } (x,y) \in J,\end{aligned}$$

where $\sigma = (1, 1)$.

For instance, $I_\theta^r f$ exists for all $r_1, r_2 \in (0, \infty)$, when $f \in L^1(J)$. Note also that when $u \in \mathcal{C}$, then $(I_\theta^r f) \in \mathcal{C}$, moreover,

$$(I_\theta^r f)(x,0) = (I_\theta^r f)(0,y) = 0;\ x \in [0,a],\ y \in [0,b].$$

Example 2.8. Let $\lambda, \omega \in (-1, 0) \cup (0, \infty)$ and $r = (r_1, r_2) \in (0, \infty) \times (0, \infty)$, then

$$I_\theta^r x^\lambda y^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda+r_1)\Gamma(1+\omega+r_2)} x^{\lambda+r_1} y^{\omega+r_2};\ \ \text{for almost all } (x,y) \in J.$$

By $1-r$ we mean $(1-r_1, 1-r_2) \in [0, 1) \times [0, 1)$. Denote by $D_{xy}^2 := \frac{\partial^2}{\partial x \partial y}$, the mixed second-order partial derivative.

Definition 2.9. [32] Let $r \in (0,1] \times (0,1]$ and $f \in L^1(J)$. The Caputo fractional-order derivative of order r of f is defined by the expression

$$\begin{aligned} {}^cD^r_\theta f(x,y) &= (I^{1-r}_\theta D^2_{xy} f)(x,y) \\ &= \frac{1}{\Gamma(1-r_1)\Gamma(1-r_2)} \int_0^x \int_0^y \frac{D^2_{st} f(s,t)}{(x-s)^{r_1}(y-t)^{r_2}} \, \mathrm{d}t \, \mathrm{d}s. \end{aligned}$$

The case $\sigma = (1,1)$ is included and we have

$$({}^cD^\sigma_\theta f)(x,y) = (D^2_{xy} f)(x,y); \text{ for almost all } (x,y) \in J.$$

Example 2.10. Let $\lambda, \omega \in (-1,0) \cup (0,\infty)$ and $r = (r_1, r_2) \in (0,1] \times (0,1]$, then

$${}^cD^r_\theta x^\lambda y^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda-r_1)\Gamma(1+\omega-r_2)} x^{\lambda-r_1} y^{\omega-r_2}; \text{ for almost all } (x,y) \in J.$$

Let $a_1 \in [0,a]$, $z = (a_1, 0)$, $J_z = (a_1, a] \times [0,b]$, $r_1, r_2 > 0$ and $r = (r_1, r_2)$. For $u \in L^1(J_z)$, the expression

$$(I^r_z u)(x,y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{a_1^+}^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} u(s,t) \, \mathrm{d}t \, \mathrm{d}s,$$

is called the left-sided mixed Riemann–Liouville integral of order r of u.

Definition 2.11. [32] For $u \in L^1(J_z)$ where $D^2_{xy} u$ is Lebesgue integrable on $[x_k, x_{k+1}] \times [0,b]$, $k = 0, \ldots, m$, the Caputo fractional-order derivative of order r of u is defined by the expression

$$({}^cD^r_z f)(x,y) = (I^{1-r}_z D^2_{xy} f)(x,y).$$

Let

$$\begin{aligned} \mu_k(x,y) = {} & \frac{u(x,0)}{f(x,0,u(x,0))} + \frac{u(x_k^+, y)}{f(x_k^+, y, u(x_k^+, y))} - \frac{u(x_k^+, 0)}{f(x_k^+, 0, u(x_k^+, 0))}; \\ & k = 0, \ldots, m. \end{aligned}$$

For the existence of solutions for the problem (1), we need the following Lemmas. Let $g \in G(x,y,u(x,y))$.

Lemma 2.12. [1] *A function $u \in AC(J_k)$; $k = 0, \ldots, m$ is said to be a solution of the differential equation*

$${}^cD^r_{\theta_k} \left(\frac{u(x,y)}{f(x,y,u(x,y))} \right) = g(x,y); \qquad (x,y) \in J_k, \tag{2}$$

if and only if $u(x,y)$ satisfies

$$u(x,y) = f(x,y,u(x,y)) \Big(\mu_k(x,y) + (I^r_{\theta_k} g)(x,y) \Big); \qquad (x,y) \in J_k. \tag{3}$$

Let $\mu := \mu_0$.

Lemma 2.13. [1] *A function u is a solution of the fractional integral equations*

$$\begin{cases} u(x,y) = f(x,y,u(x,y))[\mu(x,y) \\ \quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} g(s,t)\,\mathrm{d}t\,\mathrm{d}s \Big]; \quad \text{if}(x,y) \in J_0, \\ \\ u(x,y) = f(x,y,u(x,y))[\mu(x,y) \\ \quad + \sum_{i=1}^{k} \left(\frac{I_i(u(x_i^-,y))}{f(x_i^+,y,u(x_i^+,y))} - \frac{I_i(u(x_i^-,0))}{f(x_i^+,0,u(x_i^+,0))} \right) \\ \quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{i=1}^{k} \int_{x_{i-1}}^{x_i} \int_0^y (x_i-s)^{r_1-1}(y-t)^{r_2-1} g(s,t)\,\mathrm{d}t\,\mathrm{d}s \\ \quad + \int_{x_k}^{x} \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} g(s,t)\,\mathrm{d}t\,\mathrm{d}s \Big]; \ \text{if}(x,y) \in J_k, \ k = 1,\ldots,m, \end{cases}$$

if and only if u is a solution of the problem (1).

Remark 2.14. Using Lemma 2.13, solutions of the problem (1) are solutions of the fixed point inclusion $u \in N(u)$ where $N : \mathcal{PC} \to \mathcal{P}(\mathcal{PC})$ is the multivalued operator defined by

$$(Nu)(x,y) = \left\{ h \in \mathcal{PC} : \begin{cases} h(x,y) = f(x,y,u(x,y))[\mu(x,y) \\ \quad + \int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} g(s,t)\,\mathrm{d}t\,\mathrm{d}s \Big]; \\ g \in S_{G,u}, \ (x,y) \in J_0, \\ \\ h(x,y) = f(x,y,u(x,y))[\mu(x,y) \\ \quad + \sum_{i=1}^{k} \left(\frac{I_i(u(x_i^-,y))}{f(x_i^+,y,u(x_i^+,y))} - \frac{I_i(u(x_i^-,0))}{f(x_i^+,0,u(x_i^+,0))} \right) \\ \quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{i=1}^{k} \int_{x_{i-1}}^{x_i} \int_0^y (x_i-s)^{r_1-1}(y-t)^{r_2-1} g(s,t)\,\mathrm{d}t\,\mathrm{d}s \\ \quad + \int_{x_k}^{x} \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} g(s,t)\,\mathrm{d}t\,\mathrm{d}s \Big]; \\ g \in S_{G,u}, \ (x,y) \in J_k, \ k = 1,\ldots,m. \end{cases} \right\}.$$

Let us give the definition of Ulam–Hyers stability of a fixed point inclusion due to Rus.

Definition 2.15. [29] Let (X,d) be a metric space and $A : X \to X$ be an operator. The fixed point equation $x = A(x)$ is said to be Ulam–Hyers stable if there exists a real number $c_A > 0$ such that: for each real number $\epsilon > 0$ and each solution y^* of the inequality $d(y, A(y)) \le \epsilon$, there exists a solution x^* of the equation $x = A(x)$ such that

$$d(y^*, x^*) \le \epsilon c_A; \ x \in X.$$

In the multivalued case, we have the following definition.

Definition 2.16. [25] Let (X, d) be a metric space and $A : X \to \mathcal{P}(X)$ be a multivalued operator. The fixed point inclusion $u \in A(u)$ is said to be generalized Ulam–Hyers stable if and only if there exists $\Psi : [0, \infty) \times [0, \infty)$ increasing, continuous at 0 and $\Psi(0) = 0$ such that for each $\epsilon > 0$ and for each solution v^* of the inequality $H_d(u, A(u)) \leq \epsilon$, there exists a solution u^* of the inclusion $u \in A(u)$ such that

$$d(u^*, v^*) \leq \Psi(\epsilon); \quad x \in X.$$

From the above definition, we shall give four types of Ulam stability of the fixed point inclusion $u \in A(u)$. Let ϵ be a positive real number and $\Phi : J \to [0, \infty)$ be a continuous function.

Definition 2.17. The fixed point inclusion $u \in A(u)$ is said to be Ulam–Hyers stable if there exists a real number $c_A > 0$ such that for each $\epsilon > 0$ and for each solution u of the inequality $H_d(u(x, y), (Au)(x, y)) \leq \epsilon$; $(x, y) \in J$, there exists a solution v of the inclusion $u \in A(u)$ with

$$\|u(x, y) - v(x, y)\|_E \leq \epsilon c_A; \quad (x, y) \in J.$$

Definition 2.18. The fixed point inclusion $u \in A(u)$ is said to be generalized Ulam–Hyers stable if there exists an increasing function $\Theta_A \in C([0, \infty), [0, \infty))$, $\Theta_A(0) = 0$ such that for each $\epsilon > 0$ and for each solution u of the inequality $H_d(u(x, y), (Au)(x, y)) \leq \epsilon$; $(x, y) \in J$, there exists a solution v of the inclusion $u \in A(u)$ with

$$\|u(x, y) - v(x, y)\|_E \leq \Theta_A(\epsilon); \quad (x, y) \in J.$$

Definition 2.19. The fixed point inclusion $u \in A(u)$ is said to be Ulam–Hyers–Rassias stable with respect to Φ if there exists a real number $c_{A,\Phi} > 0$ such that for each $\epsilon > 0$ and for each solution u of the inequality $H_d(u(x, y), (Au)(x, y)) \leq \epsilon\Phi(x, y)$; $(x, y) \in J$, there exists a solution v of the inclusion $u \in A(u)$ with

$$\|u(x, y) - v(x, y)\|_E \leq \epsilon c_{A,\Phi}\Phi(x, y); \quad (x, y) \in J.$$

Definition 2.20. The fixed point inclusion $u \in A(u)$ is said to be generalized Ulam–Hyers–Rassias stable with respect to Φ if there exists a real number $c_{A,\Phi} > 0$ such that for each solution u of the inequality $H_d(u(x, y), (Au)(x, y)) \leq \Phi(x, y)$; $(x, y) \in J$, there exists a solution v of the inclusion $u \in A(u)$ with

$$\|u(x, y) - v(x, y)\|_E \leq c_{A,\Phi}\Phi(x, y); \quad (x, y) \in J.$$

Remark 2.21. It is clear that

(i) Definition 2.17 $\Rightarrow$ Definition 2.18,
(ii) Definition 2.19 $\Rightarrow$ Definition 2.20,
(iii) Definition 2.19 for $\Phi(x, y) = 1$ $\Rightarrow$ Definition 2.17.

We use the following fixed point theorem by Dhage [11] for proving the existence of solutions for our problem.

Theorem 2.22. *Let X be a Banach algebra and let $A, B : X \to X$ be two operators satisfying*

(a) *A is Lipschitz with a Lipschitz constant α,*
(a) *B is compact and upper semicontinuous, and*
(a) *$2M\alpha < 1$, where $M = \|B(X)\| := \sup\{\|Bu\| : u \in X\}$.*

Then, either

(i) *the operator inclusion $u \in Au\,Bu$ has a solution, or*
(i) *the set $\mathcal{E} = \{u \in X : \lambda u \in Au\,Bu;\ \lambda > 1\}$ is unbounded.*

3. Existence and Ulam Stability Results

In this section, we present the main results for the existence and the Ulam–Hyers–Rassias stability of the problem (1).

Definition 3.1. A function $w \in \mathcal{PC}$ such that its mixed derivative D^2_{xy} exists and is integrable on $J_k;\ k = 0, \ldots, m$, is said to be a solution of the problem (1) if and only if there exists $g \in S_{G,w}$ such that

(i) the function $(x, y) \mapsto \frac{w(x,y)}{f(x,y,w(x,y))}$ is absolutely continuous, and
(ii) w satisfies ${}^cD^r_{\theta_k}\left(\frac{w(x,y)}{f(x,y,w(x,y))}\right) = g(x, y)$ on J_k and the conditions

$$\begin{cases} w(x_k^+, y) = w(x_k^-, y) + I_k(w(x_k^-, y));\ y \in [0, b],\ k = 1, \ldots, m, \\ w(x, 0) = \varphi(x);\ x \in [0, a],\ w(0, y) = \psi(y);\ y \in [0, b], \end{cases}$$

are satisfied.

The following hypotheses will be used in the sequel.

(H_1) There exists a strictly positive function $\alpha \in \mathcal{C}$ such that

$$|f(x, y, u) - f(x, y, \overline{u})| \le \alpha(x, y)|u - \overline{u}|; \quad \text{for all } (x, y) \in J \text{ and } u, \overline{u} \in \mathbb{R},$$

(H_2) The multifunction G is L^1-Carathéodory, and $G(x, y, w)$ has compact and convex values for each $(x, y, w) \in J \times \mathbb{R}$, and there exists a positive function $h \in L^1(J) \cap L^\infty(J)$ such that

$$\|G(x, y, u)\|_{\mathcal{P}} \le h(x, y);\ a.e.\,(x, y) \in J, \quad \text{for all } u \in \mathbb{R},$$

(H_3) There exists a positive function $\beta \in \mathcal{C}$ such that

$$\left|\frac{I_k(u)}{f(x, y, u)}\right| \le \beta(x, y); \quad \text{for all } (x, y) \in J, \quad \text{and all } u \in \mathbb{R}.$$

Theorem 3.2. *Assume that hypotheses (H_1)–(H_3) hold. If*

$$L := \|\alpha\|_\infty \left[\|\mu\|_\infty + 2m\|\beta\|_\infty + \frac{2a^{r_1}b^{r_2}\|h\|_{L^\infty}}{\Gamma(1 + r_1)\Gamma(1 + r_2)}\right] < \frac{1}{2}, \tag{4}$$

then the problem 1 has at least one solution on J. Moreover, if the following hypothesis

(H_4), *there exists $\lambda_\Phi > 0$ such that, for each $(x, y) \in J$ and $u \in \mathbb{R}$ we have*

$$|f(x, y, u)| \le \lambda_\Phi \Phi(x, y),$$

holds, then the fixed point inclusion $u \in N(u)$ is generalized Ulam–Hyers–Rassias stable.

Proof. Define two operators A and B on $\mathcal{PC}$ by

$$(Au)(x,y) = f(x,y,u(x,y)); \quad (x,y) \in J,$$

$$(Bu)(x,y) = \mu(x,y) + + \int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} G(s,t,u(s,t))\,\mathrm{d}t\,\mathrm{d}s; \quad (x,y) \in J_0,$$

and

$$\begin{aligned}(Bu)(x,y) &= \mu(x,y) + \sum_{i=1}^{k} \left(\frac{I_i(u(x_i^-,y))}{f(x_i^+,y,u(x_i^+,y))} - \frac{I_i(u(x_i^-,0))}{f(x_i^+,0,u(x_i^+,0))} \right) \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{i=1}^{k} \int_{x_{i-1}}^{x_i} \int_0^y (x_i-s)^{r_1-1}(y-t)^{r_2-1} G(s,t,u(s,t))\,\mathrm{d}t\,\mathrm{d}s \\ &+ \int_{x_k}^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} G(s,t,u(s,t))\,\mathrm{d}t\,\mathrm{d}s; \quad (x,y) \in J_k; \\ & k = 1,\ldots,m.\end{aligned}$$

Clearly, A and B define the operators $A : \mathcal{PC} \to \mathcal{PC}$ and $B : \mathcal{PC} \to \mathcal{P}_{cp,cv}(\mathcal{PC})$. Solving the problem (1) is equivalent to solving the operator inclusion

$$u(x,y) \in (Au)(x,y)(Bu)(x,y); \; (x,y) \in J. \tag{5}$$

We show that operators A and B satisfy all the assumptions of Theorem 2.22. The proof will be given in several steps and claims. □

Step 1. *A is a Lipschitz operator.* Let $u_1, u_2 \in \mathcal{PC}$. Then, by (H_1), we have

$$\begin{aligned}|Au_1(x,y) - Au_2(x,y)| &= |f(x,y,u_1(x,y)) - f(x,y,u_2(x,y))| \\ &\le \alpha(x,y)|u_1(x,y) - u_2(x,y)| \\ &\le \|\alpha\|_\infty \|u_1 - u_2\|_{\mathcal{PC}}.\end{aligned}$$

Thus,

$$\|Au_1 - Au_2\|_{\mathcal{PC}} \le \|\alpha\|_\infty \|u_1 - u_2\|_{\mathcal{PC}},$$

Hence, A is a Lipschitz with a Lipschitz constant $\|\alpha\|_\infty$.

Step 2. *B is compact and upper semicontinuous with convex values on* $\mathcal{PC}$.

The proof of this step will be given in several claims.

Claim 1. *B has convex values on* $\mathcal{PC}$.

Let $w_1, w_2 \in B(u)$. Then, there exist $g_1, g_2 \in S_{G,u}$ such that for each $(x,y) \in J_0$, we have

$$w_l(x,y) = \mu(x,y) + \int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} g_l(s,t)\,\mathrm{d}t\,\mathrm{d}s; \; l \in \{1,2\},$$

and for each $(x,y) \in J_k;\ k = 1,\dots,m$, we have

$$w_l(x,y) = \mu(x,y) + \sum_{i=1}^{k}\left(\frac{I_i(u(x_i^-,y))}{f(x_i^+,y,u(x_i^+,y))} - \frac{I_i(u(x_i^-,0))}{f(x_i^+,0,u(x_i^+,0))}\right)$$
$$+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\sum_{i=1}^{k}\int_{x_{i-1}}^{x_i}\int_0^y (x_i-s)^{r_1-1}(y-t)^{r_2-1}g_l(s,t)\,\mathrm{d}t\,\mathrm{d}s$$
$$+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_k}^{x}\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}g_l(s,t)\,\mathrm{d}t\,\mathrm{d}s;\ l \in \{1,2\}.$$

Let $0 \le \lambda \le 1$. Then, for each $(x,y) \in J_0$, we have

$$[\lambda w_1 + (1-\lambda)w_2](x,y)$$
$$= \mu(x,y) + \int_0^x\int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}[\lambda g_1 + (1-\lambda)g_2](s,t)\,\mathrm{d}t\,\mathrm{d}s,$$

and, for each $(x,y) \in J_k;\ k = 1,\dots,m$, we have

$$[\lambda w_1 + (1-\lambda)w_2](x,y)$$
$$= \mu(x,y) + \sum_{i=1}^{k}\left(\frac{I_i(u(x_i^-,y))}{f(x_i^+,y,u(x_i^+,y))} - \frac{I_i(u(x_i^-,0))}{f(x_i^+,0,u(x_i^+,0))}\right)$$
$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\sum_{i=1}^{k}\int_{x_{i-1}}^{x_i}\int_0^y (x_i-s)^{r_1-1}(y-t)^{r_2-1}$$
$$\times[\lambda g_1 + (1-\lambda)g_2](s,t)\,\mathrm{d}t\,\mathrm{d}s$$
$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_k}^{x}\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}$$
$$\times[\lambda g_1 + (1-\lambda)g_2](s,t)\,\mathrm{d}t\,\mathrm{d}s.$$

Since $S_{G,u}$ is convex (because G has convex values), we have that

$$\lambda w_1 + (1-\lambda)w_2(x,y) \in B(u).$$

Claim 2. *B maps bounded sets into bounded sets of $\mathcal{PC}$.*

Let $w \in B(u)$ for some $u \in S$, where S is a bounded set of $\mathcal{PC}$. Then, there exists $g \in S_{G,u}$ such that for each $(x,y) \in J_0$

$$w(x,y) = \mu(x,y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_k}^{x}\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}g(s,t)\,\mathrm{d}t\,\mathrm{d}s,$$

and, for each $(x,y) \in J_k;\ k = 1,\dots,m$, we have,

$$w(x,y) = \mu(x,y) + \sum_{i=1}^{k}\left(\frac{I_i(u(x_i^-,y))}{f(x_i^+,y,u(x_i^+,y))} - \frac{I_i(u(x_i^-,0))}{f(x_i^+,0,u(x_i^+,0))}\right)$$
$$+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\sum_{i=1}^{k}\int_{x_{i-1}}^{x_i}\int_0^y (x_i-s)^{r_1-1}(y-t)^{r_2-1}g(s,t)\,\mathrm{d}t\,\mathrm{d}s$$
$$+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_k}^{x}\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}g(s,t)\,\mathrm{d}t\,\mathrm{d}s.$$

From (H_2) and (H_3), for each $(x,y)\in J_k;\ k=1,\dots,m$, we get

$$\|w\|_{\mathcal{P}C}\leq \|\mu\|_\infty+2m\|\beta\|_\infty+\frac{2a^{r_1}b^{r_2}\|h\|_{L^\infty}}{\Gamma(1+r_1)\Gamma(1+r_2)}:=\ell.$$

Claim 3. *B maps bounded sets into equicontinuous sets of $\mathcal{P}C$.*

Let $w\in B(u)$ for some $u\in S$, where S is a bounded set of $\mathcal{P}C$, and let $(\tau_1,y_1),\ (\tau_2,y_2)\in J$, with $\tau_1<\tau_2$ and $y_1<y_2$. Then, there exists $g\in S_{G,u}$ such that for each $(x,y)\in J_0$, we have

$$\begin{aligned}|w(\tau_2,y_2)-w(\tau_1,y_1)|&\leq|\mu(\tau_1,y_1)-\mu(\tau_2,y_2)|\\&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{\tau_1}\int_0^{y_1}[(\tau_2-s)^{r_1-1}(y_2-t)^{r_2-1}-(\tau_1-s)^{r_1-1}(y_1-t)^{r_2-1}]\\&\times|g(s,t)|\,\mathrm{d}t\,\mathrm{d}s\\&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{\tau_1}^{\tau_2}\int_{y_1}^{y_2}(\tau_2-s)^{r_1-1}(y_2-t)^{r_2-1}|g(s,t)|\,\mathrm{d}t\,\mathrm{d}s\\&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{\tau_1}\int_{y_1}^{y_2}(\tau_2-s)^{r_1-1}(y_2-t)^{r_2-1}|g(s,t)|\,\mathrm{d}t\,\mathrm{d}s\\&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{\tau_1}^{\tau_2}\int_0^{y_1}(\tau_2-s)^{r_1-1}(y_2-t)^{r_2-1}|g(s,t)|\,\mathrm{d}t\,\mathrm{d}s.\end{aligned}$$

Thus, that for each $(x,y)\in J_0$, we get

$$\begin{aligned}|w(\tau_2,y_2)-w(\tau_1,y_1)|&\leq|\mu(\tau_1,y_1)-\mu(\tau_2,y_2)|\\&\leq|\mu(\tau_1,y_1)-\mu(\tau_2,y_2)|\\&\quad+\frac{\|h\|_{L^\infty}}{1+\Gamma(r_1)\Gamma(1+r_2)}[2y_2^{r_2}(\tau_2-\tau_1)^{r_1}+2\tau_2^{r_1}(y_2-y_1)^{r_2}\\&\quad+\tau_1^{r_1}y_1^{r_2}-\tau_2^{r_1}y_2^{r_2}-2(\tau_2-\tau_1)^{r_1}(y_2-y_1)^{r_2}].\end{aligned}$$

Again, for each $(x,y)\in J_k;\ k=1,\dots,m$, we have

$$\begin{aligned}&|w(\tau_2,y_2)-w(\tau_1,y_1)|\\&\leq|\mu(\tau_1,y_1)-\mu(\tau_2,y_2)|+\sum_{i=1}^{k}\left|\frac{I_i(u(x_i^-,y_1))}{f(x_i^+,y_1,u(x_i^+,y_1))}-\frac{I_k(u(x_k^-,y_2))}{f(x_k^+,y_2,u(x_i^+,y_2))}\right|\\&\quad+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\sum_{i=1}^{k}\int_{x_{i-1}}^{x_i}\int_0^{y_1}(x_i-s)^{r_1-1}[(y_2-t)^{r_2-1}-(y_1-t)^{r_2-1}]\\&\quad\times|g(s,t)|\,\mathrm{d}t\,\mathrm{d}s\\&\quad+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\sum_{i=1}^{k}\int_{x_{i-1}}^{x_k}\int_{y_1}^{y_2}(x_i-s)^{r_1-1}(y_2-t)^{r_2-1}|g(s,t)|\,\mathrm{d}t\,\mathrm{d}s\\&\quad+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{\tau_1}\int_0^{y_1}[(\tau_2-s)^{r_1-1}(y_2-t)^{r_2-1}-(\tau_1-s)^{r_1-1}(y_1-t)^{r_2-1}]\\&\quad\times|g(s,t)|\,\mathrm{d}t\,\mathrm{d}s\\&\quad+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{\tau_1}^{\tau_2}\int_{y_1}^{y_2}(\tau_2-s)^{r_1-1}(y_2-t)^{r_2-1}|g(s,t)|\,\mathrm{d}t\,\mathrm{d}s\end{aligned}$$

$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{\tau_1}\int_{y_1}^{y_2}(\tau_2-s)^{r_1-1}(y_2-t)^{r_2-1}|g(s,t)|\,\mathrm{d}t\,\mathrm{d}s$$
$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{\tau_1}^{\tau_2}\int_0^{y_1}(\tau_2-s)^{r_1-1}(y_2-t)^{r_2-1}|g(s,t)|\,\mathrm{d}t\,\mathrm{d}s.$$

Hence, for each $(x,y)\in J_k;\ k=1,\ldots,m$, we get

$$\begin{aligned}
&|w(\tau_2,y_2)-w(\tau_1,y_1)|\\
&\le|\mu(\tau_1,y_1)-\mu(\tau_2,y_2)|+\sum_{i=1}^k\left|\frac{I_i(u(x_i^-,y_1))}{f(x_i^+,y_1,u(x_i^+,y_1))}-\frac{I_i(u(x_i^-,y_2))}{f(x_i^+,y_2,u(x_i^+,y_2))}\right|\\
&\quad+\frac{\|h\|_{L^\infty}}{\Gamma(r_1)\Gamma(r_2)}\sum_{i=1}^k\int_{x_{i-1}}^{x_i}\int_0^{y_1}(x_i-s)^{r_1-1}[(y_2-t)^{r_2-1}-(y_1-t)^{r_2-1}]\,\mathrm{d}t\,\mathrm{d}s\\
&\quad+\frac{\|h\|_{L^\infty}}{\Gamma(r_1)\Gamma(r_2)}\sum_{i=1}^k\int_{x_{i-1}}^{x_i}\int_{y_1}^{y_2}(x_i-s)^{r_1-1}(y_2-t)^{r_2-1}\,\mathrm{d}t\,\mathrm{d}s\\
&\quad+\frac{\|h\|_{L^\infty}}{1+\Gamma(r_1)\Gamma(1+r_2)}[2y_2^{r_2}(\tau_2-\tau_1)^{r_1}+2\tau_2^{r_1}(y_2-y_1)^{r_2}\\
&\quad+\tau_1^{r_1}y_1^{r_2}-\tau_2^{r_1}y_2^{r_2}-2(\tau_2-\tau_1)^{r_1}(y_2-y_1)^{r_2}].
\end{aligned}$$

As $\tau_1\to\tau_2$ and $y_1\to y_2$, the right-hand side of the above inequality tends to zero. As a consequence of Claims 1 to 3 together with the Arzelá–Ascoli theorem, we can conclude that B is compact. Moreover,

$$M=\|B(\mathcal{P}C)\|\le\|\mu\|_\infty+2m\|\beta\|_\infty+\frac{2a^{r_1}b^{r_2}\|h\|_{L^\infty}}{\Gamma(r_1+1)\Gamma(r_2+1)},$$

and so, by assumption (4), we get

$$2M\|\alpha\|_\infty\le 2L<1.$$

Step 3. *B has a closed graph.*

Let $u_n\to u_*$, $h_n\in B(u_n)$ and $h_n\to h_*$. We need to show that $h_*\in B(u_*)$. $h_n\in B(u_n)$ means that there exists $g_n\in S_{F,u_n}$ such that

$$\begin{cases}
h_n(x,y)=f(x,y,u_n(x,y))\Big[\mu(x,y)\\
\quad+\displaystyle\int_0^x\int_0^y\frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}g_n(s,t)\,\mathrm{d}t\,\mathrm{d}s\Big];\ (x,y)\in J_0,\\[2ex]
h_n(x,y)=f(x,y,u_n(x,y))\Big[\mu(x,y)\\
\quad+\displaystyle\sum_{i=1}^k\left(\frac{I_i(u_n(x_i^-,y))}{f(x_i^+,y,u_n(x_i^+,y))}-\frac{I_i(u_n(x_i^-,0))}{f(x_i^+,0,u_n(x_i^+,0))}\right)\\
\quad+\displaystyle\frac{1}{\Gamma(r_1)\Gamma(r_2)}\sum_{i=1}^k\int_{x_{i-1}}^{x_i}\int_0^y(x_i-s)^{r_1-1}(y-t)^{r_2-1}g_n(s,t)\,\mathrm{d}t\,\mathrm{d}s\\
\quad+\displaystyle\int_{x_k}^x\int_0^y\frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}g_n(s,t)\,\mathrm{d}t\,\mathrm{d}s\Big];\ (x,y)\in J_k,\ k=1,\ldots,m.
\end{cases}$$

We must show that there exists $g_* \in S_{G,u_*}$ such that,

$$\begin{cases} h_*(x,y) = f(x,y,u_*(x,y))\Big[\mu(x,y) \\ +\displaystyle\int_0^x\int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} g_*(s,t)\,\mathrm{d}t\,\mathrm{d}s\Big];\ (x,y)\in J_0, \\ \\ h_*(x,y) = f(x,y,u_*(x,y))\Big[\mu(x,y) \\ \quad +\displaystyle\sum_{i=1}^k\left(\frac{I_i(u_*(x_i^-,y))}{f(x_i^+,y,u_*(x_i^+,y))} - \frac{I_i(u_*(x_i^-,0))}{f(x_i^+,0,u_*(x_i^+,0))}\right) \\ \quad +\displaystyle\frac{1}{\Gamma(r_1)\Gamma(r_2)}\sum_{i=1}^k\int_{x_{i-1}}^{x_i}\int_0^y (x_i-s)^{r_1-1}(y-t)^{r_2-1}g_*(s,t)\,\mathrm{d}t\,\mathrm{d}s \\ \quad +\displaystyle\int_{x_k}^x\int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} g_*(s,t)\,\mathrm{d}t\,\mathrm{d}s\Big];\ (x,y)\in J_k,\ k=1,\ldots,m. \end{cases}$$

Clearly, we have

$$\|(h_n-\mu)-(h_*-\mu)\|_{\mathcal{PC}} = \|h_n-h_*\|_{\mathcal{PC}} \to 0 \ as\ n\to\infty.$$

Thus, we have $(h_n-\mu)\to(h_*-\mu)$ as $n\to\infty$. Now, consider the continuous linear operator $\mathcal{L}: L^1(J)\to\mathcal{PC};\quad g\mapsto(\mathcal{L}g)(x,y)$, such that

$$\begin{cases} (\mathcal{L}g)(x,y) = f(x,y,u(x,y))\Big[\mu(x,y) \\ \quad +\displaystyle\int_0^x\int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} g(s,t)\,\mathrm{d}t\,\mathrm{d}s\Big];\ (x,y)\in J_0, \\ \\ (\mathcal{L}g)(x,y) = f(x,y,u(x,y))\Big[\mu(x,y) \\ \quad +\displaystyle\sum_{i=1}^k\left(\frac{I_i(u(x_i^-,y))}{f(x_i^+,y,u(x_i^+,y))} - \frac{I_i(u(x_i^-,0))}{f(x_i^+,0,u(x_i^+,0))}\right) \\ \quad +\displaystyle\frac{1}{\Gamma(r_1)\Gamma(r_2)}\sum_{i=1}^k\int_{x_{i-1}}^{x_i}\int_0^y (x_i-s)^{r_1-1}(y-t)^{r_2-1}g(s,t)\,\mathrm{d}t\,\mathrm{d}s \\ \quad +\displaystyle\int_{x_k}^x\int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} g(s,t)\,\mathrm{d}t\,\mathrm{d}s\Big];\ (x,y)\in J_k,\ k=1,\ldots,m. \end{cases}$$

From Lemma 2.6, it follows that $\mathcal{L}\circ S_{G,u}$ is a closed graph operator. Moreover, we have

$$\Big(h_n(x,y)-\mu(x,y)\Big)\in\mathcal{L}(S_{G,u_n}).$$

Since $u_n\to u_*$, we have that $(h_*(x,y)-\mu(x,y))\in\mathcal{L}(S_{G,u_*})$. Therefore, there exists $g_*\in S_{G,u_*}$ such that

$$\begin{cases} h_*(x,y) = f(x,y,u_*(x,y))\Big[\mu(x,y) \\ \quad + \displaystyle\int_0^x\int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} g_*(s,t)\,\mathrm{d}t\,\mathrm{d}s\Big];\ (x,y)\in J_0, \\ \\ h_*(x,y) = f(x,y,u_*(x,y))\Big[\mu(x,y) \\ \quad + \displaystyle\sum_{i=1}^k\left(\frac{I_i(u_*(x_i^-,y))}{f(x_i^+,y,u_*(x_i^+,y))} - \frac{I_i(u_*(x_i^-,0))}{f(x_i^+,0,u_*(x_i^+,0))}\right) \\ \quad + \displaystyle\frac{1}{\Gamma(r_1)\Gamma(r_2)}\sum_{i=1}^k\int_{x_{i-1}}^{x_i}\int_0^y (x_i-s)^{r_1-1}(y-t)^{r_2-1}g_*(s,t)\,\mathrm{d}t\,\mathrm{d}s \\ \quad + \displaystyle\int_{x_k}^x\int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} g_*(s,t)\,\mathrm{d}t\,\mathrm{d}s\Big];\ (x,y)\in J_k,\ k=1,\dots,m. \end{cases}$$

Thus, the multivalued operator B has closed graph and consequently it is (u.s.c.) on $\mathcal{PC}$ in view of compactness of B.

Step 4. *the conclusion* (ii) *of Theorem* 2.22 *is not possible.*

Let $u\in\mathcal{PC}$ be any solution to (1), such that for any $\lambda\in(0,1)$ we have $u\in\lambda N(u)$. Then, there exists $g\in S_{G,u}$, such that

$$\begin{cases} u(x,y) = \lambda f(x,y,u(x,y))\Big[\mu(x,y) \\ \quad + \displaystyle\int_0^x\int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} g(s,t)\,\mathrm{d}t\,\mathrm{d}s\Big];\ (x,y)\in J_0, \\ \\ u(x,y) = \lambda f(x,y,u(x,y))\Big[\mu(x,y) \\ \quad + \displaystyle\sum_{i=1}^k\left(\frac{I_i(u(x_i^-,y))}{f(x_i^+,y,u(x_i^+,y))} - \frac{I_i(u(x_i^-,0))}{f(x_i^+,0,u(x_i^+,0))}\right) \\ \quad + \displaystyle\frac{1}{\Gamma(r_1)\Gamma(r_2)}\sum_{i=1}^k\int_{x_{i-1}}^{x_i}\int_0^y (x_i-s)^{r_1-1}(y-t)^{r_2-1}g(s,t)\,\mathrm{d}t\,\mathrm{d}s \\ \quad + \displaystyle\int_{x_k}^x\int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} g(s,t)\,\mathrm{d}t\,\mathrm{d}s\Big];\ (x,y)\in J_k,\ k=1,\dots,m. \end{cases}$$

Therefore,

$$\begin{aligned} |u(x,y)| &\le |f(x,y,u(x,y))|\left(|\mu(x,y)| + 2m\|\beta\|_\infty + \frac{2a^{r_1}b^{r_2}\|h\|_{L^\infty}}{\Gamma(1+r_1)\Gamma(1+r_2)}\right) \\ &\le \big[|f(x,y,u(x,y)) - f(x,y,0)| + |f(x,y,0)|\big] \\ &\quad \times \left(|\mu(x,y)| + 2m\|\beta\|_\infty + \frac{2a^{r_1}b^{r_2}\|h\|_{L^\infty}}{\Gamma(1+r_1)\Gamma(1+r_2)}\right) \\ &\le \big[\|\alpha\|_\infty |u(x,y)| + f^*\big]\left(\|\mu\|_\infty + 2m\|\beta\|_\infty + \frac{2a^{r_1}b^{r_2}\|h\|_{L^\infty}}{\Gamma(1+r_1)\Gamma(1+r_2)}\right), \end{aligned}$$

where $f^* = \sup\{|f(x,y,0)| : (x,y) \in J\}$. Hence,

$$\|u\|_{PC} \leq \frac{f^* L}{\|\alpha\|_\infty (1-L)} := M^*.$$

Thus, the conclusion (ii) of Theorem 2.22 does not hold for $\lambda^* = \frac{1}{\lambda} > 1$. Consequently, the problem (1) has a solution on J.

Step 5. *The Ulam–Hyers–Rassias stability.*

Now, we prove the generalized Ulam–Hyers–Rassias stability of the multivalued operator N. Let $u \in \mathcal{PC}$ be a solution of the inequality $|u - N(u)| \leq \Phi(x,y)$ on J, and let v be a solution of the fixed point inclusion $u \in N(u)$. Then, there exists $g_v \in S_{G,v}$, such that

$$\begin{cases} v(x,y) = f(x,y,v(x,y)) \Big[\mu(x,y) \\ \quad + \displaystyle\int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} g_v(s,t)\,\mathrm{d}t\,\mathrm{d}s\Big]; \quad \text{if } (x,y) \in J_0, \\ \\ v(x,y) = f(x,y,v(x,y)) \Big[\mu(x,y) \\ \quad + \displaystyle\sum_{i=1}^k \left(\frac{I_i(v(x_i^-,y))}{f(x_i^+,y,v(x_i^+,y))} - \frac{I_i(v(x_i^-,0))}{f(x_i^+,0,v(x_i^+,0))} \right) \\ \quad + \displaystyle\frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{i=1}^k \int_{x_{i-1}}^{x_i} \int_0^y (x_i - s)^{r_1-1}(y-t)^{r_2-1} g_v(s,t)\,\mathrm{d}t\,\mathrm{d}s \\ \quad + \displaystyle\int_{x_k}^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} g_v(s,t)\,\mathrm{d}t\,\mathrm{d}s\Big]; \text{ if } (x,y) \in J_k,\ k=1,\ldots,m. \end{cases}$$

Then, for each $(x,y) \in J$, it follows that

$$\begin{aligned} |u(x,y) - v(x,y)| &\leq H_d(u(x,y), (Nv)(x,y)) \\ &\leq H_d(u(x,y) - (Nu)(x,y)) + H_d((Nu)(x,y) - (Nv)(x,y)) \\ &\leq \Phi(x,y) + H_d((Nu)(x,y) - (Nv)(x,y)). \end{aligned}$$

Thus, there exists $g \in S_{G,u}$, such that for each $(x,y) \in J_0$, we have

$$\begin{aligned} |u(x,y) - v(x,y)| &\leq \Phi(x,y) + |f(x,y,u(x,y)) - f(x,y,v(x,y))| \\ &\quad \times \left| \mu(x,y) + \int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} g(s,t)\,\mathrm{d}t\,\mathrm{d}s \right| \\ &\quad + |f(x,y,v(x,y))| \\ &\quad \times \int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} |g(s,t) - g_v(s,t)|\,\mathrm{d}t\,\mathrm{d}s \\ &\leq \Phi(x,y) + \|\alpha\|_\infty |u(x,y) - v(x,y)| \\ &\quad \times \left[\|\mu\|_\infty + \frac{a^{r_1} b^{r_2} \|h\|_{L^\infty}}{\Gamma(1+r_1)\Gamma(1+r_2)} \right] \\ &\quad + \frac{2a^{r_1} b^{r_2} \|h\|_{L^\infty}}{\Gamma(1+r_1)\Gamma(1+r_2)} \lambda_\Phi \Phi(x,y) \end{aligned}$$

$$\leq \Phi(x,y) + L|u(x,y) - v(x,y)| + \frac{2L}{\|\alpha\|_\infty}\lambda_\Phi \Phi(x,y),$$

and for each $(x,y) \in J_k;\ k = 1, \ldots, m$, we get

$$\begin{aligned}
|u(x,y) - v(x,y)| &\leq \Phi(x,y) + |f(x,y,u(x,y)) - f(x,y,v(x,y))| \\
&\times \left|\mu(x,y) + \sum_{i=1}^{k}\left(\frac{I_i(u(x_i^-,y))}{f(x_i^+,y,u(x_i^+,y))} - \frac{I_i(u(x_i^-,0))}{f(x_i^+,0,u(x_i^+,0))}\right)\right. \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\sum_{i=1}^{k}\int_{x_{i-1}}^{x_i}\int_0^y (x_i - s)^{r_1-1}(y-t)^{r_2-1}g(s,t)\,\mathrm{d}t\,\mathrm{d}s \\
&+ \left.\int_{x_k}^{x}\int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}g(s,t)\,\mathrm{d}t\,\mathrm{d}s\right| \\
&+ |f(x,y,v(x,y))| \\
&\times \left[\sum_{i=1}^{k}\left|\frac{I_i(u(x_i^-,y))}{f(x_i^+,y,u(x_i^+,y))} - \frac{I_i(v(x_i^-,y))}{f(x_i^+,y,v(x_i^+,y))}\right|\right. \\
&+ \sum_{i=1}^{k}\left|\frac{I_i(u(x_i^-,0))}{f(x_i^+,0,u(x_i^+,0))} - \frac{I_i(v(x_i^-,0))}{f(x_i^+,0,v(x_i^+,0))}\right| \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\sum_{i=1}^{k}\int_{x_{i-1}}^{x_i}\int_0^y (x_i - s)^{r_1-1}(y-t)^{r_2-1}|g(s,t) - g_v(s,t)|\,\mathrm{d}t\,\mathrm{d}s \\
&+ \left.\int_{x_k}^{x}\int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}|g(s,t) - g_v(s,t)|\,\mathrm{d}t\,\mathrm{d}s\right] \\
&\leq \Phi(x,y) + L|u(x,y) - v(x,y)| + \frac{2L}{\|\alpha\|_\infty}\lambda_\Phi \Phi(x,y).
\end{aligned}$$

Hence, by (4) for each $(x,y) \in J_k;\ k = 0, \ldots, m$, we get

$$\begin{aligned}
|u(x,y) - v(x,y)| &\leq \frac{1}{1-L}\left(1 + \frac{2L\lambda_\Phi}{\|\alpha\|_\infty}\right)\Phi(x,y) \\
&:= c_{N,\Phi}\Phi(x,y).
\end{aligned}$$

Consequently, the fixed point inclusion $u \in N(u)$ is generalized Ulam–Hyers–Rassias stable.

4. More Existence and Ulam Stability Results

Now, we present (without proof) some existence and Ulam stability results to the following problem

$$\begin{cases} {}^cD^r_{\theta_k}\left(\frac{u(x,y)}{f(x,y,u(x,y))}\right) \in G(x,y,u(x,y));\ (x,y) \in J := [0,a] \times [0,b], \\ u(x,0) = \varphi(x);\ x \in [0,a], \quad u(0,y) = \psi(y);\ y \in [0,b], \end{cases} \tag{6}$$

where $a, b > 0,\ \theta = (0,0),\ {}^cD^r_\theta$ is the Caputo's fractional derivative of order $r = (r_1, r_2) \in (0,1] \times (0,1],\ f : J \times \mathbb{R} \to \mathbb{R}^*$ is a given continuous function, $G : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ and $\varphi \in AC([0,a]),\ \psi \in AC([0,b])$ with $\varphi(0) = \psi(0)$.

Remark 4.1. Solutions of the problem (6) are solutions of the fixed point inclusion $u \in \overline{N}(u)$ where $\overline{N} : \mathcal{C} \to \mathcal{P}(\mathcal{C})$ is the operator defined by

$$(\overline{N}u)(x,y) = \left\{ h \in \mathcal{C} : \begin{cases} h(x,y) = f(x,y,u(x,y))[\mu(x,y) \\ + \int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} g(s,t)\,\mathrm{d}t\,\mathrm{d}s\Big] ;\ g \in S_{G,u} \end{cases} \right\}.$$

Theorem 4.2. *Assume that hypotheses* (H_1) *and* (H_2) *hold. If*

$$\|\alpha\|_\infty \left[\|\mu\|_\infty + \frac{a^{r_1} b^{r_2} \|h\|_{L^\infty}}{\Gamma(1+r_1)\Gamma(1+r_2)} \right] < \frac{1}{2}, \tag{7}$$

then the problem (6) *has at least one solution on* J. *Moreover, if the hypothesis* (H_4) *holds, then the fixed point inclusion* $u \in \overline{N}(u)$ *is generalized Ulam–Hyers–Rassias stable.*

5. An Example

Consider the following partial functional discontinuous differential inclusions of the form

$$\begin{cases} {}^c D^r_{\theta_k} \left(\frac{u(x,y)}{f(x,y,u(x,y))} \right) \in G(x,y,u(x,y)), (x,y) \in [0,1] \times [0,1],\ x \neq \frac{1}{2}, k=0,1, \\ u\left(\frac{1}{2}^+, y\right) = u\left(\frac{1}{2}^-, y\right) + I_1\left(u\left(\frac{1}{2}^-, y\right)\right), \quad y \in [0,1], \\ u(x,0) = \varphi(x), \quad u(0,y) = \psi(y); \quad x,y \in [0,1], \end{cases} \tag{8}$$

where $\theta_1 = (\frac{1}{2}, 0)$, $f : [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}^*$, $G : [0,1] \times [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ and $I_1 : \mathbb{R} \to \mathbb{R}$ are defined by

$$f(x,y,u) = \frac{1}{10e^{x+y+10}(1+|u|)}, \quad G(x,y,u) = \begin{cases} \left\{ \frac{1}{e^{x+y+8}(1+u^2)} \right\}; \text{ if } \quad u < 0, \\ \left[0, \frac{1}{e^{x+y+8}} \right]; \text{ if } \quad u = 0, \\ \{0\}; \text{ if } \quad u > 0, \end{cases}$$

and

$$I_1(u) = \frac{(8+e^{-10})^2}{512e^{10}(1+|u|)^2}.$$

The functions $\varphi, \psi : [0,1] \to \mathbb{R}$ are defined by

$$\varphi(x) = \begin{cases} \frac{x^2}{2} e^{-10}; & \text{if } x \in [0, \frac{1}{2}], \\ x^2 e^{-10}; & \text{if } x \in (\frac{1}{2}, 1], \end{cases}$$

and

$$\psi(y) = y e^{-10}, \quad \text{for all } y \in [0,1].$$

We can see that the solutions of the problem (8) are solutions of the fixed point inclusion $u \in A(u)$ where $A : PC([0,1] \times [0,1], \mathbb{R}) \to \mathcal{P}(PC([0,1] \times [0,1], \mathbb{R}))$ is the multivalued operator defined by

$$(Au)(x,y) = \left\{ h \in PC([0,1]\times[0,1],\mathbb{R}) : \begin{cases} h(x,y) = f(x,y,u(x,y))[\mu(x,y) \\ \quad + I_\theta^r G(x,y,u(x,y))]; \\ (x,y) \in J_0 := [0,\frac{1}{2}] \times [0,1], \\ \\ h(x,y) = f(x,y,u(x,y))[\mu(x,y) \\ \quad + \frac{(I_1(u(\frac{1}{2}^-,y))}{f(\frac{1}{2}^+,y,u(\frac{1}{2}^+,y))} - \frac{I_1(u(\frac{1}{2}^-,0)))}{f(\frac{1}{2}^+,0,u(\frac{1}{2}^+,0))} \\ \quad + I_\theta^r G(\frac{1}{2},y,u(\frac{1}{2},y)) + I_{\theta_1}^r G(x,y,u(x,y))]; \\ (x,y) \in J_1 := (\frac{1}{2},1] \times [0,1]. \end{cases} \right\}.$$

We show that the functions φ, ψ, f, G and I_1 satisfy all the hypotheses of Theorem 3.2. Clearly, the function f is continuous and satisfies (H_1) with $\alpha(x,y) = \frac{1}{10e^{x+y+10}}$. Then, $\|\alpha\|_\infty = \frac{1}{10e^{10}}$. Also, the multifunction G satisfies (H_2) with $h(x,y) = \frac{1}{e^{x+y+8}}$, and so $\|h\|_{L^\infty} = 1/e^8$. The condition (H_3) holds with $\beta(x,y) = \frac{81e^{x+y}}{512}$. This gives $\|\beta\|_\infty = \frac{81e^2}{512}$. A simple computation gives $\|\mu\|_\infty < 4e$. The condition (4) holds. Indeed, $\Gamma(1+r_i) > \frac{1}{2}$; $i = 1,2$, and a simple computation shows that

$$\begin{aligned} 2L &= 2\|\alpha\|_\infty \left[\|\mu\|_\infty + 2m\|\beta\|_\infty + \frac{2a^{r_1}b^{r_2}\|h\|_{L^\infty}}{\Gamma(r_1+1)\Gamma(r_2+1)} \right] \\ &< \frac{1}{5e^{10}} \left[4e + \frac{81e^2}{256} + \frac{8}{e^8} \right] < 1. \end{aligned}$$

Finally, we can see that the hypothesis (H_4) is satisfied with $\Phi(x,y) = \dfrac{1}{e^{x+y+8}}$ and $\lambda_\Phi = 1$. Consequently, Theorem 3.2 implies that the problem (8) has a solution defined on $[0,1]\times[0,1]$, and the fixed point inclusion $u \in A(u)$ is generalized Ulam–Hyers–Rassias stable.

References

[1] Abbas, S., Agarwal, R.P., Benchohra, M.: Impulsive discontinuous partial hyperbolic differential equations of fractional order on Banach algebras. Electron. J. Differ. Equ. **2010**(91), 17

[2] Abbas, S., Benchohra, M.: Partial hyperbolic differential equations with finite delay involving the Caputo fractional derivative. Commun. Math. Anal. **7**, 62–72 (2009)

[3] Abbas, S., Benchohra, M.: Impulsive partial hyperbolic functional differential equations of fractional order with state-dependent delay. Frac. Calc. Appl. Anal. **13**(3), 225–244 (2010)

[4] Abbas, S., Benchohra, M.: Upper and lower solutions method for impulsive partial hyperbolic differential equations with fractional order. Nonlinear Anal. Hybrid Syst. **4**, 604–613 (2010)

[5] Abbas, S., Benchohra, M.: Impulsive partial hyperbolic differential inclusions of fractional order. Demonstratio Math. XLIII **4**, 775–797 (2010)

[6] Abbas, S., Benchohra, M., Gorniewicz, L.: Fractional order impulsive partial hyperbolic differential inclusions with variable times. Discuss. Math. Differ. Incl. Control Optim. **31**(1), 91–114 (2011)

[7] Abbas, S., Benchohra, M., N'Guérékata, G.M.: Topics in Fractional Differential Equations. Springer, New York (2012)

[8] Abbas, S., Benchohra, M., N'Guérékata, G.M., Slimani, B.A.: Darboux problem for fractional order discontinuous hyperbolic partial differential equations in Banach algebras. Complex Var. Elliptic Equ. **57**(2–4), 337–350 (2012)

[9] Bota-Boriceanu, M.F., Petrusel, A.: Ulam–Hyers stability for operatorial equations and inclusions. Anal. Univ. I. Cuza Iasi **57**, 65–74 (2011)

[10] Castro, L.P., Ramos, A.: Hyers–Ulam–Rassias stability for a class of Volterra integral equations. Banach J. Math. Anal. **3**, 36–43 (2009)

[11] Dhage, B.C.: Existence results for neutral functional differential inclusions in Banach algebras. Nonlinear Anal. **64**, 1290–1306 (2006)

[12] Hilfer, R.: Applications of Fractional Calculus in Physics. World Scientific, Singapore (2000)

[13] Hyers, D.H.: On the stability of the linear functional equation. Proc. Natl. Acad. Sci. **27**, 222–224 (1941)

[14] Hyers, D.H., Isac, G., Rassias, T.H.M.: Stability of Functional Equations in Several Variables. Birkhauser, Boston (1998)

[15] Jung, S.-M.: Hyers–Ulam–Rassias Stability of Functional Equations in Mathematical Analysis. Hadronic Press, Palm Harbor (2001)

[16] Jung, S.-M.: Hyers–Ulam–Rassias Stability of Functional Equations in Nonlinear Analysis. Springer, New York (2011)

[17] Jung, S.-M.: A fixed point approach to the stability of a Volterra integral equation. Fixed Point Theory Appl. **2007**, 1 (2007) (article ID 57064)

[18] Hu, S.H., Papageorgiou, N.: Handbook of Multivalued Analysis, Theory I. Kluwer, Dordrecht (1997)

[19] Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam (2006)

[20] Lakshmikantham, V., Bainov, D.D., Simeonov, P.S.: Theory of Impulsive Differential Equations. World Scientific, Singapore (1989)

[21] Lasota, A., Opial, Z.: An application of the KakutaniKy Fan theorem in the theory of ordinary differential equations. Bull. Acad. Pol. Sci. Ser. Sci. Math. Astronom. Phys. **13**, 781–786 (1965)

[22] Li, X., Wang, J.: Ulam–Hyers–Rassias stability of semilinear differential equations with impulses. Electron. J. Differ. Equ. **2013**(172), 8

[23] Miller, K.S., Ross, B.: An Introduction to the Fractional Calculus and Differential Equations. Wiley, New York (1993)

[24] Petru, T.P., Bota, M.-F.: Ulam–Hyers stabillity of operational inclusions in complete gauge spaces. Fixed Point Theory **13**, 641–650 (2012)

[25] Petru, T.P., Petrusel, A., Yao, J.-C.: Ulam0-Hyers stability for operatorial equations and inclusions via nonself operators. Taiwan. J. Math. **15**, 2169–2193 (2011)

[26] Podlubny, I.: Fractional Differential Equations. Academic Press, San Diego (1999)

[27] Rassias, T.H.M.: On the stability of linear mappings in Banach spaces. Proc. Am. Math. Soc. **72**, 297–300 (1978)

[28] Rus, I.A.: Ulam stability of ordinary differential equations. Stud. Univ. Babes-Bolyai, Math. LIV **4**, 125–133 (2009)

[29] Rus, I.A.: Remarks on Ulam stability of the operatorial equations. Fixed Point Theory **10**, 305–320 (2009)

[30] Samoilenko, A.M., Perestyuk, N.A.: Impulsive Differential Equations. World Scientific, Singapore (1995)

[31] Ulam, S.M.: A Collection of Mathematical Problems. Interscience Publishers, New York (1968)

[32] Vityuk, A.N., Golushkov, A.V.: Existence of solutions of systems of partial differential equations of fractional order. Nonlinear Oscil. **7**(3), 318–325 (2004)

[33] Vityuk, A.N., Mykhailenko, A.V.: The Darboux problem for an implicit fractional-order differential equation. J. Math. Sci. **175**(4), 391–401 (2011)

[34] Wang, J., Feckan, M., Zhou, Y.: Ulam's type stability of impulsive ordinary differential equations. J. Math. Anal. Appl. **395**, 258–264 (2012)

[35] Wang, J., Lv, L., Zhou, Y.: Ulam stability and data dependence for fractional differential equations with Caputo derivative, Electron. J. Qual. Theory Differ. Equ. **2011**(63), 10

[36] Wang, J., Lv, L., Zhou, Y.: New concepts and results in stability of fractional differential equations. Commun. Nonlinear Sci. Numer. Simul. **17**, 2530–2538 (2012)

[37] Wang, J., Zhou, Y., Feckan, M.: Nonlinear impulsive problems for fractional differential equations and Ulam stability. Comput. Math. Appl. **64**, 3389–3405 (2012)

[38] Wei, W., Li, X., Li, X.: New stability results for fractional integral equation. Comput. Math. Appl. **64**, 3468–3476 (2012)

Saïd Abbas
10–2320 de Salaberry
Montréal, QC H3M 1K9
Canada
e-mail: abbasmsaid@yahoo.fr

Mouffak Benchohra
Laboratoire de Mathématiques
Université de Sidi Bel-Abbès
B.P. 89, 22000 Sidi Bel-Abbès
Algeria
e-mail: benchohra@yahoo.com; benchohra@univ-sba.dz

and

Department of Mathematics
Faculty of Science
King Abdulaziz University
P.O. Box 80203 Jeddah 21589,
Saudi Arabia

Received: September 20, 2013.
Revised: July 15, 2014.
Accepted: September 8, 2014.