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Approximations of a Complex Brownian Motion by Processes Constructed from a Lévy Process

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Abstract. In this paper, we show an approximation in law of the complex Brownian motion by processes constructed from a stochastic process with independent increments. We give sufficient conditions to the characteristic function of the process with independent increments that ensure the existence of such an approximation. We apply these results to Lévy processes. Finally we extend these results to the m-dimensional complex Brownian motion.

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1. Introduction and Main Result

The purpose of this paper is to research a weak approximation of a complex Brownian motion. The most typical processes taken as approximations to Gaussian processes are usually based on Donsker approximations (the functional central limit theorem) or on Kac-Stroock type approximations. In this paper, we will deal with this last type of approximations.

Kac [7] described the solution of the telegrapher's equation in terms of a Poisson process. Later, Stroock [9] showed the weak convergence of this solution to a Brownian motion. More precisely, given $\{N_t, t \geq 0\}$ a standard Poisson process, the laws of the processes x_ε

$$\left\{ x_\varepsilon(t) = \varepsilon \int_0^{\frac{t}{\varepsilon^2}} (-1)^{N_s} ds, \quad t \in [0,T] \right\}$$

converge weakly towards the law of a standard Brownian motion in the space of continuous functions on $[0,T]$.

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These results have been extended to obtain approximations of other processes as, among others: m-dimensional Brownian motion [5], stochastic partial differential equations driven by Gaussian white noise [2], fractional stochastic differential equations [4], multiple Wiener integrals [3] or complex Brownian motion [1].

Although all these cases are built beginning with a Poisson process, a detailed study of the proofs shows that the authors use only some properties of the Poisson process that can be found in a bigger class of processes as Lévy processes. Actually, we will deal with approximations of the complex Brownian motion built from a unique stochastic process with independent increments. Let us recall that $\{B_t,\, t \in [0,T]\}$ is a complex Brownian motion if it takes values on $\mathbb{C}$ and its real part and its imaginary part are two independent standard Brownian motions.

We consider the processes

$$\left\{ x_\varepsilon^\theta(t) = c(\theta)\varepsilon \int_0^{\frac{2t}{\varepsilon^2}} \mathrm{e}^{i\theta X_s} ds, \quad t \in [0,T] \right\}, \tag{1}$$

where $\{X_t, t \geq 0\}$ is a stochastic process with independent increments and $c(\theta)$ is a constant, depending on θ, that we will determine later. Let us recall that our approximations can be written as

$$x_\varepsilon^\theta(t) = \varepsilon c(\theta) \int_0^{\frac{2t}{\varepsilon^2}} \cos(\theta X_s)\mathrm{d}s + i\varepsilon c(\theta) \int_0^{\frac{2t}{\varepsilon^2}} \sin(\theta X_s)\mathrm{d}s.$$

When X is a Poisson process in [1] it has been proved that for $\theta \neq 0$ and $\theta \neq \pi$ the limit is a complex Brownian motion. Furthermore, for $\theta = \pi$ we obtain an alternative version of Stroock's results since

$$\mathrm{e}^{i\theta X_s} = (-1)^{X_s}.$$

The aim of this paper is to study the weak limits of the processes (1) when ε tends to zero, showing that Lévy processes can be used to approximate a complex Brownian motion.

Having this type of approximation ensures the robustness of the process limit, in our case the Brownian motion, to be used as a model in practical situations. In addition, we can obtain expressions that can be useful for simulation.

In Sect. 2, we recall some basic facts about Lévy processes and we present the classical methodology to obtain weak approximations of Gaussian processes. Section 3 is devoted to give the main results of the paper. First we give some conditions on the characteristic functions of the process X that ensures the weak convergence of (1) to a complex Brownian motion. Then, we discuss when the characteristic functions of Lévy processes satisfy such conditions. Finally, in Sect. 4, we study the m-dimensional case, proving that we can obtain a m-dimensional complex Brownian motion from a unique Lévy process.

Throughout the paper K denotes positive constants, not depending on ε, which may change from one expression to another one. The real part and the imaginary part of a complex number will be denoted by $Re[\cdot]$ and $Im[\cdot]$.

2. Preliminaries

2.1. Lévy Processes

Set $\{X_s,\, s \geq 0\}$ a Lévy process, that is, X has stationary and independent increments, is continuous in probability, is càdlàg and $X_0 = 0$, and it is defined on a complete probability space $(\Omega, \mathcal{F}, P)$. Some examples of Lévy processes are, among others, Brownian motion, Poisson Process, jump-diffusion processes, stable processes or subordinators.

Consider $\phi_{X_t}(u)$ its characteristic function. Remember that it can be written as

$$\phi_{X_t}(u) = E\left(\mathrm{e}^{iuX_t}\right) = \mathrm{e}^{-t\psi_X(u)},$$

where $\psi_X(u)$ is called the Lévy exponent of X.

It is well known that the Lévy exponent can be expressed, by the Lévy-Khintchine formula, as

$$\psi_X(u) = -aiu + \frac{1}{2}\sigma^2 u^2 - \int_{\mathbb{R}\setminus\{0\}} (\mathrm{e}^{iux} - 1 - iuxI_{|x|<1})\eta(\mathrm{d}x), \tag{2}$$

where $a \in \mathbb{R}$, $\sigma \geq 0$ and η is a Lévy measure, that is, $\int_{\mathbb{R}\setminus\{0\}} \min\{x^2, 1\}\eta(\mathrm{d}x) < \infty$.

For notation and simplicity along the paper we set

$$a(u) := Re[\psi_X(u)] = \frac{1}{2}\sigma^2 u^2 - \int_{\mathbb{R}\setminus\{0\}} (\cos(ux) - 1)\eta(\mathrm{d}x), \tag{3}$$

and

$$b(u) = Im[\psi_X(u)] = -au - \int_{\mathbb{R}\setminus\{0\}} (\sin(ux) - uxI_{|x|<1})\eta(\mathrm{d}x). \tag{4}$$

Notice that $a(-u) = a(u)$ and $b(-u) = -b(u)$.

We refer the reader to [8] for more information about Lévy processes.

2.2. Weak Approximations of the Complex Brownian Motion

For any $\varepsilon > 0$, set $\{x_\varepsilon(t), t \in [0, T]\}$ a complex stochastic process with $x_\varepsilon(0) = 0$. Consider P_ε the image law of x_ε in the Banach space $\mathcal{C}([0, T], \mathbb{C})$ of continuous functions on $[0, T]$.

To prove that P_ε converges weakly as ε tends to zero towards the law on $\mathcal{C}([0, T], \mathbb{C})$ of a complex Brownian motion we have to check that the family P_ε is tight and that the law of all possible weak limits of P_ε is the law of two independent standard Brownian motions.

The tightness of the family P_ε can be proved checking that the laws corresponding to the real part and the imaginary part of the processes x_ε are tight. Using the Billingsley criterium (see Theorem 12.3 of [6]) and that our processes are null on the origin, it suffices to prove that there exists a constant K such that for any $s < t$

$$\sup_{\varepsilon} \left(E((Re[x_\varepsilon(t) - x_\varepsilon(s)])^4) + E((Im[x_\varepsilon(t) - x_\varepsilon(s)])^4)\right) \leq K(t - s)^2. \tag{5}$$

The second part of the proof consists in the identification of the limit law. Let $\{P_{\varepsilon_n}\}_n$ be a subsequence of $\{P_\varepsilon\}_\varepsilon$ (that we will also denote by

$\{P_\varepsilon\}$) weakly convergent to some probability P. We want to see that the canonical process $X = \{X_t(x) =: x(t)\}$ is a complex Brownian motion under the probability P, that is, the real part and the imaginary part of this process are two independent Brownian motions. Using Paul Lévy's theorem it suffices to prove that under P, the real part and the imaginary part of the canonical process are both martingales with respect to the natural filtration, $\{\mathcal{F}_t\}$, with quadratic variations $< Re[X], Re[X] >_t = t$, $< Im[X], Im[X] >_t = t$ and covariation $< Re[X], Im[X] >_t = 0$.

To see that under P the real part and the imaginary part of the canonical process X are martingales with respect to its natural filtration $\{\mathcal{F}_t\}$, we have to prove that for any $s_1 \le s_2 \le \cdots \le s_n \le s < t$ and for any bounded continuous function $\varphi : \mathbb{C}^n \longrightarrow \mathbb{R}$,

$$E_P\big[\varphi(X_{s_1}, ..., X_{s_n})(Re[X_t] - Re[X_s])\big] = 0,$$
$$E_P\big[\varphi(X_{s_1}, ..., X_{s_n})(Im[X_t] - Im[X_s])\big] = 0.$$

Since $P_\varepsilon \stackrel{w}{\Rightarrow} P$, and taking into account (5), we have that,

$$\lim_{\varepsilon\to 0} E_{P_\varepsilon}\big[\varphi(x(s_1), ..., x(s_n))(Re[x(t)] - Re[x(s)])\big]$$
$$= E_P\big[\varphi(x(s_1), ..., x(s_n))(Re[x(t)] - Re[x(s)])\big],$$

and we get the same with the imaginary part. So, it suffices to see that

$$\lim_{\varepsilon\to 0} E\big(\varphi(x_\varepsilon(s_1), ..., x_\varepsilon(s_n))\big)\big(Re[x_\varepsilon(t)] - Re[x_\varepsilon(s)]\big)\big) = 0, \tag{6}$$
$$\lim_{\varepsilon\to 0} E\big(\varphi(x_\varepsilon(s_1), ..., x_\varepsilon(s_n))\big)\big(Im[x_\varepsilon(t)] - Im[x_\varepsilon(s)]\big)\big) = 0. \tag{7}$$

To deal with the quadratic variation, it is enough to prove that for any $s_1 \le s_2 \le \cdots \le s_n \le s < t$ and for any bounded continuous function $\varphi : \mathbb{C}^n \longrightarrow \mathbb{R}$,

$$\lim_{\varepsilon\to 0} E\big[\varphi(x_\varepsilon(s_1)..., x_\varepsilon(s_n))\big((Re[x_\varepsilon(t)] - Re[x_\varepsilon(s)])^2 - (t-s)\big)\big] = 0, \tag{8}$$
$$\lim_{\varepsilon\to 0} E\big[\varphi(x_\varepsilon(s_1)..., x_\varepsilon(s_n))\big((Im[x_\varepsilon(t)] - Im[x_\varepsilon(s)])^2 - (t-s)\big)\big] = 0. \tag{9}$$

Finally to prove that $< Re[X], Im[X] >_t = 0$, it suffices to check that for any $s_1 \le s_2 \le \cdots \le s_n \le s < t$ and for any bounded continuous function $\varphi : \mathbb{C}^n \longrightarrow \mathbb{R}$,

$$\lim_{\varepsilon\to 0} E\big[\varphi(x_\varepsilon(s_1)..., x_\varepsilon(s_n))(Re[x_\varepsilon(t)] - Re[x_\varepsilon(s)])(Im[x_\varepsilon(t)] - Im[x_\varepsilon(s)])\big] = 0. \tag{10}$$

3. Approximations to a Complex Brownian Motion

We built our approximations from a stochastic process X with independent increments. We will deal with X using the study of its characteristic function ϕ_X. Let us introduce a set of useful hypothesis (H^θ) for the characteristic function ϕ_X of a process X:

$(H^\theta 1)$ for any $0 \le s < t$ there exists a constant $K(\theta)$ such that for any $\varepsilon > 0$,

$$\varepsilon^2 \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \int_{\frac{2s}{\varepsilon^2}}^{y} |\phi_{X_y - X_x}(\theta)| \mathrm{d}x \mathrm{d}y \le K(\theta)(t-s),$$

$(H^\theta 2)$ for any $0 \le s < t$ there exists a constant $c(\theta)$ such that

$$\lim_{\varepsilon \to 0} \varepsilon^2 c(\theta)^2 \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \int_{\frac{2s}{\varepsilon^2}}^{x} [\phi_{X_x - X_y}(\theta) + \phi_{X_x - X_y}(-\theta)] \mathrm{d}y \mathrm{d}x = 2(t-s),$$

$(H^\theta 3)$ for any $0 \le s < t$

- $\lim_{\varepsilon \to 0} \varepsilon^2 \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \int_{\frac{2s}{\varepsilon^2}}^{y} |\phi_{X_y - X_x}(\theta)| |\phi_{X_x - X_{\frac{2s}{\varepsilon^2}}}(2\theta)| \mathrm{d}x \mathrm{d}y = 0.$
- $\lim_{\varepsilon \to 0} \varepsilon \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} |\phi_{X_x - X_{\frac{2s}{\varepsilon^2}}}(\theta)| \mathrm{d}x = 0,$

This set of hypothesis gives some sufficient conditions on the characteristic function of the process $\{X_s, s \ge 0\}$ to get the convergence to a complex Brownian motion, as we will see in the next Theorem. Furthermore, in Theorem 3.3 we check that Lévy processes satisfy such hypothesis.

Theorem 3.1. *Let $\{X_s,\ s \ge 0\}$ be a stochastic process with independent increments and characteristic function ϕ_X. Set $C_X = \{\theta,$ such that ϕ_X satisfies $(H^\theta)\}$.*

Define for any $\varepsilon > 0$ and $\theta \in C_X$

$$\left\{ x_\varepsilon^\theta(t) = \varepsilon c(\theta) \int_0^{\frac{2t}{\varepsilon^2}} \mathrm{e}^{i\theta X_s} ds, \quad t \in [0,T] \right\}$$

where $c(\theta)$ is the constant given by hypothesis $(H^\theta 2)$.

Consider P_ε^θ the image law of x_ε^θ in the Banach space $\mathcal{C}([0,T],\mathbb{C})$ of continuous functions on $[0,T]$. Then, P_ε^θ converges weakly as ε tends to zero, towards the law on $\mathcal{C}([0,T],\mathbb{C})$ of a complex Brownian motion.

Remark 3.2. These kinds of kernels can be used to obtain approximations in law to Gaussian processes that can be characterized using a representation with respect to the Brownian motion. For instance, they could be used to get approximations for stochastic partial differential equations driven by Gaussian white noise, fractional stochastic differential equations or multiple Wiener integrals.

Proof of Theorem 3.1: We will follow the method explained in Subsect. 2.2.

Step 1: Tightness We have to check (5), that is, that there exists a constant $K(\theta)$ such that for any $s < t$

$$\sup_\varepsilon \left(E(\varepsilon c(\theta) \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \cos(\theta N_x) \mathrm{d}x)^4 + E(\varepsilon c(\theta) \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \sin(\theta N_x) \mathrm{d}x)^4 \right)$$
$$\le K(\theta)(t-s)^2.$$

From the properties of the complex numbers we have that

$$\begin{aligned}
&E\left(\varepsilon c(\theta)\int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}}\cos(\theta X_x)\right)\mathrm{d}x)^4+E\left(\varepsilon c(\theta)\int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}}\sin(\theta X_x)\right)\mathrm{d}x)^4\\
&\quad\le 2E|x_\varepsilon^\theta(t)-x_\varepsilon^\theta(s)|^4\\
&\quad=2c(\theta)^4\varepsilon^4E\left(\int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}}\mathrm{e}^{i\theta X_v}\mathrm{d}v\int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}}\mathrm{e}^{-i\theta X_u}\mathrm{d}u\right)^2\\
&\quad=2c(\theta)^4\varepsilon^4\int_{[\frac{2s}{\varepsilon^2},\frac{2t}{\varepsilon^2}]^4}E\left(\mathrm{e}^{i\theta[(X_{v_1}-X_{u_1})+(X_{v_2}-X_{u_2})]}\right)\mathrm{d}v_1\mathrm{d}v_2\mathrm{d}u_1\mathrm{d}u_2. \qquad (11)
\end{aligned}$$

Using that for $x_1 < x_2 < x_3 < x_4$ and $\rho_i \in \{0,1\}$ for $i = 1,2,3,4$ with $\sum_{i=1}^4 \rho_i = 2$ we can write

$$\begin{aligned}
&(-1)^{\rho_4}X_{x_4}+(-1)^{\rho_3}X_{x_3}+(-1)^{\rho_2}X_{x_2}+(-1)^{\rho_1}X_{x_1}\\
&\quad=(-1)^{\rho_4}(X_{x_4}-X_{x_3})+\big((-1)^{\rho_4}+(-1)^{\rho_3}\big)(X_{x_3}-X_{x_2})\\
&\qquad+\big((-1)^{\rho_4}+(-1)^{\rho_3}+(-1)^{\rho_2}\big)(X_{x_2}-X_{x_1}),
\end{aligned}$$

and the last expression (11) can be written as the sum of 24 integrals of the type

$$\begin{aligned}
&2c(\theta)^4\varepsilon^4\int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}}\int_{\frac{2s}{\varepsilon^2}}^{x_4}\int_{\frac{2s}{\varepsilon^2}}^{x_3}\int_{\frac{2s}{\varepsilon^2}}^{x_2}E\left(\mathrm{e}^{i\theta[c_1(X_{x_4}-X_{x_3})+c_2(X_{x_3}-X_{x_2})+c_3(X_{x_2}-X_{x_1})]}\right)\\
&\quad\times\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4. \qquad (12)
\end{aligned}$$

where $c_1 \in \{1,-1\}$, $c_2 \in \{-2,0,2\}$ and $c_3 \in \{1,-1\}$. Notice that since the process X has independent increments, we have that

$$\begin{aligned}
&E\left(\mathrm{e}^{i\theta[c_1(X_{x_4}-X_{x_3})+c_2(X_{x_3}-X_{x_2})+c_3(X_{x_2}-X_{x_1})]}\right)\\
&\quad=E\left(\mathrm{e}^{i\theta c_1(X_{x_4}-X_{x_3})}\right)E\left(\mathrm{e}^{i\theta c_2(X_{x_3}-X_{x_2})}\right)E\left(\mathrm{e}^{i\theta c_3(X_{x_2}-X_{x_1})]}\right),
\end{aligned}$$

and we obtain,

$$\begin{aligned}
&\left|E\left(\mathrm{e}^{i\theta[c_1(X_{x_4}-X_{x_3})+c_2(X_{x_3}-X_{x_2})+c_3(X_{x_2}-X_{x_1})]}\right)\right|\\
&\quad\le|\phi_{X_{x_4}-X_{x_3}}(c_1\theta)||\phi_{X_{x_2}-X_{x_1}}(c_3\theta)|\\
&\quad\le|\phi_{X_{x_4}-X_{x_3}}(\theta)||\phi_{X_{x_2}-X_{x_1}}(\theta)|,
\end{aligned}$$

where we have used that for any random variable Z, $|\phi_Z(-u)| = |\phi_Z(u)|$.

So, each one of the 24 integrals of the type (12) is bounded by

$$c(\theta)^4\varepsilon^2\int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}}\int_{\frac{2s}{\varepsilon^2}}^{x_4}|\phi_{X_{x_4}-X_{x_3}}(\theta)|\mathrm{d}x_3\mathrm{d}x_4\varepsilon^2\int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}}\int_{\frac{2s}{\varepsilon^2}}^{x_2}|\phi_{X_{x_2}-X_{x_1}}(\theta)|\mathrm{d}x_1\mathrm{d}x_2.$$

Clearly, hypothesis ($H^\theta 1$) completes the proof of this step.

Step 2: Martingale property It is enough to check (6) and (7). So, it suffices to see that

$$E\left(\varphi(x_\varepsilon^\theta(s_1),\ldots,x_\varepsilon^\theta(s_n))\varepsilon c(\theta)\int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}}\cos(\theta X_x)\mathrm{d}x\right)$$

and,

$$E\left(\varphi(x_\varepsilon^\theta(s_1),...,x_\varepsilon^\theta(s_n))\varepsilon c(\theta)\int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}}\sin(\theta X_x)\mathrm{d}x\right)$$

converge to zero when ε tends to zero.

Thus, it is enough to prove that

$$\left|E\left(\varphi(x_\varepsilon^\theta(s_1),...,x_\varepsilon^\theta(s_n))\varepsilon c(\theta)\int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}}\mathrm{e}^{i\theta X_x}\mathrm{d}x\right)\right|$$

converges to zero when ε tends to zero.

But this expression is equal to

$$\begin{aligned}&\left|E\left(\varphi(x_\varepsilon^\theta(s_1),...,x_\varepsilon^\theta(s_n))\mathrm{e}^{i\theta X_{\frac{2s}{\varepsilon^2}}}\right)\varepsilon c(\theta)\int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}}E\left(\mathrm{e}^{i\theta\left(X_x-X_{\frac{2s}{\varepsilon^2}}\right)}\right)\mathrm{d}x\right|\\&\quad\le K\varepsilon c(\theta)\left|\int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}}E\left(\mathrm{e}^{i\theta\left(X_x-X_{\frac{2s}{\varepsilon^2}}\right)}\right)\mathrm{d}x\right|\\&\quad\le K\varepsilon c(\theta)\int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}}\left|\phi_{X_x-X_{\frac{2s}{\varepsilon^2}}}(\theta)\right|\mathrm{d}x,\end{aligned}$$

that from $(H^\theta 3)$ converges to zero when ε tends to zero.

Step 3: Quadratic variations It is enough to check (8) and (9), that is that for any $s_1\le s_2\le\cdots\le s_n\le s<t$ and for any bounded continuous function $\varphi:\mathbb{C}^n\longrightarrow\mathbb{R}$,

$$a_\varepsilon := E\big[\varphi(x_\varepsilon^\theta(s_1)...,x_\varepsilon^\theta(s_n))\big((Re[x_\varepsilon^\theta(t)]-Re[x_\varepsilon^\theta(s)])^2-(t-s)\big)\big]$$

and

$$b_\varepsilon := E\big[\varphi(x_\varepsilon^\theta(s_1)...,x_\varepsilon^\theta(s_n))\big((Im[x_\varepsilon^\theta(t)]-Im[x_\varepsilon^\theta(s)])^2-(t-s)\big)\big]$$

converge to zero when ε tends to zero.

To prove that a_ε and b_ε converge to zero, when ε goes to zero, it is enough to show that $a_\varepsilon+b_\varepsilon$ and $a_\varepsilon-b_\varepsilon$ converge to zero. But,

$$\begin{aligned}&a_\varepsilon+b_\varepsilon\\&\quad= E\big[\varphi(x_\varepsilon^\theta(s_1)...,x_\varepsilon^\theta(s_n))\big(|x_\varepsilon^\theta(t)-x_\varepsilon^\theta(s)|^2-2(t-s)\big)\big]\\&\quad= E\left[\varphi(x_\varepsilon^\theta(s_1)...,x_\varepsilon^\theta(s_n))\big(\varepsilon^2c(\theta)^2\int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}}\int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}}\mathrm{e}^{i\theta(X_v-X_u)}\mathrm{d}v\mathrm{d}u-2(t-s)\big)\right]\\&\quad= E\big[\varphi(x_\varepsilon^\theta(s_1)...,x_\varepsilon^\theta(s_n)]\big)E\left(\varepsilon^2c(\theta)^2\int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}}\int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}}\mathrm{e}^{i\theta(X_v-X_u)}\mathrm{d}v\mathrm{d}u-2(t-s)\right)\\&\quad= E\big[\varphi(x_\varepsilon^\theta(s_1)...,x_\varepsilon^\theta(s_n))\big]\big)\left[E\big(\varepsilon^2c(\theta)^2\int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}}\int_{\frac{2s}{\varepsilon^2}}^{v}\mathrm{e}^{i\theta(X_v-X_u)}\mathrm{d}u\mathrm{d}v\big)\right.\\&\qquad\left.+E\left(\varepsilon^2c(\theta)^2\int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}}\int_{\frac{2s}{\varepsilon^2}}^{u}\mathrm{e}^{-i\theta(X_u-X_v)}\mathrm{d}v\mathrm{d}u\right)-2(t-s)\big)\right]\end{aligned}$$

$$= E\big[\varphi(x_\varepsilon^\theta(s_1)\ldots,x_\varepsilon^\theta(s_n)\big]\big)$$
$$\times\left[\varepsilon^2 c(\theta)^2\int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}}\int_{\frac{2s}{\varepsilon^2}}^{x}[\phi_{X_x-X_y}(\theta)+\phi_{X_x-X_y}(-\theta)]\mathrm{d}y\mathrm{d}x-2(t-s)\right].$$

Clearly, $(H^\theta 2)$ yields that $\lim_{\varepsilon\to 0}(a_\varepsilon+b_\varepsilon)=0$.

It remains to see that $a_\varepsilon-b_\varepsilon$ converges to zero. Indeed

$$\begin{aligned}
a_\varepsilon-b_\varepsilon &= E\Big[\varphi(x_\varepsilon^\theta(s_1)\ldots,x_\varepsilon^\theta(s_n))\\
&\quad\times\Big(\varepsilon c(\theta)\int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}}\cos(\theta X_x)\mathrm{d}x\Big)^2-\Big(\varepsilon c(\theta)\int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}}\sin(\theta X_x)\mathrm{d}x\Big)^2\Big]\\
&= \frac{1}{2}E\Big[\varphi(x_\varepsilon^\theta(s_1)\ldots,x_\varepsilon^\theta(s_n))\\
&\quad\times\left(\varepsilon c(\theta)\int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}}\mathrm{e}^{i\theta X_x}\mathrm{d}x\right)^2+\left(\varepsilon c(\theta)\int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}}\mathrm{e}^{-i\theta X_x}\mathrm{d}x\right)^2\Big],
\end{aligned}\tag{13}$$

where in the last step we have used that $2(\alpha^2-\beta^2)=(\alpha+\beta i)^2+(\alpha-\beta i)^2$. We will show that this two last terms go to zero. For the first we have that,

$$\begin{aligned}
&\frac{1}{2}E\left[\varphi(x_\varepsilon^\theta(s_1)\ldots,x_\varepsilon^\theta(s_n))\Big(\varepsilon c(\theta)\int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}}\mathrm{e}^{i\theta X_x}\mathrm{d}x\Big)^2\right]\\
&= E\left[\varphi(x_\varepsilon^\theta(s_1)\ldots,x_\varepsilon^\theta(s_n))\varepsilon^2c(\theta)^2\int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}}\int_{\frac{2s}{\varepsilon^2}}^{y}\mathrm{e}^{i\theta(X_x+X_y)}\mathrm{d}x\mathrm{d}y\right]\\
&= E\Bigg[\varphi(x_\varepsilon^\theta(s_1)\ldots,x_\varepsilon^\theta(s_n))\\
&\quad\times\varepsilon^2c(\theta)^2\int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}}\int_{\frac{2s}{\varepsilon^2}}^{y}\mathrm{e}^{i\theta(X_y-X_x)+2i\theta(X_x-X_{\frac{2s}{\varepsilon^2}})+2i\theta X_{\frac{2s}{\varepsilon^2}}}\mathrm{d}x\mathrm{d}y\Bigg]\\
&= E\left[\varphi(x_\varepsilon^\theta(s_1)\ldots,x_\varepsilon^\theta(s_n))\mathrm{e}^{2i\theta X_{\frac{2s}{\varepsilon^2}}}\right]\\
&\quad\times\varepsilon^2c(\theta)^2\int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}}\int_{\frac{2s}{\varepsilon^2}}^{y}\phi_{X_y-X_x}(\theta)\cdot\phi_{X_x-X_{\frac{2s}{\varepsilon^2}}}(2\theta)\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

Notice that this last expression can be bounded by

$$K\varepsilon^2c(\theta)^2\int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}}\int_{\frac{2s}{\varepsilon^2}}^{y}|\phi_{X_y-X_x}(\theta)||\phi_{X_x-X_{\frac{2s}{\varepsilon^2}}}(2\theta)|\mathrm{d}x\mathrm{d}y$$

that from $(H^\theta 3)$ converges to zero when ε goes to zero. Following the same computations, and using that, in general, for any random variable Z, $|\phi_Z(-u)|$

$= |\phi_Z(u)|$ we obtain the same bound and the convergence to zero, for the second term of expression (13).

Step 4: Quadratic covariation It is enough to check (10). Using that

$$\alpha\beta = \frac{1}{4} i[(\alpha - \beta i)^2 - (\alpha + \beta i)^2],$$

the term in the left side of (10) is equal to

$$\begin{aligned} &E\big(\varphi(x_\varepsilon^\theta(s_1), ..., x_\varepsilon^\theta(s_n))\big) \left(\varepsilon c(\theta) \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \cos(\theta X_x) \mathrm{d}x\right) \left(\varepsilon c(\theta \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \sin(\theta X_x) \mathrm{d}x)\right) \\ &\quad = \frac{1}{4} i E \left[\varphi(x_\varepsilon^\theta(s_1) ..., x_\varepsilon^\theta(s_n))\right. \\ &\quad\quad \times \left. \left(\varepsilon c(\theta) \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \mathrm{e}^{-i\theta X_x} \mathrm{d}x\right)^2 - \left(\varepsilon c(\theta) \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \mathrm{e}^{i\theta X_x} \mathrm{d}x\right)^2\right]. \end{aligned}$$

We have already shown in the study of (13) that this term goes to zero. □

Let us state now the main result of the paper. We prove that the approximations built from a Lévy process converge to a complex Brownian motion.

Theorem 3.3. *Define for any $\varepsilon > 0$*

$$\left\{x_\varepsilon^\theta(t) = \varepsilon c(\theta) \int_0^{\frac{2t}{\varepsilon^2}} \mathrm{e}^{i\theta X_s} ds, \quad t \in [0, T]\right\}$$

where $\{X_s,\ s \geq 0\}$ is a Lévy process with Lévy exponent ψ_X and

$$c(u) = \sqrt{\frac{|\psi_X(u)|^2}{2Re[\psi_X(u)]}}.$$

Consider P_ε^θ the image law of x_ε^θ in the Banach space $\mathcal{C}([0, T], \mathbb{C})$ of continuous functions on $[0, T]$. Then, for θ such that $Re[\psi_X(\theta)]Re[\psi_X(2\theta)] \neq 0$, P_ε^θ converges weakly as ε tends to zero, towards the law on $\mathcal{C}([0, T], \mathbb{C})$ of a complex Brownian motion.

Proof. The result follows as a particular case of Theorem 3.1. It suffices to check that the characteristic function ϕ_X of the Lévy process X satisfies (H^θ) for any θ such that $a(\theta)a(2\theta) \neq 0$ [recall definitions (3) and (4)].

Proof of $(H^\theta 1)$: We can write

$$\begin{aligned} \varepsilon^2 \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \int_{\frac{2s}{\varepsilon^2}}^{y} |\phi_{X_y - X_x}(\theta))| \mathrm{d}x \mathrm{d}y &= \varepsilon^2 \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \int_{\frac{2s}{\varepsilon^2}}^{y} \mathrm{e}^{-(y-x)a(\theta)} \mathrm{d}x \mathrm{d}y \\ &\leq \frac{2}{a(\theta)}(t - s). \end{aligned}$$

Using that $a(\theta) > 0$ we complete the proof of $(H^\theta 1)$.

Proof of $(H^\theta 2)$: Note first that

$$\begin{aligned}
&\varepsilon^2 c(\theta)^2 \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \int_{\frac{2s}{\varepsilon^2}}^{x} \phi_{X_x - X_y}(\theta) \mathrm{d}y \mathrm{d}x \\
&\quad = \varepsilon^2 c(\theta)^2 \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \int_{\frac{2s}{\varepsilon^2}}^{x} \mathrm{e}^{-(x-y)(a(\theta)+b(\theta)i)} \mathrm{d}y \mathrm{d}x \\
&\quad = \varepsilon^2 \frac{c(\theta)^2}{a(\theta)+b(\theta)i} \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \left(1 - \mathrm{e}^{-(x-\frac{2s}{\varepsilon^2})(a(\theta)+b(\theta)i)}\right) \mathrm{d}x \\
&\quad = o(\varepsilon) + 2(t-s) \frac{c(\theta)^2}{a(\theta)+b(\theta)i}.
\end{aligned}$$

Following the same computations and taking into account that $a(-\theta) = a(\theta)$, and that $b(-\theta) = -b(\theta)$ we obtain that

$$\begin{aligned}
&\varepsilon^2 c(\theta)^2 \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \int_{\frac{2s}{\varepsilon^2}}^{x} \phi_{X_x - X_y}(-\theta) \mathrm{d}y \mathrm{d}x \\
&\quad = o(\varepsilon) + 2(t-s) \frac{c(\theta)^2}{a(\theta)-b(\theta)i}.
\end{aligned}$$

So

$$\begin{aligned}
&\varepsilon^2 c(\theta)^2 \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \int_{\frac{2s}{\varepsilon^2}}^{x} [\phi_{X_x - X_y}(\theta) + \phi_{X_x - X_y}(-\theta)] \mathrm{d}y \mathrm{d}x \\
&\quad = o(\varepsilon) + 2(t-s) \left(\frac{c(\theta)^2}{a(\theta)+b(\theta)i} + \frac{c(\theta)^2}{a(\theta)-b(\theta)i} \right) \\
&\quad = o(\varepsilon) + 2(t-s),
\end{aligned}$$

and $(H^\theta 2)$ is clearly true.

Proof of $(H^\theta 3)$: Notice that

$$\begin{aligned}
&K\varepsilon^2 \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \int_{\frac{2s}{\varepsilon^2}}^{y} |\phi_{X_y - X_x}(\theta)| |\phi_{X_x - X_{\frac{2s}{\varepsilon^2}}}(2\theta)| \mathrm{d}x \mathrm{d}y \\
&\quad = K\varepsilon^2 \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \int_{\frac{2s}{\varepsilon^2}}^{y} \mathrm{e}^{-(y-x)a(\theta)} \mathrm{e}^{-(x-\frac{2s}{\varepsilon^2})a(2\theta)} \mathrm{d}x \mathrm{d}y \\
&\quad \leq K\varepsilon^2 \frac{1}{a(\theta)} \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \mathrm{e}^{-(x-\frac{2s}{\varepsilon^2})a(2\theta)} \mathrm{d}x \\
&\quad \leq K\varepsilon^2 \frac{1}{a(\theta)a(2\theta)},
\end{aligned}$$

that converges to zero when ε goes to zero.

To prove the second steep of $(H^\theta 3)$ notice that

$$K\varepsilon \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} |\phi_{X_x - X_{\frac{2s}{\varepsilon^2}}}(\theta)| \mathrm{d}x = K\varepsilon \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \mathrm{e}^{-(x-\frac{2s}{\varepsilon^2})a(\theta)} \mathrm{d}x \leq \frac{K}{a(\theta)} \varepsilon,$$

that converges to zero when ε tends to zero. □

Remark 3.4. Given a Lévy process with characteristic function given by the Lévy-Khintchine formula (2), the condition $Re[\psi_X(\theta)] = 0$ is equivalent to

$$\frac{1}{2}\sigma^2\theta^2 - \int_{\mathbb{R}\setminus\{0\}} (\cos(\theta x) - 1)\eta(\mathrm{d}x) = 0,$$

that is, $\sigma = 0$ and

$$\int_{\mathbb{R}\setminus\{0\}} (\cos(\theta x) - 1)\eta(\mathrm{d}x) = 0.$$

So, the condition $Re[\psi_X(\theta)]Re[\psi_X(2\theta)] \neq 0$ can be written as $\sigma \neq 0$ or

$$\left(\int_{\mathbb{R}\setminus\{0\}} (\cos(\theta x) - 1)\eta(\mathrm{d}x)\right)\left(\int_{\mathbb{R}\setminus\{0\}} (\cos(2\theta x) - 1)\eta(\mathrm{d}x)\right) \neq 0.$$

Remark 3.5. When we consider $\{X_t, t \geq 0\}$ a standard Poisson process it is well-known that it is a Lévy process with Lévy exponent

$$\psi_X(u) = -(\cos(u) - 1) - i\sin(u)$$

that corresponds to the Lévy-Khintchine formula (2) with $a = 0, \sigma = 0$ and $\eta = \delta_{\{1\}}$. Then the condition $Re[\psi_X(\theta)]Re[\psi_X(2\theta)] \neq 0$ yields that $\theta \neq k\pi$ for any $k \geq 1$.

When $\theta = (2k+1)\pi$, we have that

$$x_\varepsilon^\theta(t) = c((2k+1)\pi)\varepsilon \int_0^{\frac{2t}{\varepsilon^2}} cos((2k+1)\pi X_s)ds = \varepsilon \int_0^{\frac{2t}{\varepsilon^2}} (-1)^{X_s} ds, \quad (14)$$

that is a real process that can not converge to a complex Brownian motion. Nevertheless part of the same proof done in Theorem 3.1 (steps 1 and 2 and study of a_ε, note that $b_\varepsilon = 0$) works to prove that the processes defined by (14) converge weakly to a standard Brownian motion.

On the other hand, when $\theta = 2k\pi$, we have that

$$x_\varepsilon^\theta(t) = c(2k\pi)\varepsilon \int_0^{\frac{2t}{\varepsilon^2}} cos(2k\pi X_s)ds = 0.$$

4. The m-Dimensional Case

The aim of this section is to extend this result to a m-dimensional case for any $m \geq 1$. We will give the extensions of Theorem 3.1 and 3.3.

We define for any $\varepsilon > 0$ and for any $1 \leq j \leq m$

$$\left\{x_\varepsilon^{\theta_j}(t) = \varepsilon \int_0^{\frac{2t}{\varepsilon^2}} \mathrm{e}^{i\theta_j X_s} ds,\ t \in [0,T]\right\},$$

where $\{X_s,\ s \geq 0\}$ is a stochastic process with independent increments and we consider

$$\left\{x_\varepsilon^\theta(t) = \left(x_\varepsilon^{\theta_1}, \ldots, x_\varepsilon^{\theta_m}\right)(t),\ t \in [0,T]\right\}.$$

To simplify computations and notation we will denote by θ the m values $\theta_1, \theta_2, \ldots, \theta_m$. Since we have to control more quadratic covariations we will need to introduce new hypothesis on θ, $(\bar{H}^{\theta_j,\theta_h})$ for a characteristic function ϕ_X: $(\bar{H}^{\theta_j,\theta_h})$ For any $c_1 \in \{-1,1\}$

$$\lim_{\varepsilon\to 0} \varepsilon^2 \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \int_{\frac{2s}{\varepsilon^2}}^{y} |\phi_{X_y - X_x}(\theta_j)||\phi_{X_x - X_{\frac{2s}{\varepsilon^2}}}(\theta_j + c_1\theta_h)|\mathrm{d}x\mathrm{d}y = 0.$$

Then, the extension of Theorem 3.1, reads as follows:

Theorem 4.1. *Let $\{X_s,\ s \ge 0\}$ be a stochastic process with independent increments and characteristic function ϕ_X. Set $C_X^m = \{\theta \in \mathbb{R}^m$, such that ϕ_X satisfies (H^{θ_j}) for any $j = 1,\dots,m$ and satisfies $(\bar{H}^{\theta_j,\theta_h})$ for any $h \ne j\}$.*

Define for any $\varepsilon > 0$ and for any $1 \le j \le m$

$$\left\{ x_\varepsilon^{\theta_j}(t) = \varepsilon c(\theta_j) \int_0^{\frac{2t}{\varepsilon^2}} \mathrm{e}^{i\theta_j X_s} ds, \quad t \in [0,T] \right\},$$

where $c(\theta_j)$ is the constant given by hypothesis $(H^\theta 2)$.

Consider P_ε^θ the image law of $x_\varepsilon^\theta = \left(x_\varepsilon^{\theta_1},\dots,x_\varepsilon^{\theta_m}\right)$ in the Banach space $\mathcal{C}([0,T],\mathbb{C}^m)$ of continuous functions on $[0,T]$. Then, if $\theta \in C_X^m$, P_ε^θ converges weakly as ε tends to zero towards the law on $\mathcal{C}([0,T],\mathbb{C}^m)$ of a m-dimensional complex Brownian motion.

Proof. The proof follows applying the computations done for the one-dimensional case combined to the method used in [5]. We will only give some hints of the proof.

Notice that the proof of the tightness, the martingale property of each component and the quadratic variations can be done following exactly the proof of the one-dimensional case. So, only to study all the covariations remains. As it can be seen in Section 3.1 in [5], it suffices to prove that for $j \ne h$ and for any $s_1 \le s_2 \le \cdots \le s_k \le s < t$ and for any bounded continuous function $\varphi : \mathbb{C}^{mk} \longrightarrow \mathbb{R}$,

$$E\left(\varphi\left(x_\varepsilon^\theta(s_1),\dots,x_\varepsilon^\theta(s_k)\right)\left(\varepsilon\int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} c(\theta_j)\cos(\theta_j X_x)\mathrm{d}x\right)\right.$$
$$\left.\times\left(\varepsilon\int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} c(\theta_h)\cos(\theta_h X_y)\mathrm{d}y\right)\right),$$

$$E\left(\varphi\left(x_\varepsilon^\theta(s_1),\dots,x_\varepsilon^\theta(s_k)\right)\left(\varepsilon\int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} c(\theta_j)\sin(\theta_j X_x)\mathrm{d}x\right)\right.$$
$$\left.\times\left(\varepsilon\int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} c(\theta_h)\sin(\theta_h X_y)\mathrm{d}y\right)\right)$$

and

$$E\left(\varphi\left(x_\varepsilon^\theta(s_1),\dots,x_\varepsilon^\theta(s_k)\right)\left(\varepsilon\int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} c(\theta_j)\cos(\theta_j X_x)\mathrm{d}x\right)\right.$$
$$\left.\times\left(\varepsilon\int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} c(\theta_h)\sin(\theta_h X_y)\mathrm{d}y\right)\right)$$

converge to zero when ε tends to zero. But, using that $\cos(\theta) = \frac{e^{i\theta}+e^{-i\theta}}{2}$ and $\sin(\theta) = \frac{e^{i\theta}-e^{-i\theta}}{2i}$, and the symmetry between x and y (interchanging the roles of j and h), it is enough to show that

$$\lim_{\varepsilon\to 0} \varepsilon^2 \left| E\left(\varphi\left(x_\varepsilon^\theta(s_1),\dots,x_\varepsilon^\theta(s_k)\right)\int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}}\int_{\frac{2s}{\varepsilon^2}}^{y} e^{i(c_1\theta_j X_x + c_2\theta_h X_y)} dx dy\right)\right| = 0, \tag{15}$$

for any $c_1, c_2 \in \{-1,1\}$. But,

$$\begin{aligned}
&\left| E\left(\varphi\left(x_\varepsilon^\theta(s_1),\dots,x_\varepsilon^\theta(s_k)\right)\int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}}\int_{\frac{2s}{\varepsilon^2}}^{y} e^{i(c_1\theta_j X_x + c_2\theta_h X_y)} dx dy\right)\right| \\
&\quad = \Bigg| E(\varphi\left(x_\varepsilon^\theta(s_1),\dots,x_\varepsilon^\theta(s_k)\right) \\
&\qquad \times \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}}\int_{\frac{2s}{\varepsilon^2}}^{y} e^{ic_2\theta_h(X_y-X_x)} e^{i(c_1\theta_j+c_2\theta_h)(X_x - X_{\frac{2s}{\varepsilon^2}})} e^{i(c_1\theta_j+c_2\theta_h)X_{\frac{2s}{\varepsilon^2}}} dx dy)\Bigg| \\
&\quad \le K \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}}\int_{\frac{2s}{\varepsilon^2}}^{y} |\phi_{(X_y-X_x)}(c_2\theta_h)||\phi_{(X_x-X_{\frac{2s}{\varepsilon^2}})}(c_1\theta_j + c_2\theta_h)| dx dy, \\
&\quad \le K \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}}\int_{\frac{2s}{\varepsilon^2}}^{y} |\phi_{(X_y-X_x)}(\theta_h)||\phi_{(X_x-X_{\frac{2s}{\varepsilon^2}})}(\theta_j + c_3\theta_h)| dx dy,
\end{aligned}$$

for $c_3 \in \{-1,1\}$ and (15) follows from $(\bar{H}^{\theta_j,\theta_h})$. □

Finally, we state the extension of Theorem 3.3.

Theorem 4.2. *Assume now that* $\{X_s,\, s \ge 0\}$ *is a Lévy process with Lévy exponent* ψ_X *and set*

$$c(u) = \sqrt{\frac{|\psi_X(u)|^2}{2Re[\psi_X(u)]}}.$$

Consider P_ε^θ *the image law of* x_ε^θ *in the Banach space* $\mathcal{C}([0,T],\mathbb{C}^m)$ *of continuous functions on* $[0,T]$*. Then, for* θ *such that* $Re[\psi_X(\theta_j)]Re[\psi_X(2\theta_j)] \neq 0$ *for all* $j \in \{1,\dots,m\}$ *and* $Re[\psi_X(\theta_j + c_1\theta_h)] \neq 0$ *for all* $j,h \in \{1,\dots,m\}$ *and* $c_1 \in \{-1,1\}$, P_ε^θ *converges weakly as* ε *tends to zero, towards the law on* $\mathcal{C}([0,T],\mathbb{C}^m)$ *of a* m*-dimensional complex Brownian motion.*

Proof. As in the one-dimensional case it suffices to check that the characteristic function ϕ_X of the Lévy process satisfies (H^{θ_j}) for any $j = 1,\dots,m$ and satisfies $(\bar{H}^{\theta_j,\theta_h})$ for any $j \neq h$. Only the second part remains to be seen and can be easily checked that, for $c_1 \in \{-1,1\}$

$$\begin{aligned}
&\int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}}\int_{\frac{2s}{\varepsilon^2}}^{y} |\phi_{(X_y-X_x)}(\theta_j)||\phi_{(X_x-X_{\frac{2s}{\varepsilon^2}})}(\theta_j + c_1\theta_h)| dx dy \\
&\quad \le K\varepsilon^2 \frac{1}{a(\theta_j)a(\theta_j + c_1\theta_h)}.
\end{aligned}$$

□

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