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Riesz Spaces of Normal Semicontinuous Functions

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Abstract. The real-valued bounded normal upper semicontinuous functions on a topological space X were introduced by Dilworth (Trans Am Math Soc 68:427–438, 1950). The aim of this paper is to show that the set of all normal upper (lower) semicontinuous functions on a completely regular topological space X can be endowed with an algebraic structure and lattice operations such that it becomes a Dedekind complete Riesz space that is the Dedekind completion of $C_b(X)$, the Riesz space of all bounded continuous functions on X. The Dedekind completion of $C(X)$ is also obtained.

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1. Introduction

It is well known that the Riesz spaces $C(X)$ or $C_b(X)$ of all continuous or bounded continuous functions on a topological space X are not Dedekind complete, that is, an arbitrary order bounded subset of $C(X)$ or $C_b(X)$ need not have a supremum in $C(X)$ or $C_b(X)$, respectively ([14], Sect. 43). Not as well known is the Dedekind completion of these Riesz spaces. As a justification of this remark we give a list of papers on the Dedekind completion of $C(X)$ or $C_b(X)$ and note that the list is far from complete: [2,4–12,16,17].

The first who attempted to construct the Dedekind completion of $C_b(X)$ was Dilworth in 1950. In his famous paper [8] he introduced a new type of real-valued bounded discontinuous functions, called normal upper semicontinuous, which are characterized by the property that $S(I(f)) = f$, where $S(f)$ and $I(f)$ are the upper and lower limit functions of f. (See Sect. 2 for precise definitions. A dual notion of normal lower semicontinuous function can be defined by changing the above condition in $I(S(f)) = f$). Dilworth proved

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the following theorem : *If X is a completely regular topological space, then the Dedekind completion of the lattice $C_b(X)$ is isomorphic with the lattice of all real-valued normal upper semicontinuous functions on X* ([8], Theorem 4.1). To prove this result he used the general theory of Dedekind completion of an ordered set by cuts [15], or, more precisely, by normal sets, and remarked that the normal subsets of $C_b(X)$ correspond to the normal semicontinuous functions ([8], Lemmas 4.2 and 4.3). As shown in the statement of the theorem which has been quoted above, Dilworth studied the Dedekind completion of $C_b(X)$ only in terms of its lattice structure, and not from the point of view of the Riesz spaces.

Using a topological characterization of a normal upper semicontinuous function ([8], Theorem 3.2) Dilworth noted that the pointwise supremum of two normal upper semicontinuous functions is also normal upper semicontinuous, but the dual statement for infimum fails to hold. In Theorem 4.2 [8] he showed how to compute the supremum and the infimum of a bounded collection of normal upper semicontinuous functions on X, but nothing was said about the algebraic structure of the set of normal upper semicontinuous functions.

The aim of this paper is to show that Dilworth's result is also valid in Riesz spaces setting. For this aim we introduce an algebraic structure on the set of all normal upper semicontinuous functions on X and replace the pointwise infimum of two functions with a suitable one. In this manner the set of all normal upper semicontinuous functions becomes a Dedekind complete Riesz space that is the Dedekind completion of $C_b(X)$. A dual result for normal lower semicontinuous functions is also proved. Finally, we show how to construct the Dedekind completion of $C(X)$ in a similar manner.

The terminology for Riesz spaces used in this paper is that of [14].

2. Notation and Preliminaries

By X we denote a completely regular topological space ([1], Definition 2.45). The set of all bounded real-valued functions on X is denoted by $\mathcal{B}(X)$. Endowed with the pointwise order $\mathcal{B}(X)$ is a Dedekind complete Riesz space and $C_b(X)$ is a Riesz subspace of it. Since for every function $f \in \mathcal{B}(X)$ there is a positive real constant M such that $|f| \leq M\mathbf{1}_X$ (where $\mathbf{1}_X$ is the constant function equal with 1 on the whole X), the Riesz ideal generated by $C_b(X)$ in $\mathcal{B}(X)$ is the whole $\mathcal{B}(X)$.

In the study of the normal semicontinuous functions the following nonlinear operators are usually used.

The Baire Operators For every $f \in \mathcal{B}(X)$ we denote by $I(f)$ the lower limit function of f and by $S(f)$ the upper limit function of f, that is,

$$I(f) : X \longrightarrow \mathbb{R}, \quad I(f)(x) = \sup_{V \in \mathcal{V}_x} \inf_{y \in V} f(y), \quad x \in X,$$

$$S(f) : X \longrightarrow \mathbb{R}, \quad S(f)(x) = \inf_{V \in \mathcal{V}_x} \sup_{y \in V} f(y), \quad x \in X,$$

where $\mathcal{V}_x$ denotes the set of all neighborhoods of the point x in X.

Obviously, $I(f) \leq f \leq S(f)$ for every $f \in \mathcal{B}(X)$, and f is bounded on X if and only if $I(f)$ and $S(f)$ are bounded on X. So we have two nonlinear operators $I, S : \mathcal{B}(X) \longrightarrow \mathcal{B}(X)$. I is called the *lower Baire operator* and S is called the *upper Baire operator* in honor of R. Baire who used these operators for the first time in his book [3] under the name $m(f)$ for $I(f)$ and $M(f)$ for $S(f)$.

The operators I and S are *monotone* and *idempotent* and their compositions $I \circ S$ and $S \circ I$ have the same properties.

The Operators $\ell(\cdot)$ *and* $u(\cdot)$ For every $f \in \mathcal{B}(X)$ we also can associate two new functions $\ell(f)$ and $u(f)$ defined as follows:

$$\ell(f) = \bigvee \{g : g \in C_b(X),\ g \leq f\}, \tag{2.1}$$

$$u(f) = \bigwedge \{g : g \in C_b(X),\ g \geq f\}, \tag{2.2}$$

where $\bigvee$ and $\bigwedge$ denote the pointwise supremum and infimum, respectively.

Obviously, for every f in $\mathcal{B}(X)$ we have $u(-f) = -\ell(f)$ and

$$\ell(f) \leq I(f) \leq S(f) \leq u(f).$$

If X is a *completely regular* topological space the following equalities hold ([8], Lemma 4.1):

$$\ell(f) = I(f) \quad \text{and} \quad u(f) = S(f). \tag{2.3}$$

In the following will use the operators ℓ and u since their properties have been studied in detail by Kaplan in [12] and [13]. Let us note that Kaplan defined these operators on the second dual of $C(X)$, with X compact, but, in general, their properties depend only on the order between functions and not on the structure of the functions. So the properties of ℓ and u also hold in our setting. Hereunder we will list a small part of these properties, namely those that we need for our proofs. (See [12] or [13], Ch.10, for the proofs of these properties.)

Obviously, $f \leq g$ implies $\ell(f) \leq \ell(g)$, $u(f) \leq u(g)$, and for any $\lambda \geq 0$ we have $\ell(\lambda f) = \lambda \ell(f)$ and $u(\lambda f) = \lambda u(f)$.

The operator ℓ is supra-additive, u is sub-additive and for any functions $f, g \in \mathcal{B}(X)$ we have

$$\ell(f) + \ell(g) \leq \ell(f+g) \leq \ell(f) + u(g) \leq u(f+g) \leq u(f) + u(g). \tag{2.4}$$

Consequently, we obtain

$$\ell(f) - u(g) \leq \ell(f-g) \leq \begin{matrix} u(f) - u(g) \\ \ell(f) - \ell(g) \end{matrix} \leq u(f-g) \leq u(f) - \ell(g). \tag{2.5}$$

The operators ℓ and u have the following lattice properties:

$$\ell(f \wedge g) = \ell(f) \wedge \ell(g), \tag{2.6}$$

$$u(f \vee g) = u(f) \vee u(g). \tag{2.7}$$

A function $f : X \longrightarrow \mathbb{R}$ is called lower semicontinuous (upper semicontinuous) if for every real number λ the set $\{x \in X : f(x) > \lambda\}$ ($\{x \in X : f(x) < \lambda\}$) is open. The set of all real-valued bounded lower (upper) semicontinuous functions on X is denoted by $\mathcal{L}_{sc}(X)$ ($\mathcal{U}_{sc}(X)$). Let us note that for every $f \in \mathcal{B}(X)$ we have $\ell(f) \le f \le u(f)$, and $\ell(f) = f \Leftrightarrow f \in \mathcal{L}_{sc}(X)$, $u(f) = f \Leftrightarrow f \in \mathcal{U}_{sc}(X)$. Therefore, the lower semicontinuous functions are the fixed points of the operator ℓ, and the upper semicontinuous functions are the fixed points of the operator u, that is, $\mathcal{L}_{sc}(X) = \{f \in \mathcal{B}(X) : \ell(f) = f\}$ and $\mathcal{U}_{sc}(X) = \{f \in \mathcal{B}(X) : u(f) = f\}$.

Obviously, if $f, g \in \mathcal{L}_{sc}(X)$ and $\lambda \ge 0$, then $f + g$, $f \wedge g$, λf are in $\mathcal{L}_{sc}(X)$. Similarly, if $f, g \in \mathcal{U}_{sc}(X)$ and $\lambda \ge 0$, then $f + g$, $f \vee g$, λf are in $\mathcal{U}_{sc}(X)$. The operator ℓ restricted to $\mathcal{L}_{sc}(X)$ and the operator u restricted to $\mathcal{U}_{sc}(X)$ become additive.

3. Normal Semicontinuous Functions

A function $f \in \mathcal{B}(X)$ is called *normal upper semicontinuous* [8] if it is upper semicontinuous and $f = u(\ell(f))$. If $f \in \mathcal{B}(X)$ is lower semicontinuous and $f = \ell(u(f))$, then f is called *normal lower semicontinuous.* The set of all normal upper semicontinuous functions is denoted by

$$\mathcal{NU}_{sc}(X) = \{f \in \mathcal{U}_{sc}(X) : u(\ell(f)) = f\},$$

and the set of all normal lower semicontinuous functions is denoted by

$$\mathcal{NL}_{sc}(X) = \{f \in \mathcal{L}_{sc}(X) : \ell(u(f)) = f\}.$$

In the following, to avoid the use of a large number of brackets we will write $u\ell(f)$ instead $u(\ell(f))$ and so on.

The following example shows the difference between an upper semicontinuous function and a normal upper semicontinuous function.

Example. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $f(x) = \begin{cases} \sin\frac{1}{x}, & x \neq 0, \\ \lambda, & x = 0, \end{cases}$ where λ is a real number. Then:

(i) f is upper semicontinuous if and only if $\lambda \ge 1$.
(ii) f is normal upper semicontinuous if and only if $\lambda = 1$.

First we note that the pointwise sum of two normal upper semicontinuous functions is upper semicontinuous, but, in general, not normal. The next example illustrates such a situation.

Example. The following two functions $f, g : \mathbb{R} \longrightarrow \mathbb{R}$,

$$f(x) = \begin{cases} \sin\dfrac{1}{x}, & x \neq 0, \\ 1, & x = 0, \end{cases} \quad \text{and} \quad g(x) = \begin{cases} \cos\dfrac{1}{x}, & x \neq 0, \\ 1, & x = 0, \end{cases}$$

are normal upper semicontinuous, but their pointwise sum

$$(f+g)(x) = \begin{cases} \sqrt{2}\cos\left(\dfrac{1}{x} - \dfrac{\pi}{4}\right), & x \neq 0, \\ 2, & x = 0, \end{cases}$$

is upper semicontinuous at $x = 0$, but it is not normal. To obtain a normal function we must change the value at $x = 0$ from 2 in $\sqrt{2}$.

3.1. The Sum of Two Normal Upper Semicontinuous Functions

For $f, g \in \mathcal{NU}_{sc}(X)$ we define their sum by putting

$$f\overline{\oplus}g = u\ell(f+g). \tag{3.1}$$

Since $u\ell$ is an idempotent operator the function $f\overline{\oplus}g$ is in $\mathcal{NU}_{sc}(X)$ and

$$f\overline{\oplus}g = u\ell(f+g) \le uu(f+g) = f+g.$$

Obviously, $f\overline{\oplus}g = g\overline{\oplus}f$ and $f\overline{\oplus}0 = f$. The fact that we have $(f\overline{\oplus}g)\,\overline{\oplus}h = f\overline{\oplus}\,(g\overline{\oplus}h)$ results from the following lemma.

Lemma 3.1. *For $f, g, h \in \mathcal{NU}_{sc}(X)$ we have*

$$(f\overline{\oplus}g)\,\overline{\oplus}h = u\ell(f+g+h).$$

Proof. Indeed, first we note that

$$(f\overline{\oplus}g)\,\overline{\oplus}h = u\ell(f\overline{\oplus}g+h) \le u\ell(f+g+h).$$

Conversely, by using the inequalities (2.4), we have

$$u\ell(f+g+h) \le u\,[\ell(f+g)+u(h)] \le u\,[\ell(f+g)] + u\,[u(h)] = f\overline{\oplus}g + h.$$

Then, since the operator $u\ell$ is monotone and idempotent, we obtain

$$u\ell(f+g+h) \le u\ell\,(f\overline{\oplus}g+h) = (f\overline{\oplus}g)\,\overline{\oplus}h,$$

and the proof is complete. □

For every $f \in \mathcal{NU}_{sc}(X)$, $-f \in \mathcal{NL}_{sc}(X)$, but $u\ell(-f) \in \mathcal{NU}_{sc}(X)$ and $u\ell(-f) = u\,(-f) = -\ell(f)$. Using the fact that $\ell(f) \le f$ and the inequalities (2.5) we have

$$0 \le u\ell\,[f-\ell(f)] \le u\,[\ell(f)-\ell\ell(f)] = 0,$$

and then

$$f\overline{\oplus}u(-f) = u\ell\,[f+u(-f)] = u\ell\,[f-\ell(f)] = 0.$$

This last equality shows that, for any $f \in \mathcal{NU}_{sc}(X)$, $u(-f)$ is the additive inverse with respect to the addition defined in (3.1). Therefore, the following proposition holds.

Proposition 3.2. *$(\mathcal{NU}_{sc}(X), \overline{\oplus})$ is an Abelian group.*

3.2. Scalar Multiplication

For λ a real number and $f \in \mathcal{NU}_{sc}(X)$ we define

$$\lambda\overline{\odot}f = u\ell(\lambda f). \tag{3.2}$$

Since $u\ell$ is idempotent we have $\lambda\overline{\odot}f \in \mathcal{NU}_{sc}(X)$. Let us also note that

$$\lambda\overline{\odot}f = \begin{cases} \lambda f, & \lambda \ge 0, \\ \lambda\ell(f), & \lambda < 0. \end{cases} \tag{3.3}$$

Proposition 3.3. *The scalar multiplication defined in $\mathcal{NU}_{sc}(X)$ by (3.2) has the following properties:*

(i) $1\overline{\odot}f = f$, *for all $f \in \mathcal{NU}_{sc}(X)$.*
(ii) $\lambda\overline{\odot}\,(\mu\overline{\odot}f) = (\lambda\mu)\,\overline{\odot}f$, *for all $\lambda, \mu \in \mathbb{R}$ and all $f \in \mathcal{NU}_{sc}(X)$.*

(iii) $\lambda \overline{\odot} (f \overline{\oplus} g) = (\lambda \overline{\odot} f) \overline{\oplus} (\lambda \overline{\odot} g)$, *for all* $\lambda \in \mathbb{R}$ *and all* $f, g \in \mathcal{NU}_{sc}(X)$.
(iv) $(\lambda + \mu) \overline{\odot} f = (\lambda \overline{\odot} f) \overline{\oplus} (\mu \overline{\odot} f)$, *for all* $\lambda, \mu \in \mathbb{R}$ *and all* $f \in \mathcal{NU}_{sc}(X)$.

Proof. We will prove only (iii) and (iv).

(iii) For $\lambda \geq 0$ is easy to see that

$$(\lambda \overline{\odot} f) \overline{\oplus} (\lambda \overline{\odot} g) = (\lambda f) \overline{\oplus} (\lambda g) = u\ell (\lambda f + \lambda g) = \lambda u\ell (f + g) = \lambda \overline{\odot} (f \overline{\oplus} g).$$

For $\lambda < 0$, using (3.3), we have $\lambda \overline{\odot} (f \overline{\oplus} g) = \lambda \ell(f \overline{\oplus} g) = \lambda \ell u\ell(f + g)$ and

$$(\lambda \overline{\odot} f) \overline{\oplus} (\lambda \overline{\odot} g) = (\lambda \ell(f)) \overline{\oplus} (\lambda \ell(g)) = u\ell [\lambda \ell(f) + \lambda \ell(g)] = \lambda \ell u [\ell(f) + \ell(g)].$$

So, in order to have the equality in (iii) we must prove that

$$\ell u\ell(f + g) = \ell u [\ell(f) + \ell(g)]. \tag{3.4}$$

Since $\ell(f) + \ell(g) \leq \ell(f + g)$ and ℓu is a monotone operator, we have

$$\ell u [\ell(f) + \ell(g)] \leq \ell u\ell(f + g).$$

To prove the converse inequality we remark first that

$$\ell u [\ell(f + g) - \ell(f) - \ell(g)] = 0.$$

Indeed, by using the inequalities (2.5), we have

$$\begin{aligned} 0 &\leq \ell u [\ell(f + g) - \ell(f) - \ell(g)] \leq \ell u [\ell(f) + u(g) - \ell(f) - \ell(g)] \\ &= \ell u [g - \ell(g)] \leq \ell [u(g) - \ell\ell(g)] = \ell [g - \ell(g)] \leq u(g) - u\ell(g) = g - g = 0, \end{aligned}$$

and then, with (2.4), we obtain

$$\begin{aligned} \ell u\ell(f + g) &= \ell u [\ell(f + g) - \ell(f) - \ell(g) + \ell(f) + \ell(g)] \\ &\leq \ell \{u [\ell(f + g) - \ell(f) - \ell(g)] + u [\ell(f) + \ell(g)]\} \\ &\leq \underbrace{\ell u [\ell(f + g) - \ell(f) - \ell(g)]}_{=0} + uu [\ell(f) + \ell(g)] = u [\ell(f) + \ell(g)]. \end{aligned}$$

Since ℓ is monotone and idempotent, from the above inequality, we obtain the desired converse inequality $\ell u\ell(f + g) \leq \ell u [\ell(f) + \ell(g)]$, and so (3.4) is proved. Therefore, (iii) holds.

(iv) If $\lambda, \mu \geq 0$ or $\lambda, \mu < 0$ the proof is straightforward.

Let $\lambda > 0$, $\mu < 0$ and $\lambda + \mu > 0$. Then $(\lambda \overline{\odot} f) \overline{\oplus} (\mu \overline{\odot} f) = \lambda f \overline{\oplus} \mu \ell(f) = u\ell [\lambda f + \mu \ell(f)]$ and $(\lambda + \mu) \overline{\odot} f = (\lambda + \mu) f$. So, in this case, we must prove that

$$(\lambda + \mu) f = u\ell [\lambda f + \mu \ell(f)]. \tag{3.5}$$

In order to use the inequalities (2.5) we denote $\mu = -\beta$, with $\beta > 0$. Thus (3.5) becomes

$$(\lambda - \beta) f = u\ell [\lambda f - \beta \ell(f)]. \tag{3.6}$$

To prove this equality we note first that

$$\begin{aligned} u\ell [\lambda f - \beta \ell(f)] &\leq u [\lambda \ell(f) - \beta \ell\ell(f)] = u [(\lambda - \beta) \ell(f)] \\ &= (\lambda - \beta) u\ell(f) = (\lambda - \beta) f. \end{aligned}$$

This proves one inequality in (3.6). On the other hand, we have

$$\begin{aligned} u\ell\,[\lambda f - \beta\ell(f)] &\geq u\,[\lambda\ell(f) - \beta u\ell(f)] = u\,[\lambda\ell(f) - \beta f] \\ &\geq \lambda u\ell(f) - \beta u(f) = (\lambda - \beta)\,f, \end{aligned}$$

and thus (3.6) is proved.

Now let $\lambda > 0$, $\mu < 0$ and $\lambda + \mu < 0$. As above we must prove that

$$(\lambda - \beta)\,\ell(f) = u\ell\,[\lambda f - \beta\ell(f)]. \tag{3.7}$$

First we have

$$\begin{aligned} u\ell\,[\lambda f - \beta\ell(f)] &\leq u\,[\lambda\ell(f) - \beta\ell\ell(f)] = u\,[(\lambda - \beta)\,\ell(f)] \\ &= (\lambda - \beta)\,\ell\ell(f) = (\lambda - \beta)\,\ell(f). \end{aligned}$$

On the other hand

$$\begin{aligned} u\ell\,[\lambda f - \beta\ell(f)] &\geq u\,[\lambda\ell(f) - \beta u\ell(f)] = u\,[\lambda\ell(f) - \beta f] \\ &\geq \lambda\ell\ell(f) - \beta\ell(f) = (\lambda - \beta)\,\ell(f), \end{aligned}$$

and so (3.7) is proved.

A similar proof can be given if $\lambda < 0$ and $\mu > 0$, and so (iv) holds for all $\lambda, \mu \in \mathbb{R}$. □

Proposition 3.2 and Proposition 3.3 show that the following theorem holds.

Theorem 3.4. *The set $\mathcal{NU}_{sc}(X)$ of all normal upper semicontinuous functions on a completely regular topological space X endowed with the addition defined by $f \overline{\oplus} g = u\ell(f + g)$ and the scalar multiplication defined by $\lambda \overline{\odot} f = u\ell(\lambda f)$ is a linear space.*

4. The Riesz Space $\mathcal{NU}_{sc}(X)$

Obviously, the linear space $(\mathcal{NU}_{sc}(X), \overline{\oplus}, \overline{\odot})$ endowed with the pointwise order relation between functions is an ordered linear space, that is, the following properties hold:

(i) $f \leq g \Rightarrow f \overline{\oplus} h = u\ell(f + h) \leq u\ell(g + h) = g \overline{\oplus} h$, for all $f, g, h \in \mathcal{NU}_{sc}(X)$.
(ii) $f \geq 0$ and $\lambda \geq 0 \Rightarrow \lambda \overline{\odot} f \geq 0$.

Now we study the lattice properties of this ordered linear space. Let f, g be two functions in $\mathcal{NU}_{sc}(X)$. Dilworth showed that $f \vee g \in \mathcal{NU}_{sc}(X)$, but $f \wedge g \notin \mathcal{NU}_{sc}(X)$ [8]. In order to obtain a Riesz space we define the lattice operations in $\mathcal{NU}_{sc}(X)$ by putting for the supremum of two functions f and g the pointwise supremum

$$f \bigvee_{\mathcal{NU}} g = f \vee g,$$

and define the infimum in the following manner

$$f \bigwedge_{\mathcal{NU}} g = u\ell(f \wedge g).$$

If $f, g \in \mathcal{NU}_{sc}(X) \subset \mathcal{U}_{sc}(X)$, then $f = u(f) = u\ell(f)$ and $g = u(g) = u\ell(g)$. By using the equality (2.7), we have

$$\begin{aligned} f \vee g &= u(f) \vee u(g) = u(f \vee g) \geq u\ell(f \vee g) \\ &\geq u\left[\ell(f) \vee \ell(g)\right] = u\ell(f) \vee u\ell(g) = f \vee g. \end{aligned}$$

Therefore $f \vee g = u\ell(f \vee g)$, and we have another proof of Dilworth's result that the pointwise supremum of two normal upper semicontinuous functions is a normal upper semicontinuous function [8].

Let us note that $f \bigwedge_{\mathcal{NU}} g$ is the greatest lower bound of the set $\{f, g\}$ in $\mathcal{NU}_{sc}(X)$. Indeed,

$$f \bigwedge_{\mathcal{NU}} g = u\ell(f \wedge g) \leq \begin{matrix} u\ell(f) = f, \\ u\ell(g) = g, \end{matrix}$$

that is, $f \bigwedge_{\mathcal{NU}} g$ is a lower bound of the set $\{f, g\}$ in $\mathcal{NU}_{sc}(X)$. If h is a function in $\mathcal{NU}_{sc}(X)$ such that $h \leq f, g$, then $h \leq f \wedge g$ and $h = u\ell(h) \leq u\ell(f \wedge g) = f \bigwedge_{\mathcal{NU}} g$.

Proposition 4.1. (i) *The set $\mathcal{NU}_{sc}(X)$ is a lattice with the lattice operations*

$$f \bigvee_{\mathcal{NU}} g = f \vee g, \qquad f \bigwedge_{\mathcal{NU}} g = u\ell(f \wedge g).$$

(ii) *The set $\mathcal{NU}_{sc}(X)$ is a Dedekind complete lattice in which for any non-empty order bounded subset $\{f_\gamma\}_{\gamma \in \Gamma}$ we have*

$$\bigvee_{\mathcal{NU}} f_\gamma = u(\vee_\gamma f_\gamma), \quad \bigwedge_{\mathcal{NU}} f_\gamma = u\ell(\wedge_\gamma f_\gamma). \tag{4.1}$$

For the proof of (4.1) see [8], Theorem 4.2. In conclusion, the following theorem holds.

Theorem 4.2. *$(\mathcal{NU}_{sc}(X), \overline{\oplus}, \overline{\odot}, \leq, \bigvee_{\mathcal{NU}}, \bigwedge_{\mathcal{NU}})$ is a Dedekind complete Riesz space.*

5. The Riesz Space $\mathcal{NL}_{sc}(X)$

In a similar manner we can prove a dual result for $\mathcal{NL}_{sc}(X)$, the set of all normal lower semicontinuous functions on X.

Theorem 5.1. *The set $\mathcal{NL}_{sc}(X)$ endowed with the addition*

$$f \underline{\oplus} g = \ell u(f + g), \tag{5.1}$$

and the scalar multiplication

$$\lambda \underline{\odot} f = \ell u(\lambda f) = \begin{cases} \lambda f, & \lambda \geq 0, \\ \lambda u(f), & \lambda < 0, \end{cases} \tag{5.2}$$

the pointwise order and the lattice operations

$$f \bigvee_{\mathcal{NL}} g = \ell u(f \vee g), \quad f \bigwedge_{\mathcal{NL}} g = f \wedge g,$$

is a Dedekind complete Riesz space.

In this space the supremum and the infimum of any nonempty order bounded subset $\{f_\gamma\}_{\gamma\in\Gamma}$ are computed with the following formulae:

$$\bigvee_{\mathcal{NL}} f_\gamma = \ell u(\underset{\gamma}{\vee} f_\gamma), \quad \bigwedge_{\mathcal{NL}} f_\gamma = \ell(\underset{\gamma}{\wedge} f_\gamma). \tag{5.3}$$

In the rest of this paper by $\mathcal{NU}_{sc}(X)$ and $\mathcal{NL}_{sc}(X)$ we will understand the Riesz spaces defined in Theorem 4.2 and Theorem 5.1, respectively. Actually these Riesz spaces are order isomorphic as is shown in the next theorem.

Theorem 5.2. *The Riesz spaces $\mathcal{NU}_{sc}(X)$ and $\mathcal{NL}_{sc}(X)$ are order isomorphic by the following operators*

$$\ell : \mathcal{NU}_{sc}(X) \longrightarrow \mathcal{NL}_{sc}(X), \quad u : \mathcal{NL}_{sc}(X) \longrightarrow \mathcal{NU}_{sc}(X),$$

which are inverses of each other.

Proof. In order to show that ℓ is a linear operator we must prove that

$$\ell\,(f \overline{\oplus} g) = \ell(f) \underline{\oplus} \ell(g), \quad f, g \in \mathcal{NU}_{sc}(X), \tag{5.4}$$

$$\ell\,(\lambda \overline{\odot} f) = \lambda \underline{\odot} \ell(f), \quad \lambda \in \mathbb{R},\ f \in \mathcal{NU}_{sc}(X). \tag{5.5}$$

By using the definitions (3.1) and (5.1) for the addition in $\mathcal{NU}_{sc}(X)$ and $\mathcal{NL}_{sc}(X)$, respectively, the equality (5.4) becomes $\ell u \ell(f+g) = \ell u\,[\ell(f)+\ell(g)]$, which holds by (3.4). The equality (5.5) results immediately from the definitions (3.2) and (5.2) of the scalar multiplications.

The operator ℓ is a lattice morphism since for every f and g in $\mathcal{NU}_{sc}(X)$ we have

$$\ell\left(f \bigvee_{\mathcal{NU}} g\right) = \ell(f) \bigvee_{\mathcal{NL}} \ell(g), \quad \ell\left(f \bigwedge_{\mathcal{NU}} g\right) = \ell(f) \bigwedge_{\mathcal{NL}} \ell(g). \tag{5.6}$$

Indeed, the first equality in (5.6) means $\ell\,(f \vee g) = \ell u\,[\ell(f) \vee \ell(g)]$, and it holds by (2.7). The second equality in (5.6) means $\ell u \ell(f \wedge g) = \ell(f) \wedge \ell(g)$. By using (2.6) and the fact that ℓ and u are monotone and idempotent operators, we have

$$\begin{aligned} \ell(f) \wedge \ell(g) &= \ell(f \wedge g) \le \ell u \ell(f \wedge g) = \ell u\,[\ell(f) \wedge \ell(g)] \\ &\le \ell\,[u\ell(f) \wedge u\ell(g)] = \ell\,(f \wedge g) = \ell(f) \wedge \ell(g). \end{aligned}$$

Since $u\ell(f) = f$, for every $f \in \mathcal{NU}_{sc}(X)$, and $\ell u(f) = f$, for every $f \in \mathcal{NL}_{sc}(X)$, ℓ and u are inverses of each other. □

6. The Dedekind Completion of $C_b(X)$ and $C(X)$ with Normal Semicontinuous Functions

Let L and K be two Riesz spaces, with K Dedekind complete. We recall that the Dedekind complete Riesz space K is called a *Dedekind completion* of the Riesz space L, if L is embedded in K as a Riesz subspace, which we identify with L, and for every $f \in K$ we have

$$\sup\{g : g \in L,\ g \le f\} = f = \inf\{g : g \in L,\ g \ge f\}.$$

The Dedekind complete Riesz space K is denoted by L^δ ([14], Definition 32.1).

Theorem 6.1. *Let X be a completely regular topological space. Then the Dedekind completion of $C_b(X)$ is $\mathcal{NU}_{sc}(X)$, that is, $C_b(X)^\delta = \mathcal{NU}_{sc}(X)$.*

Proof. Obviously $C_b(X)$ is a Riesz subspace of the Dedekind complete Riesz space $\mathcal{NU}_{sc}(X)$. Every function f in $\mathcal{NU}_{sc}(X)$ satisfies the equalities $u\ell(f) = f = u(f)$. Using the definitions (2.1) and (2.2) of the operators u and ℓ, and the formulae (4.1) for computing the supremum and the infimum of a set of functions in $\mathcal{NU}_{sc}(X)$, we can write these equalities as follows:

$$f = u\ell(f) = u\left(\bigvee\{g \in C_b(X) : g \le f\}\right) = \bigvee_{\mathcal{NU}}\{g \in C_b(X) : g \le f\},$$

and

$$f = u\ell(f) = u\ell u(f) = u\ell\left(\bigwedge\{g \in C_b(X) : g \ge f\}\right) = \bigwedge_{\mathcal{NU}}\{g \in C_b(X) : g \ge f\}.$$

Therefore we have

$$\bigvee_{\mathcal{NU}}\{g \in C_b(X) : g \le f\} = f = \bigwedge_{\mathcal{NU}}\{g \in C_b(X) : g \ge f\},$$

which shows that $\mathcal{NU}_{sc}(X)$ is the Dedekind completion of $C_b(X)$. □

Theorem 6.1 and Theorem 5.2 show that the following corollary holds.

Corollary 6.2. *If X is a completely regular topological space, then*

$$C_b(X)^\delta = \mathcal{NL}_{sc}(X).$$

The construction developed above for $C_b(X)^\delta$ can also be made for $C(X)^\delta$. For this aim we must do some minor changes. First we replace $\mathcal{B}(X)$ with the Riesz space $\mathcal{B}_c(X)$ of all real-valued C-bounded functions on X. (A function $f : X \longrightarrow \mathbb{R}$ is called C-bounded if there exists $g \in C(X)$ such that $|f| \le g$). $\mathcal{B}_c(X)$ is the Riesz ideal generated by $C(X)$ in the Riesz space of all the real-valued functions defined on X. Thus the operators $\ell, u : \mathcal{B}_c(X) \longrightarrow \mathcal{B}_c(X)$ are well defined. The equalities (2.3), that is, $\ell(f) = I(f)$ and $u(f) = S(f)$, also hold for every C-bounded function f, as has been shown in [7], Proposition 4.1. We denote by $\mathcal{NU}^{cb}_{sc}(X) = \{f \in \mathcal{B}_c(X) : u\ell(f) = f\}$, the set of all C-bounded functions that are normal upper semicontinuous, and by $\mathcal{NL}^{cb}_{sc}(X) = \{f \in \mathcal{B}_c(X) : \ell u(f) = f\}$, the set of all C-bounded functions that are normal lower semicontinuous. Then, like above, we can endow these sets with a Riesz space structure. The obtained Riesz spaces are Dedekind complete and order isomorphic $\mathcal{NU}^{cb}_{sc}(X) \cong \mathcal{NL}^{cb}_{sc}(X)$.

After these explanations we can state the following theorem, which shows that a result of Horn ([11], Theorem 11) also holds in the Riesz spaces setting.

Theorem 6.3. *If X is a completely regular topological space, then*

$$C(X)^\delta = \mathcal{NU}^{cb}_{sc}(X) \cong \mathcal{NL}^{cb}_{sc}(X).$$

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