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PC-Almost Automorphic Solution of Impulsive Fractional Differential Equations

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Abstract. In this paper, we introduce a PC-almost automorphic function and establish the composition theorem, which is an important result from application point of view. As an application, we study the existence of almost automorphic solution to impulsive fractional functional differential equations with the assumption that the forcing term is almost automorphic. The results are established by fixed point methods and α-resolvent family of bounded linear operators. At the end, some examples are given to illustrate our analytic findings.

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Keywords. Fractional operators, α-resolvent family of bounded linear operators, PC-almost automorphic function, fixed point theorems.

1. Introduction

In this paper, we establish existence of PC-almost automorphic solution of impulsive fractional semilinear differential equations (1.1, 1.2, 1.3) using α-resolvent family of bounded linear operators, Sadovskii's fixed point theorem and Schaefer fixed point theorem. Uniqueness of the solution is established by using Banach's contraction principle. Problems, under consideration in this paper, are given below point by point.

1.1. Problem Description

We consider the following impulsive fractional differential equations:

P-1: Impulsive fractional differential equation of order $\alpha \in (1, 2)$.

$$\begin{aligned} {}^{C}D^{\alpha}x(t) &= Ax(t) + D_t^{\alpha-1}(f(t, x(t))),\ t \in \mathbb{R},\ t \neq t_k, \\ \Delta x(t)|_{t=t_k} &= I_k(x(t_k^-)), k = 1, 2, \ldots, \\ x(0) &= x_0, \end{aligned} \tag{1.1}$$

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where $f \in C(\mathbb{R} \times X, X), I_k : X \to X$ and $x_0 \in X$. $0 = t_0 < t_1 < t_2 < \cdots < t_n < \cdots, \Delta x(t)|_{t=t_k} = x(t_k^+) - x(t_K^-), x(t_k^+) = \lim_{h\to 0} x(t + h)$ and $x(t_k^-) = \lim_{h\to 0} x(t - h)$.

P-2: Impulsive fractional functional semilinear evolution equation of order $\alpha \in (1, 2)$.

$$\begin{aligned} D_t^\alpha x(t) &= Ax(t) + D_t^{\alpha-1}(g(t, x_t)), \quad t \in \mathbb{R}, t \neq t_k \\ \Delta x(t)|_{t=t_k} &= J_k(x(t_k^-)), \quad k = 1, 2, \ldots, \\ x(s) &= \phi(s) \quad s \in [-r, 0], \end{aligned} \tag{1.2}$$

where $g : \mathbb{R} \times \mathcal{C}_r \to X, \mathcal{C}_r = PC([-r, 0], X), J_k : X \to X$, and the operator $A : D(A) \subset X \to X$ is a linear close and densely defined of sectorial type in a separable Banach space X with norm $\|\cdot\|_X$.

P-3: Impulsive fractional functional semilinear evolution equation of order $\alpha \in (1, 2)$.

$$\begin{aligned} D_t^\alpha x(t) &= Ax(t) + D_t^{\alpha-1}(h(t, x(t), x_t)), \quad t \in \mathbb{R}, t \neq t_k \\ \Delta x(t)|_{t=t_k} &= J_k(x(t_k^-)), \quad k = 1, 2, \ldots, \\ x(s) &= \phi(s) \quad s \in [-r, 0], \end{aligned} \tag{1.3}$$

where $\mathbb{R} \times X \times \mathcal{C}_r \to X, \mathcal{C}_r = PC([-r, 0], X), J_k : X \to X$, and the operator $A : D(A) \subset X \to X$ is a linear close and densely defined of sectorial type in a separable Banach space X with norm $\|\cdot\|_X$.

1.2. Almost Automorphic Solution

Since the introduction of almost periodic functions by Bohr [4], there have been various important generalizations of this concept. One important generalization is the concept of almost automorphic functions which was introduced by Bochner. This concept is no exception and many mathematicians applied this in the field of fractional differential equations, for more details one can see the [14,15] and references therein. It can be argued that many phenomena exhibit regularity behavior periodicity. These kinds of phenomena can be modeled by considering more general notion such as almost periodic, almost automorphic, pseudo-almost automorphic and asymptotically almost automorphic. One of the very important question in the field of differential equations is that if the force function possesses a special characteristic then whether the solution possesses the same characteristic or not. In this work, we introduce a PC-almost automorphic function and establish the composition theorem, which is an important result from application point of view. As an application, we study the existence of PC-almost automorphic solution to impulsive fractional functional differential equations with the assumption that the forcing term is almost automorphic.

1.3. Impulsive Differential Equations

Impulsive differential equation provides a realistic framework of modeling systems in fields like population dynamics, control theory, physics, biology and medicine, when the dynamics undergo some abrupt changes at certain moments of time like earthquake, harvesting, shock and so forth. Milman and

Myshkis [17] first introduced impulsive differential equations in 1960. Followed by their work (Milman and Myshkis), there are several monographes and papers written by many authors like Bainov and Simeonov [2], Benchohra et al. [3], Lakshmikantham et al. [13], Samoilenko and Perestyuk [23] and Mahto et al. [16]. In several fields like biology, population dynamics and so forth problems with hereditary are best modeled by delay/functional differential equations [10]. The problems with impulsive effects and hereditary properties could be modeled by impulsive functional differential equations.

1.4. Fractional Differential Equations

Last few decades have witnessed tremendous works on fractional differential equations. In the beginning, researchers were focusing mainly on theoretical study of these equations. But, recently, fractional differential equations attracted many mathematicians and scientists because of their usefulness in the various problems arising from engineering and physical sciences. It has been shown that many physical systems can be represented more accurately through fractional derivative formulation [19]. Therefore, the theory has been applied to many fields, for example, in the field of viscoelasticity, feedback amplifiers, electrical circuits, control theory, electro analytical chemistry, fractional multi-poles, neuron modeling encompassing different branches of physics, chemistry and biological sciences. For more details, one can see one excellently written book by Podlubny [20]. Many physical processes appear to exhibit fractional-order behavior that may vary with time or space. The fractional calculus has allowed the operations of integration and differentiation to any arbitrary order. The order may take any real or imaginary value. The existence and uniqueness of the solutions of fractional differential equations have been shown by many authors [1,11,12,15,20].

There are many evolutionary systems, which encounter delay, impulse effects and anomalous characteristics in together. The combination of all these three characteristics, in modeling the dynamics of those systems, allows to consider an impulsive fractional functional differential equation. In spite of rich applicability, the analysis and the application of impulsive fractional functional differential equations is in infancy. Guo et al. [8] discussed natural formula of solution of impulsive fractional functional differential equation and establish existence and uniqueness of the solution by virtue of Schauder fixed-point theorem and Banach contraction principle. Stamova et al. [22] established stability solution of impulsive fractional functional differential equations using comparison principle.

2. Preliminaries

This section is devoted to basic definitions and results, which are necessary to follow this work. Let us denote $B(X)$, the Banach space of all linear and bounded operators on X endowed with the norm $\|\cdot\|_{B(X)}$ and $\mathcal{C} = \mathcal{C}(\mathbb{R}, X)$ the set of all continuous functions from $\mathbb{R}$ to X. It is important to define sectorial operator to define the mild solution of any fractional abstract equations. So, let us begin this section with this definition.

Definition 2.1. Sectorial operator: A closed linear operator A is said to be sectorial of type ω and angle θ if there exists $0<\theta<\frac{\pi}{2}, M_1>0$ and $\omega \in \mathbb{R}$ such that its resolvent exists outside the sector

$$\omega + S_\theta := \{\omega+\lambda : \lambda \in \mathbb{C},\ |arg(-\lambda)|<\theta\},$$

and satisfies

$$\|(\lambda - A)^{-1}\|_{B(X)} \leq \frac{M_1}{|\lambda-\omega|},\ \lambda \notin \omega + S_\theta.$$

Sectorial operators are well studied in the literature, for more details, one could see [9]. It is easy to verify that an operator A is sectorial of type ω if and only if $\lambda I - A$ is sectorial of type 0.

Definition 2.2. Let A be a closed linear and densely defined operator with domain $D(A)$ defined on a Banach space X and $\alpha>0$. A is the generator of an α-resolvent family if there exists $\omega \geq 0$ and an strongly continuous function $S_\alpha : \mathbb{R}_+ \to B(X)$ such that $\{\lambda^\alpha : \Re(\lambda)>\omega\}$ and

$$(\lambda^\alpha - A)^{-1}x = \int_0^\infty \mathrm{e}^{-\lambda t} S_\alpha(t)x\mathrm{d}t.$$

In this case, $S_\alpha(t)$ is α-resolvent family generated by A.

Cuevas and Lizama [15] have shown that the Eq. (1.3) can be thought of a limiting case of the equation

$$z'(t) = \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} Az(s)\mathrm{d}s + h(t,x(t),x_t),\ t\geq 0,$$
$$z(s)=\phi(s), s\in[-r,0], \tag{2.1}$$

in the sense that the solutions are asymptotic to each other as $t\to\infty$. If the operator A is sectorial of type ω with $\theta \in [0, \pi(1-\frac{\alpha}{2}))$, then the problem (2.1) is well posed (see [15]). Thus using variation of parameter formula, one can obtain

$$z(t) = S_\alpha(t)(\phi(0)) + \int_0^t S_\alpha(t-s)h(s,x(s),x_s)\mathrm{d}s, \quad t\geq 0, \tag{2.2}$$

where

$$S_\alpha(t) = \frac{1}{2\pi i}\int_\gamma \mathrm{e}^{\lambda t}\lambda^{\alpha-1}(\lambda^\alpha I - A)^{-1}\mathrm{d}\lambda, \quad t\geq t_0.$$

Here the path γ lies outside the sector $\omega + S_\theta$. Further, if $S_\alpha(t)$ is integrable then the solution is given by

$$x(t) = \int_{-\infty}^t S_\alpha(t-s)h(s,x(s),x_s)\mathrm{d}s. \tag{2.3}$$

Subtracting Eq. (2.3) from Eq. (2.2), we get

$$z(t)-x(t) = S_\alpha(t)(\phi(0)) - \int_t^\infty S_\alpha(s)h(t-s,x(t-s),x_{t-s}).$$

Hence for $f \in L^{p'}(\mathbb{R}^+ \times X, X)$, where $p' \in [1,\infty)$, we have $v(t)-u(t)\to 0$ as $t\to\infty$.

Recently, an important result has been proved by Cuesta [5] (Theorem 1) that if A is a sectorial operator of type $\omega < 0$ for some $M > 0$ and $\theta \in [0, \pi(1-\frac{\alpha}{2}))$, then there exists $C > 0$ such that

$$\|S_\alpha(t)\|_{B(X)} \leq \frac{CM}{1+|\omega|t^\alpha}$$

for $t \geq 0$.

For reader's convenience, we define the following class of spaces:

- $\mathcal{PC}(\mathbb{R}, X) = \Big\{\phi : \mathbb{R} \to X : \phi$ is continuous for every $t \notin \{t_k\}, \lim_{h\to 0} \phi(t_k + h) = \phi(t_k^+), \lim_{h\to 0} \phi(t_k - h) = \phi(t_k^-)$ exist and $\phi(t_k) = \phi(t_k^-)\Big\}$.
- $\mathcal{PC}(\mathbb{R} \times X, X) = \Big\{\phi : \mathbb{R} \times X \to X : \phi$ is continuous for every $t \notin \{t_k\}, \lim_{h\to 0} \phi(t_k + h, x) = \phi(t_k^+, x), \lim_{h\to 0} \phi(t_k - h, x) = \phi(t_k^-, x)$ exist and$\phi(t_k, x) = \phi(t_k^-, x)\Big\}$.
- $\mathcal{AA}_p(\mathbb{R}, X) = \Big\{\phi \in \mathcal{PC}(\mathbb{R}, X) : \phi$ is almost automorphic function$\Big\}$
- $\mathcal{AA}_p(\mathbb{R} \times X, X) = \Big\{\phi \in \mathcal{PC}(\mathbb{R} \times X, X) : \phi$ is almost automorphic function$\Big\}$
- $\mathcal{AAS}(\mathbb{Z}, X) = \Big\{\phi : \mathbb{Z} \to X$ is an almost automorphic sequence$\Big\}$

The definition of PC-almost automorphic operator has been given by N'Guéré kata and Pankov [7]. Now we state the following definitions in the framework of impulsive systems.

Definition 2.3. A function $f \in \mathcal{PC}(\mathbb{R}, X)$ is called a PC-almost automorphic if

(i) sequence of impulsive moments $\{t_k\}$ is a PC-almost automorphic sequence,
(ii) for every real sequence (s_n), there exists a sub-sequence (s_{n_k}) such that

$$g(t) = \lim_{n\to\infty} f(t + s_{n_k})$$

is well defined for each $t \in \mathbb{R}$ and

$$\lim_{n\to\infty} g(t - s_{n_k}) = f(t)$$

for each $t \in \mathbb{R}$.

Denote by $\mathcal{AA}_p(\mathbb{R}, X)$ the set of all such functions.

Definition 2.4. A function $f \in \mathcal{PC}(\mathbb{R}\times X, X)$ is called PC-almost automorphic in t uniformly for x in compact subsets of X if

(i) sequence of impulsive moments $\{t_k\}$ is an almost automorphic sequence,
(ii) for every compact subset K of X and every real sequence (s_n), there exists a sub-sequence (s_{n_k}) such that

$$g(t, x) = \lim_{n\to\infty} f(t + s_{n_k}, x)$$

is well defined for each $t \in \mathbb{R}$, $x \in K$ and

$$\lim_{n\to\infty} g(t - s_{n_k}, x) = f(t, x)$$

for each $t \in \mathbb{R}$, $x \in K$.

Denote by $\mathcal{AA}_p(\mathbb{R} \times X, X)$ the set of all such functions.

Definition 2.5. A sequence of continuous functions, $I_k : X \to X$ is almost automorphic, if for integer sequence $\{k_n^{'}\}$, there exists a sub-sequence $\{k_n\}$ such that

$$\lim_{n\to\infty} I_{(k+k_n)}(x) = I_k^*(x)$$

and

$$\lim_{n\to\infty} I^*_{(k-k_n)}(x) = I_k(x)$$

for each k and $x \in X$.

Definition 2.6. A bounded sequence $x : \mathbb{Z}^+ \to X$ is called an almost automorphic sequence, if for every real sequence $(k_n^{'})$, there exists a sub-sequence (k_n) such that

$$y(k) = \lim_{n\to\infty} x(k + k_n)$$

is well defined for each $m \in \mathbb{Z}$ and

$$\lim_{n\to\infty} y(k - k_n) = x(k)$$

for each $k \in \mathbb{Z}^+$. We denote $\mathcal{AAS}(\mathbb{Z}, X)$, the set of all such sequences.

Definition 2.7 *(Solution).* A piece-wise continuous function $x \in PC([t_0, \infty), X)$ and which satisfies the integral equation

$$\begin{aligned} x(t) = S_\alpha(t)x_0 + \int_{t_0}^{t} S_\alpha(t-s)f(s, x(s))\mathrm{d}s, \quad t \in [t_0, t_1] = S_\alpha(t)x_0 \\ + \sum_{t_0<t_k<t} S_\alpha(t-t_k)I_k(x(t_k)) + \sum_{t_0<t_k<t} \int_{t_{k-1}}^{t_k} S_\alpha(t_k-s)f(s, x(s))\mathrm{d}s \\ + \int_{t_k}^{t} S_\alpha(t-s)f(s, x(s))\mathrm{d}s, \quad t \in (t_k, t_{k+}], k = 1, 2, \ldots, \end{aligned} \tag{2.4}$$

is solution of the problem (1.1).

By taking $t_0 \to -\infty$, the solution (2.4) becomes

$$x(t) = \sum_{t_k<t} S_\alpha(t-t_k)I_k(x(t_k)) + \int_{-\infty}^{t} S_\alpha(t-s)f(s, x(s))\mathrm{d}s. \tag{2.5}$$

Similarly, the solutions of problems (1.2) and (1.3) are

$$x(t) = \sum_{t_k<t} S_\alpha(t-t_k)I_k(x(t_k)) + \int_{-\infty}^{t} S_\alpha(t-s)g(s, x_s)\mathrm{d}s. \tag{2.6}$$

and

$$x(t) = \sum_{t_k<t} S_\alpha(t-t_k)I_k(x(t_k)) + \int_{-\infty}^{t} S_\alpha(t-s)h(s,x(s),x_s)\mathrm{d}s. \quad (2.7)$$

respectively.

Definition 2.8 (*Definition 11.1* [24]). Kuratowskii non-compactness measure: Let M be a bounded set in metric space (X, d), then Kuratowskii non-compactness measure, $\mu(M)$ is defined as $\inf\{\epsilon : M$ covered by a finite many sets such that the diameter of each set $\le \epsilon\}$.

Definition 2.9 (*Definition 11.6* [24]). Condensing map: Let $\Phi : X \to X$ be a bounded and continuous operator on Banach space X such that $\mu(\Phi(B)) < \mu(B)$ for all bounded set $B \subset D(\Phi)$, where μ is the Kuratowskii non-compactness measure, then Φ is called condensing map.

Definition 2.10. Compact map: A map $f : X \to X$ is said to be compact if the image of every bounded subset of X under f is pre-compact (closure is compact).

Theorem 2.11 ([21]). *Let B be a convex, bounded and closed subset of a Banach space X and $\Phi : B \to B$ be a condensing map. Then, Φ has a fixed point in B.*

Lemma 2.12 (Example 11.7, [24]). *A map $\Phi = \Phi_1 + \Phi_2 : X \to X$ is k-contraction with $0 \le k < 1$ if*

(a) Φ_1 *is k-contraction, i.e.,* $\|\Phi_1(x) - \Phi_1(y)\|_X \le k\|x - y\|_X$ *and*
(b) Φ_2 *is compact,*

and hence Φ is a condensing map.

3. Existence of PC-Almost Automorphic Solution

In this section, we establish composition theorem for PC-almost automorphic functions. As an application, we study the existence of PC-almost automorphic solution of impulsive fractional functional differential equations. We first prove the existence of PC-almost automorphic solution of Eq. (1.1) by using Sadovskii's fixed-point theorem. Secondly, the existence of the PC-almost automorphic solutions of Eqs. (1.2, 1.3) using Schaefer's fixed-point theorem. The uniqueness of each solution of these three Eqs. (1.1, 1.2, 1.3) is established using Banach's contraction principle.

Lemma 3.1. *Let $I_k : X \to X$ is a sequence of almost automorphic functions and $K \in X$ be a compact subset. If I_k satisfies Lipschitz condition on X, i.e.,*

$$\|I_k(x) - I_k(y)\| \le L\|x - y\|, \forall x, y \in X, \forall k,$$

then the sequence $\{I_k(x) : x \in K\}$ *is almost periodic.*

Lemma 3.2. *Let* $I_k : X \to X$ *is a sequence of almost automorphic functions and* $\phi \in \mathcal{AA}_p(\mathbb{R}, X)$*. If* I_k *satisfies Lipschitz condition on* X*, i.e.,*

$$\|I_k(x) - I_k(y)\| \le L\|x - y\|, \forall x, y \in X, \forall k,$$

then the sequence $\{I_k(\phi(t_k))\}$ *is almost automorphic.*

Proof. From definition of PC-almost automorphic function and almost automorphic sequence, we get

$$\begin{aligned} \|I_{k+k_n}(x(t_{k+k_n})) - I_k^*(x(t_k))\| &\le \|I_{k+k_n}(x(t_{k+k_n})) - I_{k+k_n}(x(t_k))\| \\ &\quad + \|I_{k+k_n}(x(t_k)) - I_k^*(x(t_k))\| \\ &\le L\|x(t_{k+k_n}) - x(t_k)\| \\ &\quad + \|I_{k+k_n}(x(t_k)) - I_k^*(x(t_k))\|. \end{aligned} \tag{3.1}$$

Using Lemma 3.1 and the above expression (3.1), we see that the sequence $\{I_k(\phi(t_k))\}$ is almost automorphic. □

Lemma 3.3. *Composition theorem: Let* $f \in \mathcal{AA}_p(\mathbb{R} \times X, X)$ *and uniformly continuous on any compact set of* $\mathcal{AA}_p(\mathbb{R}, X)$*. If* $\phi \in \mathcal{AA}_p(\mathbb{R}, X)$*, then* $f(\cdot, \phi(\cdot)) \in \mathcal{AA}_p(\mathbb{R} \times X, X)$*.*

Proof. The range of ϕ is relatively compact in $\mathcal{AA}_p(\mathbb{R}, X)$, means $K = \{\phi(t) : t \in \mathbb{R}\}$ is compact. Compactness of K follows from the fact that it is bounded and close in X. Now, from the definition of almost automorphy of f and ϕ for any sub-sequence $\{s_{n_k}\}$ of $\{s_n\}$ there exist functions g and ψ such that

$$\lim_{k\to\infty} f(t + s_{n_k}, x) = g(t, x) \quad \text{and} \quad \lim_{k\to\infty} g(t - s_{n_k}, x) = f(t, x) \forall t \in \mathbb{R}.$$

Using definition of convergence of sequences, for every $\epsilon > 0$ there exists positive integer K_1, K_2 such that

$$\|f(t + s_{n_k}, x) - g(t, x)\| < \frac{\epsilon}{2} \quad \forall k \ge K_1, \forall t \in \mathbb{R} \text{ uniformly in x,} \tag{3.2}$$

and

$$\|\phi(t + s_{n_k}) - \psi(t)\| < \frac{\epsilon}{2} \quad \forall k \ge K_2, \forall t \in \mathbb{R}. \tag{3.3}$$

Using Eqs. (3.2, 3.3), we see that

$$\begin{aligned} \|f(t + s_{n_k}, \phi(t + s_{n_k})) - g(t, \phi(t))\| &\le \|f(t + s_{n_k}, \phi(t+s_{n_k})) - g(t, \phi(t+s_{n_k}))\| \\ &\quad + \|f(t, \phi(t + s_{n_k})) - g(t, \phi(t))\| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

And hence, lemma. □

Lemma 3.4. *Let* $S_\alpha(t)$ *be a strongly continuous family of bounded linear operators such that* $\|S_\alpha(t)\| \le \frac{CM}{1+|\omega|t^\alpha}$*. If* $f : \mathbb{R} \to X$ *is a PC-almost automorphic function then* $\int_{-\infty}^{t} S_\alpha(t - s) f(s) \mathrm{d}s + \sum_{t>t_k} S_\alpha(t - t_k) I_k(x(t_k))$ *is PC-almost automorphic.*

Proof. Let $\{t_{n_k}\}$ be an sub-sequence of an arbitrary sequence $\{t_n\}$. Since f is PC-almost automorphic, there exists g such that

$$\lim_{k\to\infty} f(t+t_{n_k}) = g(t) \quad \forall t \in \mathbb{R}$$

and

$$\lim_{k\to\infty} g(t-t_{n_k}) = f(t) \quad \forall t \in \mathbb{R}.$$

We define

$$F(t) = \int_{-\infty}^{t} S_\alpha(t-s)f(s)\mathrm{d}s + \sum_{t>t_k} S_\alpha(t-t_k)I_k(x(t_k))$$

and

$$G(t) = \int_{-\infty}^{t} S_\alpha(t-s)g(s)\mathrm{d}s + \sum_{t>t_k} S_\alpha(t-t_k)I_k(x(t_k)).$$

Using continuity of $S_\alpha(t)$ and Lebesgue's dominated convergence theorem, wee see that

$$\int_{-\infty}^{t} S_\alpha(t-s)f(s+t_{n_k})\mathrm{d}s \to \int_{-\infty}^{t} S_\alpha(t-s)g(s)\mathrm{d}s. \tag{3.4}$$

Also,

$$\begin{aligned}\sum_{t+t_{n_k}>t_k} S_\alpha(t+t_{n_k}-t_k)I_k(x(t_k)) &= \sum_{t>t_k} S_\alpha(t-t_k)I_k(x(t_k+t_{n_k})) \\ &\to \sum_{t>t_k} S_\alpha(t-t_k)I_k^*(x(t_k)). \end{aligned}\tag{3.5}$$

So, using Eqs. (3.4, 3.5) we see that

$$\lim_{k\to\infty} F(t+t_{n_k}) = G(t) \quad \forall t \in \mathbb{R}$$

Similarly, we can prove that

$$\lim_{k\to\infty} G(t-t_{n_k}) = F(t) \quad \forall t \in \mathbb{R}$$

And hence the result. □

3.1. Impulsive Fractional Differential Equation

Consider the PC-almost automorphic forcing term f of Eq. (1.1) in a perturbed form $f = f_1 + f_2$ which satisfies the following assumptions:

(Af.1) f_1 is bounded and Lipschitz, in particular, there exists a Lesbegue 1-integrable function $M_1 : \mathbb{R} \to \mathbb{R}_+$, a positive constant L_1, such that $\|f_1(t,x)\|_X \le M_1(t)$ and $\|f_1(t,x) - f_1(t,y)\|_X \le L_1\|x-y\|_X$ for all $(t,x),(t,y) \in \mathbb{R}\times X$,

(Af.2) f_2 is compact and bounded, in particular, there exists a Lesbegue 1-integrable function $M_2 : \mathbb{R} \to \mathbb{R}_+$ such that$\|f_2(t,x)\|_X \le M_2(t)$ for all $(t,x) \in \mathbb{R}\times X$,

(AI.3) $I_k \in C(X,X)$ is a sequence of almost automorphic function and satisfies $\|I_k(x) - I_k(y)\|_X \le L_2\|x-y\|_X$ for some positive constant L_2.

Theorem 3.5. *Existence of solution using Sadovskii's fixed-point theorem under the assumptions* $(Af.1), (Af.2), (AI.3)$, *the Eq.* (1.1) *has a PC-almost automorphic solution.*

Proof. Let B_λ be the closed bounded and convex subset of $\mathcal{AA}_p(\mathbb{R}, X)$, where B_λ is defined as $B_\lambda = \{x \in \mathcal{AA}_p(\mathbb{R}, X) : \ \|x\| \leq \lambda\}$.

Define the operator $F : B_\lambda \to \mathcal{AA}_p(\mathbb{R}, X)$ as follows:

$$Fx(t) = \int_{-\infty}^{t} S_\alpha(t-s) f(s, x(s)) \mathrm{d}s + \sum_{t>t_k} S_\alpha(t-t_k) I_k(x(t_k)).$$

Let us consider

$$F_1 x(t) = \int_{-\infty}^{t} S_\alpha(t-s) f_1(s, x(s)) \mathrm{d}s + \sum_{t>t_k} S_\alpha(t-t_k) I_k(x(t_k))$$

$$F_2 x(t) = \int_{-\infty}^{t} S_\alpha(t-s) f_2(s, x(s)) \mathrm{d}s.$$

Now, we need to establish the following results for existence of almost automorphic solution:

(i) F is well defined,
(ii) F is self-mapping,
(ii) F_1 is continuous and contraction,
(iv) F_2 is compact,
(v) F is condensing.

Step 1 F is well defined. Using composition theorem (3.3), we see that for $x \in \mathcal{AA}_p(\mathbb{R}, X), s \to f(s, x(s))$ is almost automorphic and hence bounded. Integrability of $\frac{1}{1+|\omega|t^\alpha}$ on $\mathbb{R}_+$ for $\alpha > 1$ guarantees the existence of $Fx(t)$. And further using Lemma 3.4, we get $Fx \in \mathcal{AA}_p(\mathbb{R}, X)$. Hence F is well defined.

Step 2 F is self-mapping.

$$\|Fx(t)\| \leq \| \int_{-\infty}^{t} S_\alpha(t-s) f(s, x(s)) \mathrm{d}s\| + \| \sum_{t>t_k} S_\alpha(t-t_k) I_k(x(t_k))\|$$

$$(1+|\omega|t^\alpha)\|Fx(t)\| \leq CM_1 \| \int_{-\infty}^{t} \frac{1+|\omega|t^\alpha}{1+|\omega|(t-s)^\alpha} M^*(s) \mathrm{d}s\| + (1+|\omega|t^\alpha) \times \| \sum_{t>t_k} S_\alpha(t-t_k)(I_k(x(t_k))\|,$$

where $M^*(t) = M_1(t) + M_2(t)$

$$(1+|\omega|t^\alpha)\|Fx(t)\| \leq CM2^\alpha \| \int_{-\infty}^{t} (1+|\omega|s^\alpha) M^*(s) \mathrm{d}s\| + (1+|\omega|t^\alpha) \times \| \sum_{t>t_k} S_\alpha(t-t_k) I_k(x(t_k))\|,$$

$$(\because \frac{1+|\omega|t^\alpha}{1+|\omega|(t-s)^\alpha} \leq 2^\alpha (1+|\omega|s^\alpha))$$

$$\begin{aligned}(1+|\omega|t^{\alpha})\|Fx(t)\| &\leq CM2^{\alpha}(1+|\omega|t^{\alpha})\|M^{*}\|_{1}\\ &\quad +(1+|\omega|t^{\alpha})\|\sum_{t>t_k} S_{\alpha}(t-t_k)I_k(x(t_k))\|\|Fx(t)\|\\ &\leq CM2^{\alpha}(1+|\omega|t^{\alpha})\|M^{*}\|_{1}\\ &\quad +\|\sum_{t>t_k} S_{\alpha}(t-t_k)(I_k(x(t_k))-I_k(0))\|\\ &\quad +\|\sum_{t>t_k} S_{\alpha}(t-t_k)I_k(0)\|\\ \|Fx(t)\| &\leq C_1+\lambda L_2\sum_{t>t_k}\frac{1}{1+|\omega|(t-t_k)^{\alpha}}\\ &\leq \lambda, C_1=CM2^{\alpha}(1+|\omega|t^{\alpha})\|M^{*}\|_1+\|\sum_{t>t_k} S_{\alpha}(t-t_k)I_k(0)\|.\end{aligned}$$

Also, from Lemma 3.4, Fx is PC-almost automorphic. So, $F(B_{\lambda}) \subset B_{\lambda}$. Hence, F is self-mapping.

Step 3 F_1 is continuous.

$$\begin{aligned}&\|F_1x^n(t)-F_1x(t)\|\\ &\quad=\|\int_{-\infty}^{t} S_{\alpha}(t-s)f_1(s,x^n(s))\mathrm{d}s+\sum_{t>t_k} S_{\alpha}(t-t_k)I_k(x^n(t_k))\\ &\qquad-\int_{-\infty}^{t} S_{\alpha}(t-s)\times f_1(s,x(s))\mathrm{d}s+\sum_{t>t_k} S_{\alpha}(t-t_k)I_k(x(t_k))\|\\ &\quad\leq\|\int_{-\infty}^{t} S_{\alpha}(t-s)(f_1(s,x^n(s))-f_1(s,x(s)))\mathrm{d}s\|\\ &\qquad+\|\sum_{t>t_k} S_{\alpha}(t-t_k)(I_k(x^n(t_k))-I_k(x(t_k)))\|\\ &\quad\leq\Big(\frac{\pi CML_1|\omega|^{-\frac{1}{\alpha}}}{\alpha\sin(\frac{\pi}{\alpha})}+L_2\sum_{t>t_k}\frac{1}{1+|\omega|(t-t_k)^{\alpha}}\Big)\|x^n-x\|.\end{aligned}$$

Therefore, as $n\to\infty, Fx^n\to Fx$, hence the F is continuous on B_{λ}.

Step 4 F_1 is contraction.

$$\begin{aligned}\|F_1x(t)-F_1y(t)\| &= \|\int_{-\infty}^{t} S_{\alpha}(t-s)(f_1(s,x(s))-f_1(s,y(s)))\mathrm{d}s\\ &\quad+\sum_{t>t_k} S_{\alpha}(t-t_k)\times(I_k(x(t_k))-I_k(x(t_k)))\|\\ &=\Big(\int_{-\infty}^{t}\|S_{\alpha}(t-s)\|L(s)\mathrm{d}s+l\sum_{t>t_k}\|S_{\alpha}(t-t_k)\|\Big)\|x-y\|\\ &\leq\Big(\frac{\pi CML_1|\omega|^{-\frac{1}{\alpha}}}{\alpha\sin(\frac{\pi}{\alpha})}+L_2\sum_{t>t_k}\frac{1}{1+|\omega|(t-t_k)^{\alpha}}\Big)\|x-y\|.\end{aligned}$$

Therefore, F_1 is contraction on B_λ provided, provided that $\left(\frac{\pi CML_1|\omega|^{-\frac{1}{\alpha}}}{\alpha\sin(\frac{\pi}{\alpha})}+L_2\sum_{t>t_k}\frac{1}{1+|\omega|(t-t_k)^\alpha}\right)<1$.

Step 5 F_2 is compact.

For $t_k<\tau_1<\tau_2\leq t_{k+1}$.

$$\begin{aligned}
&\|F_2x(\tau_2)-F_2x(\tau_1)\|\\
&=\|\int_{-\infty}^{\tau_2}S_\alpha(\tau_2-s)f_2(s,x(s))\mathrm{d}s-\int_{-\infty}^{\tau_1}S_\alpha(\tau_1-s)f_2(s,x(s))\mathrm{d}s\|\\
&\leq\|\int_{-\infty}^{\tau_1}(S_\alpha(\tau_2-s)-S_\alpha(\tau_1-s))f_2(s,x(s))\mathrm{d}s\|\\
&\quad+\|\int_{\tau_1}^{\tau_2}S_\alpha(\tau_2-s)f_2(s,x(s))\mathrm{d}s\|\\
&\leq\|\int_{-\infty}^{\tau_1}S_\alpha(\tau_1-s)(f_2(\tau_2-s,x(\tau_2-s))-f_2(\tau_1-s,x(\tau_1-s)))\mathrm{d}s\|\\
&\quad+\|\int_{\tau_1}^{\tau_2}S_\alpha(\tau_2-s)f_2(s,x(s))\mathrm{d}s\|\\
&\leq\|f_2(\tau_2-\cdot,x(\tau_2-\cdot))-f_2(\tau_1-\cdot,x(\tau_1-\cdot))\|\frac{\pi CM|\omega|^{-\frac{1}{\alpha}}}{\alpha\sin(\frac{\pi}{\alpha})}\\
&\quad+\|\int_{\tau_1}^{\tau_2}S_\alpha(\tau_2-s)f_2(s,x(s))\mathrm{d}s\|.
\end{aligned}$$

We can easily establish that the right-hand side of the above expression does not depend on x and $\to 0$ as $\tau_2\to\tau_1$. Thus using Arzela–Ascoli theorem for equi-continuous functions (Diethelm, Theorem D.10 [6]), we conclude that $F_2(B_\lambda)$ is relatively compact and hence F_2 is completely continuous. And thus F_2 is compact.

Step 6 F is condensing.

As $F=F_1+F_2, F_1$ is continuous, contraction and F_2 is compact, so using the Lemma 2.12, F is condensing map on B_r.

And hence using the Theorem 2.11, we conclude that Eq. (1.1) has a PC-almost automorphic solution in B_λ. □

Theorem 3.6. *If f is bounded and Lipschitz, in particular, $\|f(t,x)-f(t,y)\|_X\leq L_1^*\|x-y\|_X$ for all $(t,x),(t,y)\in R$, then the problem (1.1) has a unique solution in B_λ, provided that*

$$\left(\frac{\pi CML_1^*|\omega|^{-\frac{1}{\alpha}}}{\alpha\sin(\frac{\pi}{\alpha})}+L_2\sum_{t>t_k}\frac{1}{1+|\omega|(t-t_k)^\alpha}\right)<1.$$

Proof. We define an operator $F:\mathcal{AA}_p(\mathbb{R},X)\to\mathcal{AA}_p(\mathbb{R},X)$ as follows:

$$Fx(t)=\int_{-\infty}^{t}S_\alpha(t-s)f(s,x(s))\mathrm{d}s+\sum_{t>t_k}S_\alpha(t-t_k)I_k(x(t_k)).$$

Step 1 F is well defined. The proof is similar to the proof in Step 1 of Theorem 3.5.
Step 2 F is self-mapping. The proof is similar to the proof in Step 2 of Theorem 3.5.
Step 3 F is continuous. The proof is similar to the proof in Step 3 of Theorem 3.5.
Step 4 F is contraction. The proof is similar to the proof in Step 4 of Theorem 3.5.

Now by applying Banach's contraction principle, we get the operator has a unique fixed point in $\mathcal{AA}_p(\mathbb{R}, X)$. And hence, the problem (1.1) has a unique solution in $\mathcal{AA}_p(\mathbb{R}, X)$. □

3.2. Impulsive Fractional Functional Differential Equation

Consider the PC-almost automorphic forcing term g of Eq. (1.2) and which satisfies the following assumptions.

(Ag.1) $g \in C(I \times \mathcal{C}_r, X)$ also g is bounded, in particular, there exists a positive constant M_3, such that $\|f(t,\phi)\|_X \leq M_3(1 + \|\phi\|_r)$ for all $(t,\phi) \in \mathbb{R} \times \mathcal{C}_r$,

(AJ.3) $J_k \in C(X, X)$ is a sequence of almost automorphic function and satisfies $\|J_k(x) - J_k(y)\|_X \leq L_2^* \|x - y\|_X$ for some positive constant L_2^* and $\forall x, y \in X$.

Theorem 3.7. *Existence of solution using Schaefer's fixed-point theorem: Under the assumptions* $(Ag.1), (AJ.3)$, *the Eq.* (1.2) *has a PC-almost automorphic solution.*

Proof. Define the operator $G : \mathcal{AA}_p(\mathbb{R}, X) \to \mathcal{AA}_p(\mathbb{R}, X)$ by

$$Gx(t) = \int_{-\infty}^{t} S_\alpha(t-s) g(s, x_s) \mathrm{d}s + \sum_{t > t_k} S_\alpha(t - t_k) J_k(x(t_k)).$$

We establish the following:

(i) G is well defined,
(ii) G is continuous,
(iii) G maps bounded set into bounded set,
(iv) G maps bounded set into equi-continuous set,
(v) there exists a priori bound.

Step 1 G is well defined. Using composition theorem (3.3), we see that for $x \in \mathcal{AA}_p(\mathbb{R}, X), s \to g(s, x(s))$ is almost automorphic and hence bounded. Integrability of $\frac{1}{1+|\omega|t^\alpha}$ on $\mathbb{R}_+$ for $\alpha > 1$ guarantees the existence of $Gx(t)$. And further using Lemma 3.4, we get $Gx \in \mathcal{AA}_p(\mathbb{R}, X)$. Hence G is well defined.

Step 2 G is continuous.

$$\|Gx^n(t) - Gx(t)\|$$

$$= \| \int_{-\infty}^{t} S_\alpha(t-s) g(s, x_s^n) \mathrm{d}s + \sum_{t>t_k} S_\alpha(t-t_k) J_k(x^n(t_k))$$
$$- \int_{-\infty}^{t} S_\alpha(t-s) g(s, x_s) \mathrm{d}s + \sum_{t>t_k} S_\alpha(t-t_k) J_k(x(t_k)) \|$$
$$\leq \| \int_{-\infty}^{t} S_\alpha(t-s)(g(s, x_s^n) - g(s, x_s)) \mathrm{d}s \|$$
$$+ \| \sum_{t>t_k} S_\alpha(t-t_k)(J_k(x^n(t_k)) - J_k(x(t_k))) \|$$
$$\leq \frac{\pi C M |\omega|^{-\frac{1}{\alpha}}}{\alpha \sin(\frac{\pi}{\alpha})} \|(g(\cdot, x_{(\cdot)}^n) - g(\cdot, x_{(\cdot)}))\| + L_2^* \sum_{t>t_k} \frac{1}{1+|\omega|(t-t_k)^\alpha} \|x^n - x\|.$$

Therefore, as $n \to \infty, Gx^n \to Gx$, hence the G is continuous on $\mathcal{AA}_p(\mathbb{R}, X)$.

Step 3 G maps bounded set into bounded set. It is enough to prove that for any $\delta > 0$, there exists $r =$ such that $x \in B = \{x \in \mathcal{AA}_p(\mathbb{R}, X) | \|x\| < \delta\}$ and we have $\|Gx\| \leq \lambda$. For $t \in \mathbb{R}$, we have

$$\|Gx(t)\| \leq \| \int_{-\infty}^{t} S_\alpha(t-s) g(s, x_s) \mathrm{d}s \| + \| \sum_{t>t_k} S_\alpha(t-t_k) J_k(x(t_k)) \|$$
$$\leq \frac{\pi C M M_3 (1+\|x\|) |\omega|^{-\frac{1}{\alpha}}}{\alpha \sin(\frac{\pi}{\alpha})} + L_2^* \sum_{t>t_k} \frac{1}{1+|\omega|(t-t_k)^\alpha} \|(x(t_k))\| + C_2.$$

Step 4 G maps bounded set into equi-continuous set. For $t_k < \tau_1 < \tau_2 \leq t_{k+1}$.

$$\|Gx(\tau_2) - Gx(\tau_1)\|$$
$$= \| \int_{-\infty}^{\tau_2} S_\alpha(\tau_2 - s) g(s, x_s) \mathrm{d}s - \int_{-\infty}^{\tau_1} S_\alpha(\tau_1 - s) g(s, x_s) \mathrm{d}s \|$$
$$\leq \| \int_{-\infty}^{\tau_1} (S_\alpha(\tau_2 - s) - S_\alpha(\tau_1 - s)) g(s, x_s) \mathrm{d}s \| + \| \int_{\tau_1}^{\tau_2} S_\alpha(\tau_2 - s) g(s, x_s) \mathrm{d}s \|$$
$$\leq \| \int_{-\infty}^{\tau_1} S_\alpha(\tau_1 - s)(g(\tau_2 - s, x_{(\tau_2 - s)}) - g(\tau_1 - s, x_{(\tau_1 - s)})) \mathrm{d}s \|$$
$$+ \| \int_{\tau_1}^{\tau_2} S_\alpha(\tau_2 - s) g(s, x_s) \mathrm{d}s \| \leq \| g(\tau_2 - \cdot, x_{(\tau_2 - \cdot)})$$
$$- g(\tau_1 - \cdot, x_{(\tau_1 - \cdot)}) \| \frac{\pi C M |\omega|^{-\frac{1}{\alpha}}}{\alpha \sin(\frac{\pi}{\alpha})} + \| \int_{\tau_1}^{\tau_2} S_\alpha(\tau_2 - s) g(s, x_s) \mathrm{d}s \|.$$

We can easily establish that the right-hand side of the above expression does not depend on x and $\to 0$ as $\tau_2 \to \tau_1$. Thus, using Arzela–Ascoli theorem for equi-continuous functions (Diethelm, Theorem D.10 [6]), we conclude that G is completely continuous. And thus, G is compact.

Step 5 A priori bound: now we prove that the set,

$$E(G) = \{x \in \mathcal{AA}_p(\mathbb{R}, X) | x = \lambda Gx \text{ for some } \lambda \in (0, 1)\}$$

is bounded.

We observe that for $t \in \mathbb{R}$ and $x \in E(G), x(t) = \lambda Gx(t)$ and

$$\begin{aligned}\|x(t)\| &\leq \|Gx(t)\| \\ &\leq \| \int_{-\infty}^{t} S_\alpha(t-s)g(s,x_s)\mathrm{d}s\| + \| \sum_{t>t_k} S_\alpha(t-t_k)J_k(x(t_k))\| \\ &\leq \frac{\pi CMM_3(1+\|x\|)|\omega|^{-\frac{1}{\alpha}}}{\alpha \sin(\frac{\pi}{\alpha})} + L_2 \sum_{t>t_k} \frac{1}{1+|\omega|(t-t_k)^\alpha} \|(x(t_k))\| + C_2\end{aligned}$$

$$\|x\| \leq \frac{\frac{\pi CMM_3|\omega|^{-\frac{1}{\alpha}}+C_2}{\alpha \sin(\frac{\pi}{\alpha})}}{1 - \frac{\alpha \sin(\frac{\pi}{\alpha})|\omega|^{\frac{1}{\alpha}}}{\pi CMM_3} - L_2^* \sum_{t>t_k} \frac{1}{1+|\omega|(t-t_k)^\alpha}}.$$

This shows that $E(G)$ is bounded. Hence, using Schaefer's fixed-point theorem, the Eq. (1.2) has at least one solution. □

Theorem 3.8. *If g is bounded and Lipschitz, in particular, there exists a positive constant L_3, such that $\|g(t,\phi) - g(t,\psi)\|_X \leq L_3\|\phi - \psi\|_r$ for all (t,ϕ), $(t,\psi) \in \mathbb{R} \times \mathcal{C}_r$, then the problem (1.2) has a unique solution in B_λ, provided that*

$$\left(\frac{\pi CML_3|\omega|^{-\frac{1}{\alpha}}}{\alpha \sin(\frac{\pi}{\alpha})} + L_2^* \sum_{t>t_k} \frac{1}{1+|\omega|(t-t_k)^\alpha}\right) < 1.$$

Proof. The proof is similar to the proof of Theorem 3.6. □

Consider the PC-almost automorphic forcing term h of Eq. (1.3), which satisfies the following assumptions.

(Ah.1) $h \in C(I \times X \times \mathcal{C}_r, X)$ also h is bounded, in particular, there exists positive constant M_4 such that $\|h(t,x,\phi)\|_X \leq M_4(1 + \|x\|_X + \|\phi\|_r)$ for all $(t,x,\phi) \in I \times X \times \mathcal{C}_r$.

(AJ.3) $J_k \in C(X,X)$ is a sequence of almost automorphic function and satisfies $\|J_k(x) - J_k(y)\|_X \leq L_2^*\|x-y\|_X$, for some positive constant L_2^* and $\forall x \in X$.

Theorem 3.9. *Under the assumptions $(Ah.1), (AJ.3)$,, the Eq. (1.3) has a PC-almost automorphic solution.*

Proof. The proof is similar to the proof of Theorem 3.7. □

Theorem 3.10. *If h is bounded and Lipschitz, in particular, there exists a positive constants L_4, L_5, such that $\|h(t,x,\phi) - h(t,y,\psi)\|_X \leq L_4\|x-y\|_X + L_5\|\phi - \psi\|_r$ for all $(t,x,\phi), (t,y,\psi) \in \mathbb{R} \times X \times \mathcal{C}_r$, then the problem (1.3) has a unique solution in B_λ, provided that*

$$\left(\frac{\pi CM(L_4+L_5)|\omega|^{-\frac{1}{\alpha}}}{\alpha \sin(\frac{\pi}{\alpha})} + L_2^* \sum_{t>t_k} \frac{1}{1+|\omega|(t-t_k)^\alpha}\right) < 1.$$

Proof. The proof is similar to the proof of Theorem 3.6. □

4. Examples

To illustrate the analytical findings of our work, we consider the following examples of fractional order $\alpha \in (1, 2)$:

Example 4.1. Consider the following fractional relaxation oscillation equation,

$$\begin{aligned} \partial_t^\alpha u(t,y) &= \partial_y^2 u(t,y) - \mu u(t,y) + \partial_t^{\alpha-1}(\beta u(t,y) \\ &\quad \times (cos t + cos\sqrt{2}t + \beta \mathrm{e}^{-|t|} sin(u(t,y))), \\ \Delta u(t_k, y) &= I_k(u(t_k, y)), \\ u(t,0) &= u(t,\pi) = 0, \forall t \in \mathbb{R}, y \in [0,\pi], \\ u(0,\xi) &= u_0(\xi), \xi \in [0,\pi], \mu > 0. \end{aligned} \tag{4.1}$$

Let $x(t)y = u(t, y)$ and assume that

$$f(t, x(t)) = cos t + cos\sqrt{2}t + \beta \mathrm{e}^{-|t|} sin(u(t,y))$$

is a continuous function with respect to t and satisfies Lipschitz condition in x.

Define the operators

$$\begin{aligned} Ax &= \frac{\partial^2 x}{\partial y^2} - \omega x \quad \text{with domain} \\ D(A) &= \{x \in L^2(0,\pi) : x, x^{'} \text{ are absolutely continuous and} \\ &\qquad x, x^{'}, x^{''} \in L^2(0,\pi)\}. \end{aligned}$$

It is well known that for $\alpha = 1$, the sectorial operator $A = \frac{\partial^2}{\partial y^2} - \omega$ generates an analytic semigroup and for $\alpha = 2$, the sectorial operator $A = \frac{\partial^2}{\partial y^2} - \omega$ generates a cosine family of operators. Therefore, the above problem can be posed as an abstract problem (1.1) defined on $X = L^2(0, \pi)$. We assume that our function satisfies all the assumptions of Theorem 3.5. Hence, the problem (4.1) has a PC-almost automorphic solution.

Example 4.2. Consider the following delay fractional relaxation oscillation equation,

$$\begin{aligned} \partial_t^\alpha u(t,y) &= \partial_y^2 u(t,y) - pu(t,y) + \partial_t^{\alpha-1}(cos t + cos\sqrt{2}t \\ &\quad + sin\left(\int_{-\infty}^t \mathrm{e}^{(t-s)} u(s,y)\mathrm{d}s\right), \\ \Delta u(t_k, y) &= I_k(u(t_k, y)), \\ u(t,0) &= u(t,\pi) = 0, \forall t \in \mathbb{R}, y \in [0,\pi], \\ u(0,\xi) &= u_0(\xi), \xi \in [0,\pi], \tau, p > 0, \\ u(t,y) &= \phi(t,y), t \in [-\pi, 0]. \end{aligned} \tag{4.2}$$

Let $x(t)y = u(t, y)$ and assume that

$$f(t, x_t) = cos t + cos\sqrt{2}t + sin\left(\int_{-\infty}^t \mathrm{e}^{(t-s)} u(s,y)\mathrm{d}s\right)$$

is a continuous function with respect to t and satisfies Lipschitz condition in x_t.

Define the operators

$$\begin{aligned} Ax &= \frac{\partial^2 x}{\partial y^2} - \omega x \quad \text{with domain} \\ D(A) &= \{x \in L^2(0,\pi) : x, x^{'} \text{ are absolutely continuous and} \\ &\qquad x, x^{'}, x^{''} \in L^2(0,\pi)\}. \end{aligned}$$

It is well known that for $\alpha = 1$, the sectorial operator $A = \frac{\partial^2}{\partial y^2} - \omega$ generates an analytic semigroup and for $\alpha = 2$, the sectorial operator $A = \frac{\partial^2}{\partial y^2} - \omega$ generates a cosine family of operators. Therefore, the above problem can be posed as an abstract problem (1.2) defined on $X = L^2(0,\pi)$. We assume that our function satisfies all the assumptions of Theorem 3.7. Hence the problem (4.2) has a PC-almost automorphic solution.

Example 4.3. Consider the following fractional relaxation oscillation equation,

$$\begin{aligned} \partial_t^\alpha u(t,y) &= \partial_y^2 u(t,y) - \mu u(t,y) + \partial_t^{\alpha-1}(\beta u(t,y)(cost + cos\sqrt{2}t \\ &\quad + \beta \mathrm{e}^{-|t|} sin(u(t,y)) + sin\left(\int_{-\infty}^t \mathrm{e}^{(t-s)} u(s,y)\mathrm{d}s\right), \\ \Delta u(t_k,y) &= I_k(u(t_k,y)), \\ u(t,0) &= u(t,\pi) = 0, \forall t \in \mathbb{R}, y \in [0,\pi], \\ u(0,\xi) &= u_0(\xi), \xi \in [0,\pi], \mu > 0. \end{aligned} \tag{4.3}$$

Let $x(t)y = u(t,y)$ and assume that

$$f(t,x(t),x_t) = cost + cos\sqrt{2}t + \beta \mathrm{e}^{-|t|} sin(u(t,y)) + sin\left(\int_{-\infty}^t \mathrm{e}^{(t-s)} u(s,y)\mathrm{d}s\right)$$

is a continuous function with respect to t and satisfies Lipschitz condition in $x \text{and} x_t$.

Define the operators

$$\begin{aligned} Ax &= \frac{\partial^2 x}{\partial y^2} - \omega x \quad \text{with domain} \\ D(A) &= \{x \in L^2(0,\pi) : x, x^{'} \text{ are absolutely continuous and} \\ &\qquad x, x^{'}, x^{''} \in L^2(0,\pi)\}. \end{aligned}$$

It is well known that for $\alpha = 1$, the sectorial operator $A = \frac{\partial^2}{\partial y^2} - \omega$ generates an analytic semigroup and for $\alpha = 2$, the sectorial operator $A = \frac{\partial^2}{\partial y^2} - \omega$ generates a cosine family of operators. Therefore, the above problem can be posed as an abstract problem (1.3) defined on $X = L^2(0,\pi)$. We assume that our function satisfies all the assumptions of Theorem 3.9. Hence, the problem (4.3) has a PC-almost automorphic solution.

Example 4.4. Consider a fractional relaxations oscillation equation

$$\begin{aligned} \partial_t^\alpha u(t,y) &= \partial_y^2 u(t,y) + \partial_t^{\alpha-1}(f_1(t,u(t,y)) \\ &\qquad + \int_{-r}^t h(t-s) f_2(s,u(s,y))\mathrm{d}s), \quad t\in\mathbb{R},\ y\in\Omega=[0,\pi], \\ x(t,0) &= x(t,\pi)=0 \quad t\in\mathbb{R}, \\ \Delta u(t_k,y) &= -u(t_k,y) \quad k=1,2,\ldots, \\ u(s,y) &= \phi(s,y) \quad s\in[-r,0]. \end{aligned} \tag{4.4}$$

Let $x(t)y = u(t,y)$ and assume that

$$f(t,x(t),x_t) = f_1(t,u(t,y)) + \int_{-r}^t h(t-s) f_2(s,u(s,y))\mathrm{d}s$$

is a continuous function with respect to $t \neq t_k$ and PC-almost automorphic. Also, f_1 satisfies Lipschitz condition in x and x_t.

Define the operators

$$\begin{aligned} Ax &= \frac{\partial^2 x}{\partial y^2} - \omega x \quad \text{with domain} \\ D(A) &= \{x \in L^2(0,\pi) : x, x^{'} \text{ are absolutely continuous and} \\ &\qquad x, x^{'}, x^{''} \in L^2(0,\pi)\}. \end{aligned}$$

It is well known that for $\alpha = 1$, the sectorial operator $A = \frac{\partial^2}{\partial y^2} - \omega$ generates an analytic semigroup and for $\alpha = 2$, the sectorial operator $A = \frac{\partial^2}{\partial y^2} - \omega$ generates a cosine family of operators. Therefore, the above problem can be posed as an abstract problem (1.3) defined on $X = L^2(0,\pi) = U$. We assume that our function satisfies all the assumptions of Theorem 3.9. Hence, the problem (4.4) has a PC-almost automorphic solution.

5. Discussion

Since, the introduction of almost periodic functions by Bohr [4], there have been various important generalization of this concept. One important generalization is the concept of almost automorphic functions which was introduced by Bochner. After that, we encounter several important generalization of these functions like:

i. Pseudo-almost periodic and automorphic,
ii. Weighted pseudo-almost periodic and automorphic,
iii. Stepanov almost periodic and automorphic,
iv. Stepanov type pseudo-almost periodic and automorphic,
v. Stepanov type weighted pseudo-almost periodic and automorphic,

and many more. For origin and details of these functions, one can see [4] and references therein. The application of these functions in the area of differential equations attracted many mathematicians and extensive research have been done. In recent year, the applications of these functions in the field of fractional differential equations got a lot of attention after the introduction of the α-resolvent family of bounded linear operators, $S_\alpha(t)$, for detail, we refer to [14, 15].

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