

On the First Stability Eigenvalue of Constant Mean Curvature Surfaces Into Homogeneous 3-Manifolds

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Abstract. We find out upper bounds for the first eigenvalue of the stability operator for compact constant mean curvature surfaces immersed into certain 3-dimensional Riemannian spaces, in particular into homogeneous 3-manifolds. As an application we derive some consequences for strongly stable surfaces in such ambient spaces. Moreover, we also get a characterization of Hopf tori in certain Berger spheres.

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1. Introduction

Let $\psi : \Sigma^2 \rightarrow M^3$ be a compact surface immersed into a 3-dimensional Riemannian space. We will assume that Σ is two-sided, which means that there exists a unit normal vector field N globally defined on Σ , and will denote by A its second fundamental form (with respect to N) and by H its mean curvature, $H = (1/2) \operatorname{tr}(A)$. Every smooth function $f \in C^\infty(\Sigma)$ induces a normal variation ψ_t of the immersion ψ , with variational normal field fN and first variation of the area functional $\mathcal{A}(t)$ given by

$$\delta_f \mathcal{A} = \frac{d\mathcal{A}}{dt}(0) = -2 \int_{\Sigma} f H.$$

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As a consequence, minimal surfaces ($H = 0$) are characterized as critical points of the area functional whereas constant mean curvature (CMC) surfaces are critical points of the area functional restricted to smooth functions f which satisfy the additional condition $\int_{\Sigma} f = 0$. Geometrically, such additional condition means that the variations under consideration preserve a certain volume function.

For such critical points, the stability of the corresponding variational problem is given by the second variation of the area functional,

$$\delta_f^2 \mathcal{A} = \frac{d^2 \mathcal{A}}{dt^2}(0) = - \int_{\Sigma} f J f,$$

with $Jf = \Delta f + (|A|^2 + \overline{\text{Ric}}(N, N)) f$, where Δ stands for the Laplacian operator on Σ and $\overline{\text{Ric}}$ denotes the Ricci curvature of M^3 . The surface Σ is said to be strongly stable if $\delta_f^2 \mathcal{A} \geq 0$, for every $f \in C^\infty(\Sigma)$. The operator $J = \Delta + |A|^2 + \overline{\text{Ric}}(N, N)$ is called the Jacobi or stability operator of the surface, and it is a Schrödinger operator. As it is well known, the spectrum of J

$$\text{Spec}(J) = \{\lambda_1 < \lambda_2 < \lambda_3 < \dots\}$$

consists of an increasing sequence of eigenvalues λ_k with finite multiplicities m_k and such that $\lim_{k \rightarrow \infty} \lambda_k = +\infty$. Moreover, the first eigenvalue is simple ($m_1 = 1$) and it satisfies the following min–max characterization

$$\lambda_1 = \min \left\{ \frac{- \int_{\Sigma} f J(f)}{\int_{\Sigma} f^2} : f \in C^\infty(\Sigma), f \neq 0 \right\}. \tag{1.1}$$

Observe that a real number λ is an eigenvalue of J if and only if $J(f) + \lambda f = 0$ for some smooth function $f \in C^\infty(\Sigma)$, $f \neq 0$. In terms of the spectrum, Σ is strongly stable if and only if $\lambda_1 \geq 0$.

Our objective in this paper is to establish estimates for λ_1 and some characterizations for compact CMC surfaces into 3-dimensional Riemannian spaces with sectional curvature bounded from below. In particular, we find out upper bounds for λ_1 for compact constant mean curvature surfaces into homogeneous 3-manifolds. As an application we derive some consequences for strongly stable surfaces in such ambient spaces. Moreover, we get also a characterization of Hopf tori in certain Berger spheres and into the product $S^2 \times S^1$.

Related to our results, let us recall that in 1968, Simons [11] found out an estimate for the first eigenvalue of J on any compact minimal hypersurface in the standard sphere. In particular, for minimal compact surfaces in the 3-sphere he proved that $\lambda_1 = -2$ if the surface is totally geodesic and $\lambda_1 \leq -4$ otherwise. Later on, Wu [16] characterized the equality by showing that it holds only for the minimal Clifford torus. More recently, Perdomo [9] gave a new proof of this spectral characterization by getting an interesting formula that relates the first eigenvalue λ_1 , the genus of the surface, the area and a simple invariant. Finally, Alías, Barros and Brasil [2] extended Wu and Perdomo’s results to the case of CMC hypersurfaces in the standard sphere, characterizing some CMC Clifford tori.

The standard 3-sphere is a simply connected space form (the compact one). Besides them, the most regular Riemannian 3-manifolds are the homogeneous ones and between them, the only compact are the Berger spheres and their quotients. In the last years, the CMC surfaces of the homogeneous Riemannian 3-manifolds have been deeply studied (the starting point was the work of Abresch and Rosenberg [1]) and several stability results have been proved (see [12] and [15]). See also [3] and [13] for other recent interesting characterization and classification results for CMC surfaces in such ambient spaces.

2. First Results

Let $\psi : \Sigma^2 \rightarrow M^3$ be a compact two-sided surface with constant mean curvature H immersed into a 3-dimensional Riemannian space. Let us introduce the so-called traceless second fundamental form of Σ , that is, the tensor ϕ given by $\phi = A - HI$, where I denotes the identity operator on $\mathfrak{X}(\Sigma)$. Observe that $\text{tr}(\phi) = 0$ and $|\phi|^2 = |A|^2 - 2H^2 \geq 0$, with equality if and only if Σ is totally umbilical. For that reason ϕ is also called the total umbilicity tensor of Σ . In terms of ϕ , the Jacobi operator is given by

$$J = \Delta + |\phi|^2 + 2H^2 + \overline{\text{Ric}}(N, N).$$

Now let us choose a first positive eigenfunction $\rho \in C^\infty(\Sigma)$ of the stability operator. Thus $J\rho = -\lambda_1\rho$ or, equivalently,

$$\Delta\rho = -(\lambda_1 + |A|^2 + \overline{\text{Ric}}(N, N))\rho. \tag{2.1}$$

Extending Perdomo’s ideas [9, Section 3] to our more general case, one can compute

$$\Delta \log \rho = \rho^{-1} \Delta \rho - \rho^{-2} |\nabla \rho|^2 = -(\lambda_1 + |A|^2 + \overline{\text{Ric}}(N, N)) - \rho^{-2} |\nabla \rho|^2,$$

and integrate on Σ to find

$$\alpha = \int_{\Sigma} \rho^{-2} |\nabla \rho|^2 = -\lambda_1 \text{Area}(\Sigma) - \int_{\Sigma} (|A|^2 + \overline{\text{Ric}}(N, N)),$$

where $\alpha \geq 0$ defines a simple invariant that is independent of the choice of ρ because λ_1 is simple. In other words

$$\lambda_1 = -\frac{1}{\text{Area}(\Sigma)} \left(\alpha + \int_{\Sigma} (|A|^2 + \overline{\text{Ric}}(N, N)) \right).$$

Now from the Gauss equation, we obtain a relation between the norm of the shape operator $|A|^2$, the sectional curvature \overline{K}_Σ of the tangent plane to Σ in M^3 , and the Gaussian curvature K of the surface as $|A|^2 = 2(2H^2 + \overline{K}_\Sigma - K)$ and, by the Gauss–Bonnet Theorem, the above formula becomes

$$\lambda_1 = -4H^2 - \frac{1}{\text{Area}(\Sigma)} \left(\alpha + 8\pi(g - 1) + \int_{\Sigma} (2\overline{K}_\Sigma + \overline{\text{Ric}}(N, N)) \right). \tag{2.2}$$

As a first consequence, we can establish the following result.

Theorem 2.1. *Let M^3 be a 3-dimensional Riemannian space with sectional curvature \overline{K} bounded from below by a constant c . Consider Σ^2 a compact two-sided surface immersed into M^3 with constant mean curvature H , and let λ_1 stand for the first eigenvalue of its Jacobi operator. Then*

- (i) $\lambda_1 \leq -2(H^2 + c)$, with equality if and only if Σ is totally umbilic in M^3 and the normal direction to Σ is a direction of minimum Ricci curvature of M^3 equals $2c$,
- (ii) $\lambda_1 \leq -4(H^2 + c) - 8\pi(g - 1)/\text{Area}(\Sigma)$, where g denotes the genus of Σ . Moreover, equality holds if and only if Σ has constant Gaussian curvature, $\overline{K}_\Sigma = c$ and the normal direction to Σ is a direction of minimum Ricci curvature of M^3 equals $2c$.

Proof. Taking the constant function $f = 1$ as a test function in (1.1) to estimate λ_1 , one easily gets that

$$\begin{aligned} \lambda_1 &\leq \frac{-\int_\Sigma 1J1}{\int_\Sigma 1^2} = -2H^2 - \frac{1}{\text{Area}(\Sigma)} \int_\Sigma \overline{\text{Ric}}(N, N) - \frac{1}{\text{Area}(\Sigma)} \int_\Sigma |\phi|^2 \\ &\leq -2(H^2 + c) - \frac{1}{\text{Area}(\Sigma)} \int_\Sigma |\phi|^2 \leq -2(H^2 + c) \end{aligned} \tag{2.3}$$

Moreover, if the equality holds then $\phi = 0$ on Σ and $\overline{\text{Ric}}(N, N) = 2c$. So Σ is totally umbilic and because of $2c \leq \overline{\text{Ric}}(X, X)$ for all $X \in \mathfrak{X}(M^3)$, the normal direction to Σ is a direction of minimum Ricci curvature of M^3 . Conversely, if Σ is totally umbilic and $\overline{\text{Ric}}(N, N) = 2c$ then $J = \Delta + 2H^2 + 2c$ and so $\lambda_1 = -2H^2 - 2c$.

On the other hand, from (2.2) we get that

$$\lambda_1 \leq -4(H^2 + c) - 8\pi(g - 1)/\text{Area}(\Sigma)$$

since $2\overline{K}_\Sigma + \overline{\text{Ric}}(N, N) \geq 4c$ and $\alpha \geq 0$, which proves the first statement of part (ii). Moreover, if the equality holds then $\alpha = 0$, $\overline{K}_\Sigma = c$ and $\overline{\text{Ric}}(N, N) = 2c$, so the normal direction to Σ is a direction of minimum Ricci curvature of M^3 as above. The fact that $\alpha = 0$ implies ρ is constant and from (2.1) we see that $|A|^2$ is also constant, which means that K is constant by the Gauss equation. Conversely, under such hypothesis we have $J = \Delta + 4(H^2 + c) - 2K$, hence $\lambda_1 = -4(H^2 + c) + 2K = -4(H^2 + c) - 8\pi(g - 1)/\text{Area}(\Sigma)$ by the Gauss–Bonnet formula. □

In particular, as a direct consequence of this theorem, we have the following corollary in 3-dimensional simply connected space forms: the Euclidean space \mathbb{R}^3 , the hyperbolic space $\mathbb{H}^3(c)$, and the standard sphere $\mathbb{S}^3(c)$; for this last case the result was achieved before (see [2]).

Corollary 2.2. *Let Σ^2 be a compact two-sided surface with constant mean curvature H immersed into a 3-dimensional simply connected space form $M^3(c)$, and let λ_1 stand for the first eigenvalue of its Jacobi operator. Then*

- (i) either $\lambda_1 = -2(H^2 + c)$ (and Σ is totally umbilic in $M^3(c)$), or

- (ii) $\lambda_1 \leq -4(H^2 + c)$, with equality if and only if Σ^2 is a Clifford torus in $\mathbb{S}^3(c)$.

Proof. Since $\overline{K} \equiv c$ we have $\overline{\text{Ric}}(N, N) = 2c$, so the normal direction of Σ is a direction of minimum Ricci curvature of $M^3(c)$. Now, if Σ is totally umbilic we know from Theorem 2.1 that $\lambda_1 = -2(H^2 + c)$. Otherwise, using the fact that the genus of a constant mean curvature non-totally umbilic surface in $M^3(c)$ is greater than or equal to 1 we obtain $\lambda_1 \leq -4(H^2 + c)$. Moreover, equality holds if and only if $g = 1$ and Σ has constant Gaussian curvature, so by the Gauss–Bonnet formula it must be $K = 0$. This occurs only when Σ^2 is a Clifford torus in $\mathbb{S}^3(c)$. □

We can also derive the following consequence for strongly stable CMC surfaces.

Corollary 2.3. *Let M^3 be a 3-dimensional Riemannian space with sectional curvature \overline{K} bounded from below by a constant c .*

- (i) *There exists no strongly stable CMC surface with $H^2 + c > 0$.*
- (ii) *If Σ^2 is a strongly stable CMC surface and $H^2 + c = 0$ (that is, $c = 0$ and $H = 0$ or $c < 0$ and $H^2 = -c$), then Σ^2 is topologically either a sphere or a torus.*
- (iii) *If Σ^2 is a strongly stable CMC surface and $H^2 + c < 0$ (that is, $c < 0$ and $H^2 < -c$), then*

$$\text{Area}(\Sigma) |H^2 + c| \geq 2\pi(g - 1).$$

The proof of item (i) above is a direct application of the estimate for λ_1 given in item (i) of Theorem 2.1, while items (ii) and (iii) follow from the estimate for λ_1 given in item (ii) of Theorem 2.1.

3. Surfaces in Homogeneous 3-Manifolds

From now on, we will focus our attention on the study of compact CMC surfaces into homogeneous Riemannian 3-manifolds whose isometry group has dimension 4. So, if M^3 is a homogeneous Riemannian 3-manifold, it is well known that there exists a Riemannian submersion $\Pi : M^3 \rightarrow B^2(\kappa)$, where $B^2(\kappa)$ is a 2-dimensional simply connected space form of constant curvature κ , with totally geodesic fibers and there exists a unit Killing field ξ on M^3 which is vertical respect to Π . If $\overline{\nabla}$ stands for the Levi-Civita connection of M^3 , we have

$$\overline{\nabla}_X \xi = \tau(X \wedge \xi), \tag{3.1}$$

for all vector fields X on M^3 , where \wedge is the vector product in M^3 and the constant τ is the bundle curvature (see [5] for details). As the isometry group of M^3 has dimension 4, $\kappa - 4\tau^2 \neq 0$. We will denote such manifolds and certain quotients by $\mathbb{E}^3(\kappa, \tau)$. Depending on τ and κ , we can distinguish the different cases (see for instance [4] or [10]):

- (i) When $\tau = 0$, $\mathbb{E}^3(\kappa, \tau)$ is the product $B^2(\kappa) \times \mathbb{R}$, that is, up to scaling, the spaces $\mathbb{S}^2(\kappa) \times \mathbb{R}$ for $\kappa > 0$, and $\mathbb{H}^2(\kappa) \times \mathbb{R}$ for $\kappa < 0$, and their quotients $B^2(\kappa) \times \mathbb{S}^1$.

- (ii) When $\tau \neq 0$, $\mathbb{E}^3(\kappa, \tau)$ is one of the Berger spheres $\mathbb{S}_b^3(\kappa, \tau)$ for $\kappa > 0$, the Heisenberg group $Nil_3(\tau)$ for $\kappa = 0$ and the universal cover $\widehat{Sl}(2, \mathbb{R})(\kappa, \tau)$ of the Lie group $Sl(2, \mathbb{R})(\kappa, \tau)$ for $\kappa < 0$, and their quotients $\mathbb{S}_b^3(\kappa, \tau)/\mathbb{Z}_n$ and $PSl(2, \mathbb{R}) = Sl(2, \mathbb{R})(\kappa, \tau)/\mathbb{Z}_n$, $n \geq 2$.

The submersion $\Pi : \mathbb{E}^3(\kappa, \tau) \rightarrow B^2(\kappa)$ allows us to get a very interesting example of surface. So, if γ is any regular curve in $B^2(\kappa)$ we know that $\Sigma = \Pi^{-1}(\gamma)$ is a surface in $\mathbb{E}^3(\kappa, \tau)$ with ξ a tangent vector field, i.e., $\langle N, \xi \rangle \equiv 0$. From (3.1) it follows that ξ is a parallel vector field on Σ and hence Σ is a flat surface. We will call $\Pi^{-1}(\gamma)$ a Hopf surface of $\mathbb{E}^3(\kappa, \tau)$. If γ is a closed curve, $\Pi^{-1}(\gamma)$ is a Hopf cylinder, and additionally if Π is a circle Riemannian submersion, $\Pi^{-1}(\gamma)$ is a Hopf torus. Let us observe that a Hopf torus $\Pi^{-1}(\gamma)$ has constant mean curvature if and only if the curve γ has constant curvature.

In this section, we start the study of the first stability eigenvalue of any compact CMC surface $\psi : \Sigma^2 \rightarrow \mathbb{E}^3(\kappa, \tau)$. According to (2.2), we have to consider the curvature tensor of any $\mathbb{E}^3(\kappa, \tau)$. It is well known (see [5]) that the Riemannian curvature tensor \overline{R} of $\mathbb{E}^3(\kappa, \tau)$ is given by

$$\begin{aligned} \langle \overline{R}(X, Y)Z, W \rangle &= (\kappa - 3\tau^2)\{\langle Y, Z \rangle \langle X, W \rangle - \langle X, Z \rangle \langle Y, W \rangle\} \\ &\quad + (\kappa - 4\tau^2)\{\langle X, \xi \rangle \langle Z, \xi \rangle \langle Y, W \rangle - \langle Y, \xi \rangle \langle Z, \xi \rangle \langle X, W \rangle\} \\ &\quad + \langle X, Z \rangle \langle Y, \xi \rangle \langle \xi, W \rangle - \langle Y, Z \rangle \langle X, \xi \rangle \langle \xi, W \rangle, \end{aligned}$$

for all vector fields X, Y, Z and W on $\mathbb{E}^3(\kappa, \tau)$. As a consequence, the Ricci curvature of $\mathbb{E}^3(\kappa, \tau)$ is given by

$$\overline{Ric}(X, X) = \kappa - 2\tau^2 + \langle X, \xi \rangle^2(4\tau^2 - \kappa), \tag{3.2}$$

for every unit vector field X on $\mathbb{E}^3(\kappa, \tau)$. Moreover, the sectional curvature \overline{K} of any plane P is

$$\overline{K}(P) = \tau^2 + \langle \nu, \xi \rangle^2(\kappa - 4\tau^2),$$

where ν is the normal to P .

A direct computation using the last two formulae shows that

$$2\overline{K}_\Sigma + \overline{Ric}(N, N) = \kappa + \langle N, \xi \rangle^2(\kappa - 4\tau^2).$$

Therefore, Eq. (2.2) reduces to

$$\lambda_1 = -4H^2 - \kappa - \frac{1}{\text{Area}(\Sigma)} \left(\alpha + 8\pi(g - 1) + (\kappa - 4\tau^2) \int_\Sigma \langle N, \xi \rangle^2 \right). \tag{3.3}$$

Now, as a direct application of the above equation we can get upper bounds for λ_1 for compact CMC surfaces into the different homogeneous spaces $\mathbb{E}^3(\kappa, \tau)$. Moreover, in some cases we characterize the equality.

Theorem 3.1. *Let $\psi : \Sigma^2 \rightarrow \mathbb{S}^2(\kappa) \times \mathbb{R}$ be a compact two-sided surface of constant mean curvature H , and let λ_1 stand for the first eigenvalue of its Jacobi operator. Then*

- (i) $\lambda_1 \leq -2H^2$, with equality if and only if Σ^2 is a horizontal slice $\mathbb{S}^2(\kappa) \times \{t\}$;

$$(ii) \lambda_1 < -4H^2 - \kappa - \frac{8\pi(g-1)}{\text{Area}(\Sigma)}.$$

Proof. Since $\tau = 0$ and $\kappa > 0$ from (3.2) we know that

$$\overline{\text{Ric}}(N, N) = \kappa(1 - \langle N, \xi \rangle^2) \geq 0,$$

so that (2.3) directly yields $\lambda_1 \leq -2H^2$. Moreover, if the equality holds then $\langle N, \xi \rangle^2 \equiv 1$ which implies that Σ^2 is a totally geodesic horizontal slice.

On the other hand, since $\tau = 0$ from (3.3), we get

$$\begin{aligned} \lambda_1 &= -4H^2 - \kappa - \frac{1}{\text{Area}(\Sigma)} \left(\alpha + 8\pi(g-1) + \kappa \int_{\Sigma} \langle N, \xi \rangle^2 \right) \\ &\leq -4H^2 - \kappa - \frac{8\pi(g-1)}{\text{Area}(\Sigma)}, \end{aligned}$$

where we are taking into account that $\kappa > 0$ and $\langle N, \xi \rangle^2 \geq 0$. In fact, equality holds if and only if $\alpha = 0$ and $\langle N, \xi \rangle \equiv 0$. This last condition implies that Σ^2 would be a Hopf torus but that is not possible because there are no flat compact surfaces in $\mathbb{S}^2(\kappa) \times \mathbb{R}$ (see [14]). \square

As a consequence, let us note that in $\mathbb{S}^2(\kappa) \times \mathbb{R}$ if $H \neq 0$ then $\lambda_1 < 0$ (so the surface is not strongly stable) and from here we easily get that the only constant mean curvature compact strongly stable surfaces in $\mathbb{S}^2(\kappa) \times \mathbb{R}$ are horizontal slices $\mathbb{S}^2(\kappa) \times \{t\}$.

Corollary 3.2. *The only strongly stable compact surfaces of constant mean curvature in $\mathbb{S}^2(\kappa) \times \mathbb{R}$ are horizontal slices.*

Theorem 3.3. *Let $\psi : \Sigma^2 \rightarrow \mathbb{S}^2(\kappa) \times \mathbb{S}^1$ be a compact two-sided surface of constant mean curvature H , and let λ_1 stand for the first eigenvalue of its Jacobi operator. Then*

- (i) $\lambda_1 \leq -2H^2$, with equality if and only if Σ^2 is a horizontal slice $\mathbb{S}^2(\kappa) \times \{p\}$;
- (ii) $\lambda_1 \leq -4H^2 - \kappa - \frac{8\pi(g-1)}{\text{Area}(\Sigma)}$ and equality holds if and only if Σ^2 is a Hopf torus over a constant curvature closed curve.

Proof. The same as above but in this case for the equality in item (ii) we do have CMC Hopf tori. Let us observe that from (3.3) the first eigenvalue for such tori is $\lambda_1 = -4H^2 - \kappa$ because of $g = 1$, $\langle N, \xi \rangle \equiv 0$ and $\alpha = 0$. \square

Theorem 3.4. *Let $\psi : \Sigma^2 \rightarrow \mathbb{H}^2(\kappa) \times \mathbb{R}$ be a compact two-sided surface of constant mean curvature H , and let λ_1 stand for the first eigenvalue of its Jacobi operator. Then*

- (i) $\lambda_1 < -2H^2 - \kappa$;
- (ii) $\lambda_1 < -4H^2 - 2\kappa - \frac{8\pi(g-1)}{\text{Area}(\Sigma)}$.

Proof. In this case, since $\tau = 0$ and $\kappa < 0$ from (3.2) we know that

$$\overline{\text{Ric}}(N, N) = \kappa(1 - \langle N, \xi \rangle^2) \geq \kappa,$$

so that (2.3) directly yields $\lambda_1 \leq -2H^2 - \kappa$. Moreover, the equality cannot hold; otherwise we would have $\langle N, \xi \rangle^2 \equiv 0$ which implies that Σ^2 is a cylinder which is not possible because it is compact.

On the other hand, since $\tau = 0$, $\kappa < 0$ and $\langle N, \xi \rangle^2 \leq 1$ we have from (3.3)

$$\lambda_1 \leq -4H^2 - 2\kappa - \frac{8\pi(g-1)}{\text{Area}(\Sigma)},$$

and equality holds if and only if $\alpha = 0$ and $\langle N, \xi \rangle^2 \equiv 1$. So $N = \pm\xi$ which implies that the surface would be a slice of $\mathbb{H}^2(\kappa) \times \mathbb{R}$ that is not compact. \square

As a consequence, we derive the following result which is related to Corollary 4.1 in [8] (it is important to realize that the notion of stability in Corollary 4.1 of [8], as well as in Theorem C of the same reference, is that of weakly stability although not explicitly stated).

Corollary 3.5. *Let $\psi : \Sigma^2 \rightarrow \mathbb{H}^2(\kappa) \times \mathbb{R}$ be a compact two-sided surface of constant mean curvature H .*

- (i) *There exists no strongly stable CMC surface with $H^2 \geq -\kappa/2$.*
- (ii) *If Σ^2 is a strongly stable CMC surface and $H^2 < -\kappa/2$, then*

$$\text{Area}(\Sigma) |2H^2 + \kappa| > 4\pi(g - 1).$$

We study now the cases where the bundle curvature $\tau \neq 0$.

Remark 3.6. When this happens, we know that $\{p \in \Sigma^2 : \langle N, \xi \rangle^2(p) = 1\} = \{p \in \Sigma^2 : \xi(p) = \pm N(p)\}$ has empty interior because the distribution $\langle \xi \rangle^\perp$ on $\mathbb{E}^3(\kappa, \tau)$ is not integrable (see [14]).

We begin with the most simple case, that is, the Heisenberg group $Nil_3(\tau)$ where $\kappa = 0$.

Theorem 3.7. *Let $\psi : \Sigma^2 \rightarrow Nil_3(\tau)$ be a compact two-sided surface of constant mean curvature H , and let λ_1 stand for the first eigenvalue of its Jacobi operator. Then*

- (i) $\lambda_1 < -2(H^2 - \tau^2)$;
- (ii) $\lambda_1 < -4(H^2 - \tau^2) - \frac{8\pi(g-1)}{\text{Area}(\Sigma)}$.

Proof. Since $\kappa = 0$, from (3.2) we know that

$$\overline{\text{Ric}}(N, N) = 2\tau^2(2\langle N, \xi \rangle^2 - 1) \geq -2\tau^2,$$

so that (2.3) directly yields $\lambda_1 \leq -2H^2 + 2\tau^2$. Moreover, the equality cannot hold because by the results in [13] we know that there is no totally umbilic surfaces in $Nil_3(\tau)$.

On the other hand, by (3.3) we get

$$\begin{aligned} \lambda_1 &= -4H^2 - \frac{1}{\text{Area}(\Sigma)} \left(\alpha + 8\pi(g-1) - 4\tau^2 \int_{\Sigma} \langle N, \xi \rangle^2 \right) \\ &\leq -4H^2 + 4\tau^2 - \frac{8\pi(g-1)}{\text{Area}(\Sigma)}. \end{aligned}$$

If the equality holds then $\langle N, \xi \rangle \equiv 1$ and it is not possible because of Remark 3.6. □

Corollary 3.8. *Let $\psi : \Sigma^2 \rightarrow Nil_3(\tau)$ be a compact two-sided surface of constant mean curvature H .*

- (i) *There exists no strongly stable CMC surface with $H^2 \geq \tau^2$.*
- (ii) *If Σ^2 is a strongly stable CMC surface and $H^2 < \tau^2$, then*

$$\text{Area}(\Sigma) |H^2 - \tau^2| > 2\pi(g-1).$$

Now, let κ be positive, thus, $\mathbb{E}^3(\kappa, \tau) = \mathbb{S}_b^3(\kappa, \tau)$ is a Berger sphere. For this homogeneous Riemannian manifold it is very common to consider two different cases based on the sign of $\kappa - 4\tau^2$ since the obtained results are quite different.

Theorem 3.9. *Let $\psi : \Sigma^2 \rightarrow \mathbb{S}_b^3(\kappa, \tau)$ be a compact two-sided surface of constant mean curvature H , and let λ_1 stand for the first eigenvalue of its Jacobi operator.*

- (a) *If $\kappa - 4\tau^2 > 0$ then*
 - (i) $\lambda_1 < -2(H^2 + \tau^2)$,
 - (ii) $\lambda_1 \leq -4H^2 - \kappa - \frac{8\pi(g-1)}{\text{Area}(\Sigma)}$ and equality holds if and only if Σ^2 is a Hopf torus over a constant curvature closed curve.
- (b) *If $\kappa - 4\tau^2 < 0$ then*
 - (i) $\lambda_1 < -2H^2 - \kappa + 2\tau^2$,
 - (ii) $\lambda_1 < -4H^2 - 2\kappa + 4\tau^2 - \frac{8\pi(g-1)}{\text{Area}(\Sigma)}$.

Proof. (a) The proof of item (i) uses the fact that in this case

$$\overline{\text{Ric}}(N, N) \geq 2\tau^2,$$

which by (2.3) gives $\lambda_1 \leq -2(H^2 + \tau^2)$. Besides the equality cannot hold because of the non-existence of totally umbilic surfaces in the Berger spheres [13]. On the other hand, since $\kappa - 4\tau^2 > 0$, by (3.3) we get

$$\lambda_1 \leq -4H^2 - \kappa - \frac{8\pi(g-1)}{\text{Area}(\Sigma)}.$$

If the equality holds then Σ^2 has to be a Hopf torus because of $\langle N, \xi \rangle \equiv 0$ and reciprocally, any CMC Hopf torus satisfies the equality as we have seen before.

(b) In this case,

$$\overline{\text{Ric}}(N, N) \geq \kappa - 2\tau^2,$$

and (2.3) yields $\lambda_1 \leq -2(H^2 - \tau^2) - \kappa$. Again, equality cannot happen since there exist no totally umbilic surfaces. Since $\kappa - 4\tau^2 < 0$, by (3.3) we have

$$\lambda_1 \leq -4H^2 - 2\kappa + 4\tau^2 - \frac{8\pi(g-1)}{\text{Area}(\Sigma)},$$

and equality holds if and only $\alpha = 0$ and $\langle N, \xi \rangle^2 \equiv 1$ but again it is not possible. □

The corresponding applications to strongly stable surfaces in the Berger spheres have no sense because we know by [7] that there exist no such surfaces in these ambient spaces. Finally, for the case $\kappa < 0$ we have the following result.

Theorem 3.10. *Let $\psi : \Sigma^2 \rightarrow \widetilde{Sl}(2, \mathbb{R})(\kappa, \tau)$ be a compact two-sided surface of constant mean curvature H , and let λ_1 stand for the first eigenvalue of its Jacobi operator. Then*

- (i) $\lambda_1 < -2H^2 - \kappa + 2\tau^2$,
- (ii) $\lambda_1 < -4H^2 - 2\kappa + 4\tau^2 - \frac{8\pi(g-1)}{\text{Area}(\Sigma)}$.

Proof. Since $\kappa < 0$ we get $\kappa - 4\tau^2 < 0$, so the proof is the same that the part (b) of the above theorem. □

Corollary 3.11. *Let $\psi : \Sigma^2 \rightarrow \widetilde{Sl}(2, \mathbb{R})(\kappa, \tau)$ be a compact two-sided surface of constant mean curvature H .*

- (i) *There exists no strongly stable CMC surface with $H^2 \geq \tau^2 - \kappa/2$.*
- (ii) *If Σ^2 is a strongly stable CMC surface and $H^2 < \tau^2 - \kappa/2$, then*

$$\text{Area}(\Sigma) |H^2 - \tau^2 + \kappa/2| > 2\pi(g-1).$$

Note Added in Proof. After submission of this paper, we were informed by Manzano that he and Pérez and Rodríguez have recently proved in [6, Theorem 2] that if Σ^2 is a two-sided, parabolic, complete, strongly stable surface in $\mathbb{E}(\kappa, \tau)$, then either $\mathbb{E}(\kappa, \tau) = \mathbb{S}^2(\kappa) \times \mathbb{R}$ and Σ is a horizontal slice or $H^2 \leq -\kappa/4$ and Σ is either a vertical multigraph or a vertical cylinder over a complete curve of geodesic curvature $2H$ in $\mathbb{M}^2(\kappa)$. It follows from here, using also [7], that the only strongly stable compact CMC surfaces in $\mathbb{E}(\kappa, \tau)$ are the horizontal slices in $\mathbb{S}^2(\kappa) \times \mathbb{R}$. This agrees with our Corollary 3.2 and clearly improves our Corollaries 3.5, 3.8 and 3.11.

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