Mediterr. J. Math. 11 (2014), 561–576
DOI 10.1007/s00009-013-0347-0
0378-620X/14/020561-16
published online October 11, 2013

Mediterranean Journal of Mathematics

The Eigenstructure of Operators Linking the Bernstein and the Genuine Bernstein–Durrmeyer operators

Heiner Gonska, Ioan Raşa and Elena-Dorina Stănilă

Abstract. We study the eigenstructure of a one-parameter class of operators U_n^ϱ of Bernstein–Durrmeyer type that preserve linear functions and constitute a link between the so-called genuine Bernstein–Durrmeyer operators U_n and the classical Bernstein operators B_n. In particular, for $\varrho \to \infty$ (respectively, $\varrho = 1$) we recapture results well-known in the literature, concerning the eigenstructure of B_n (respectively, U_n). The last section is devoted to applications involving the iterates of U_n^ϱ.

Mathematics Subject Classification (2010). Primary 41A36, 15A18; Secondary 41A10.

Keywords. Bernstein operators, genuine Bernstein–Durrmeyer operators, diagonalisation, eigenvalues, eigenfunctions, Stirling numbers, iterates.

1. Introduction

Denote by $C[0,1]$ the space of continuous, real-valued functions on $[0,1]$ and by Π_n the space of polynomials of degree at most $n \in \mathbb{N}_0 := \{0,1,2\ldots\}$.

Definition 1.1. Let $\varrho > 0$ and $n \in \mathbb{N}_0, n \geq 1$. Define the operator $U_n^\varrho\colon C[0,1] \to \Pi_n$ by

$$\begin{aligned} U_n^\varrho(f,x) &:= \sum_{k=0}^{n} F_{n,k}^\varrho(f) p_{n,k}(x) \\ &:= \sum_{k=1}^{n-1} \left(\int_0^1 \frac{t^{k\varrho-1}(1-t)^{(n-k)\varrho-1}}{B(k\varrho,(n-k)\varrho)} f(t)\,\mathrm{d}t \right) p_{n,k}(x) \\ &\quad + f(0)(1-x)^n + f(1)x^n, \end{aligned} \tag{1.1}$$

$f \in C[0,1], x \in [0,1]$ and $B(\cdot,\cdot)$ is Euler's Beta function. The fundamental functions $p_{n,k}$ are defined by

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad 0 \leq k \leq n, \quad k,n \in \mathbb{N}_0, \quad x \in [0,1].$$

Birkhäuser

For $\varrho = 1$ and $f \in C[0,1]$, we obtain

$$U_n^1(f,x) = U_n(f,x) = (n-1)\sum_{k=1}^{n-1}\left(\int_0^1 f(t)p_{n-2,k-1}(t)\,\mathrm{d}t\right)p_{n,k}(x) + (1-x)^n f(0) + x^n f(1), \tag{1.2}$$

where U_n are the "genuine" Bernstein–Durrmeyer operators (see [9]), while for $\varrho \to \infty$, for each $f \in C[0,1]$ the sequence $U_n^\varrho(f,x)$ converges uniformly to the Bernstein polynomial

$$B_n(f,x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)p_{n,k}(x). \tag{1.3}$$

The U_n^ϱ were introduced in [10] by Păltănea and further investigated in [7] and [8], where also the latter convergence was shown. In the present article the eigenstructure of U_n^ϱ will be investigated. We show that the eigenstructure of U_n^ϱ is similar to that of the classical Bernstein operators, described in [4], and covers it for $\varrho \to \infty$. Moreover, for $\varrho = 1$ we recapture and extend the known results concerning the eigenstructure of U_n. In the last section we investigate the iterates of U_n^ϱ.

2. The Images of the Monomials

In what follows we will use the *rising factorial function* defined by

$$(x)_j := x(x+1)\cdots(x+j-1), \qquad j = 1,2,\ldots, \quad (x)_0 := 1.$$

Basic properties of the functionals $F_{n,k}^\varrho : C[0,1] \to \mathbb{R}$ are the following

$$F_{n,k}^\varrho(e_m) = \frac{(k\varrho)_m}{(n\varrho)_m}, \qquad 0 \le k \le n, \text{ and } e_m(x) = x^m, x \in [0,1], \text{ for } m \ge 0. \tag{2.1}$$

This implies

$$U_n^\varrho(e_0) = e_0, \quad U_n^\varrho(e_1) = e_1. \tag{2.2}$$

More generally we have

Theorem 2.1. *The images of the monomials under U_n^ϱ can be written as*

$$U_n^\varrho(e_m) = \frac{1}{(n\varrho)_m}\sum_{l=0}^{m} c_{m-l}^{(m)}(n\varrho)^l B_n(e_l) \tag{2.3}$$

where the coefficients $c_j^{(m)}$, $j = 0,1,\ldots,m$, are given by the elementary symmetric sums:

$$c_0^{(m)} := 1, \quad c_m^{(m)} := 0,$$

$$c_1^{(m)} = 1 + 2 + \cdots + (m-1) = \frac{m(m-1)}{2},$$

$$
\begin{aligned}
c_2^{(m)} &= 1\cdot 2 + 1\cdot 3 + \cdots + 1\cdot (m-1) + 2\cdot 3 + \cdots + (m-2)\cdot(m-1),\\
&\dots\\
c_{m-1}^{(m)} &= 1\cdot 2\cdot 3 \cdots (m-1) = (m-1)!. \qquad (2.4)
\end{aligned}
$$

Proof.

$$
\begin{aligned}
U_n^{\varrho}(e_m, x) &= \sum_{k=0}^{n} F_{n,k}^{\varrho}(e_m) p_{n,k}(x)\\
&= \frac{1}{(n\varrho)_m} \sum_{k=0}^{n} k\varrho(k\varrho+1)\cdots(k\varrho+m-1) p_{n,k}(x)\\
&= \frac{1}{(n\varrho)_m} \sum_{k=0}^{n} [c_0^{(m)}(k\varrho)^m + c_1^{(m)}(k\varrho)^{m-1} + \cdots + c_{m-1}^{(m)} k\varrho] p_{n,k}(x)\\
&= \frac{1}{(n\varrho)_m} \left\{ c_0^{(m)} \varrho^m \sum_{k=0}^{n} k^m p_{n,k}(x) + c_1^{(m)} \varrho^{m-1} \sum_{k=0}^{n} k^{m-1} p_{n,k}(x) + \cdots \right.\\
&\qquad \left. + c_{m-1}^{(m)} \varrho \sum_{k=0}^{n} k p_{n,k}(x) \right\}\\
&= \frac{1}{(n\varrho)_m} \left\{ c_0^{(m)} \varrho^m n^m \sum_{k=0}^{n} \frac{k^m}{n^m} p_{n,k}(x) + c_1^{(m)} \varrho^{m-1} n^{m-1} \right.\\
&\qquad \left. \times \sum_{k=0}^{n} \frac{k^{m-1}}{n^{m-1}} p_{n,k}(x) + \cdots + c_{m-1}^{(m)} n\varrho \sum_{k=0}^{n} \frac{k}{n} p_{n,k}(x) \right\}\\
&= \frac{1}{(n\varrho)_m} \left\{ c_0^{(m)} \varrho^m n^m B_n(e_m, x) + c_1^{(m)} \varrho^{m-1} n^{m-1} B_n(e_{m-1}, x) + \cdots \right.\\
&\qquad \left. + c_{m-1}^{(m)} n\varrho B_n(e_1, x) \right\}\\
&= \frac{1}{(n\varrho)_m} \sum_{l=0}^{m} c_{m-l}^{(m)} (n\varrho)^l B_n(e_l, x).
\end{aligned}
$$

□

The operator U_n^{ϱ} reproduces linear polynomials, which are therefore eigenfunctions corresponding to the eigenvalue 1.

3. Diagonalisation and Description of the Eigenfunctions

We shall use the *Stirling numbers of second kind* $S(k, j)$ defined by

$$x^k = \sum_{j=0}^{k} S(k, j) x(x-1)\cdots(x-j+1).$$

The following identity holds (see [3], Theorem A [1b], p. 204):

$$S(k, j) = \frac{1}{j!} \sum_{i=0}^{j} \binom{j}{i} (-1)^{j-i} i^k,\ 0 \le j \le k. \qquad (3.1)$$

Consider the eigenfunction equation

$$U_n^{\varrho} p_{\varrho,k}^{(n)} = \lambda_{\varrho,k}^{(n)} p_{\varrho,k}^{(n)} \qquad (3.2)$$

with respect to the basis of monomials $\{e_0, e_1, \ldots, e_n\}$. Since U_n^ϱ is degree reducing, we have to solve an upper triangular system. This will be done in the proof of the next theorem.

Theorem 3.1. *The operator U_n^ϱ can be represented in diagonal form*

$$U_n^\varrho f = \sum_{k=0}^{n} \lambda_{\varrho,k}^{(n)} p_{\varrho,k}^{(n)} \mu_{\varrho,k}^{(n)}(f), \qquad \text{for all } f \in C[0,1], \tag{3.3}$$

with $\lambda_{\varrho,k}^{(n)}$ and $p_{\varrho,k}^{(n)}$ its eigenvalues and eigenfunctions and $\mu_{\varrho,k}^{(n)}$ the dual functionals to $p_{\varrho,k}^{(n)}$. The eigenvalues are given by

$$\lambda_{\varrho,k}^{(n)} = \varrho^{k-1} \frac{(n-1)(n-2)\cdots(n-k+1)}{(n\varrho+1)(n\varrho+2)\cdots(n\varrho+k-1)} = \frac{\varrho^k n!}{(n\varrho)_k (n-k)!} \tag{3.4}$$

and they satisfy

$$1 = \lambda_{\varrho,0}^{(n)} = \lambda_{\varrho,1}^{(n)} > \lambda_{\varrho,2}^{(n)} > \lambda_{\varrho,3}^{(n)} > \cdots > \lambda_{\varrho,n}^{(n)} > 0.$$

The eigenfunction for $\lambda_{\varrho,k}^{(n)}$ is a polynomial of degree k given by

$$p_{\varrho,k}^{(n)}(x) = \sum_{j=0}^{k} c^\varrho(j,k,n) x^j = x^k - \frac{k}{2} x^{k-1} + \text{ lower order terms}, \tag{3.5}$$

where the coefficients can be computed using the recurrence formula

$$\begin{aligned} c^\varrho(k,k,n) &:= 1, \\ c^\varrho(k-1,k,n) &:= -\frac{k}{2}, \\ c^\varrho(k-j,k,n) &:= \frac{(n\varrho)_k}{\varrho^{k-j}[\varrho^j(n-k+1)_j - (n\varrho+k-j)_j]} \\ &\quad \times \sum_{i=0}^{j-1} \sum_{l=k-j}^{k-i} \frac{c^\varrho(k-i,k,n)}{(n\varrho)_{k-i}} c_{k-i-l}^{(k-i)} \varrho^l S(l,k-j),\ j = 2,\ldots,k. \end{aligned} \tag{3.6}$$

Proof. The eigenvalues of U_n^ϱ are determined from the upper triangular system of equations (3.2). They can be found on the diagonal and are equal to the coefficients of the terms with the highest degree of $U_n^\varrho(e_m)$. As we have seen before, for $0 \le m \le n$, we can write

$$\begin{aligned} U_n^\varrho(e_m, x) = \frac{1}{(n\varrho)_m} \Bigg\{ & c_0^{(m)} \varrho^m \sum_{k=0}^{n} k^m p_{n,k}(x) + c_1^{(m)} \varrho^{m-1} \sum_{k=0}^{n} k^{m-1} p_{n,k}(x) \\ & + \cdots + c_{m-1}^{(m)} \varrho \sum_{k=0}^{n} k p_{n,k}(x) \Bigg\} \end{aligned}$$

and because

$$\sum_{k=0}^{n} k^m p_{n,k}(x) = n(n-1)(n-2)\cdots(n-m+1)x^m + \text{ terms of lower degree}$$

the eigenvalues are given by

$$\lambda_{\varrho,m}^{(n)} = \frac{1}{(n\varrho)_m}\varrho^m n(n-1)(n-2)\cdots(n-m+1) = \frac{\varrho^m \cdot n!}{(n\varrho)_m(n-m)!}.$$

The linear polynomials are eigenfunctions for the eigenvalues $\lambda_{\varrho,0}^{(n)} = \lambda_{\varrho,1}^{(n)} = 1$, for which $p_{\varrho,0}^{(n)}(x) = 1, p_{\varrho,1}^{(n)}(x) = x - \frac{1}{2}$ are clearly a basis which satisfies (3.5) and (3.6).

It remains to consider the 1-dimensional $\lambda_{\varrho,k}^{(n)}$-eigenspace of polynomials of exact degree $k = 2, 3, \ldots, n$.

We shall plug into (2.3)

$$B_n(e_m, x) = \sum_{j=0}^{m} a(j,m,n)x^j, \tag{3.7}$$

where

$$a(j,m,n) = \frac{S(m,j)n!}{n^m(n-j)!}, \quad 0 \le j \le m \le n, \tag{3.8}$$

as it was considered in [4] and we obtain

$$U_n^{\varrho}(e_m, x) = \frac{1}{(n\varrho)_m}\sum_{l=0}^{m} c_{m-l}^{(m)}(n\varrho)^l \sum_{r=0}^{l} a(r,l,n)x^r. \tag{3.9}$$

Express the eigenfunctions in the form

$$p_{\varrho,k}^{(n)}(x) = \sum_{s=0}^{k} c^{\varrho}(s,k,n)x^s, \quad c^{\varrho}(k,k,n) := 1. \tag{3.10}$$

Then the eigenfunction equation (3.2) gives

$$\begin{aligned}\lambda_{\varrho,k}^{(n)}\sum_{r=0}^{k} c^{\varrho}(r,k,n)x^r &= \sum_{s=0}^{k}\frac{c^{\varrho}(s,k,n)}{(n\varrho)_s}\sum_{l=0}^{s} c_{s-l}^{(s)}(n\varrho)^l\sum_{r=0}^{l} a(r,l,n)x^r \\ &= \sum_{s=0}^{k}\frac{c^{\varrho}(s,k,n)}{(n\varrho)_s}\sum_{r=0}^{s}\sum_{l=r}^{s} c_{s-l}^{(s)}(n\varrho)^l a(r,l,n)x^r \\ &= \sum_{r=0}^{k}\sum_{s=r}^{k}\frac{c^{\varrho}(s,k,n)}{(n\varrho)_s}\sum_{l=r}^{s} c_{s-l}^{(s)}(n\varrho)^l a(r,l,n)x^r.\end{aligned}$$

Equating the coefficients of x^r above gives for $0 \le r \le k$:

$$\lambda_{\varrho,k}^{(n)} c^{\varrho}(r,k,n) = \sum_{s=r}^{k}\frac{c^{\varrho}(s,k,n)}{(n\varrho)_s}\sum_{l=r}^{s} c_{s-l}^{(s)}(n\varrho)^l a(r,l,n).$$

Into this we make first the substitution $s = k - i$ and subsequently $r = k - j$ to obtain

$$\lambda_{\varrho,k}^{(n)} c^{\varrho}(k-j,k,n) = \sum_{i=0}^{j}\frac{c^{\varrho}(k-i,k,n)}{(n\varrho)_{k-i}}\sum_{l=k-j}^{k-i} c_{k-i-l}^{(k-i)}(n\varrho)^l a(k-j,l,n).$$

which, for $k > 1$, can be solved for $c^{\varrho}(k-j,k,n)$ to give

$$
\begin{aligned}
c^{\varrho}(k-j,k,n) &= \left(\lambda_{\varrho,k}^{(n)} - \frac{(n\varrho)^{k-j}}{(n\varrho)_{k-j}} a(k-j,k-j,n)\right)^{-1} \sum_{i=0}^{j-1} \frac{c^{\varrho}(k-i,k,n)}{(n\varrho)_{k-i}} \\
&\quad \times \sum_{l=k-j}^{k-i} c_{k-i-l}^{(k-i)} (n\varrho)^l a(k-j,l,n) \\
&= \left(\frac{\varrho^k \cdot n!}{(n\varrho)_k (n-k)!} - \frac{(n\varrho)^{k-j}}{(n\varrho)_{k-j}} \frac{S(k-j,k-j)\cdot n!}{n^{k-j}(n-k+j)!}\right)^{-1} \\
&\quad \times \sum_{i=0}^{j-1} \frac{c^{\varrho}(k-i,k,n)}{(n\varrho)_{k-i}} \sum_{l=k-j}^{k-i} c_{k-i-l}^{(k-i)} (n\varrho)^l \frac{S(l,k-j)n!}{n^l(n-k+j)!} \\
&= \left(\varrho^{k-j}\left[\frac{\varrho^j}{(n\varrho)_k(n-k)!} - \frac{1}{(n\varrho)_{k-j}(n-k+j)!}\right]\right)^{-1} \\
&\quad \times \sum_{i=0}^{j-1} \frac{c^{\varrho}(k-i,k,n)}{(n\varrho)_{k-i}} \sum_{l=k-j}^{k-i} c_{k-i-l}^{(k-i)} \varrho^l \frac{S(l,k-j)}{(n-k+j)!} \\
&= \frac{(n-k+j)!(n\varrho)_k}{\varrho^{k-j}[\varrho^j(n-k+1)_j - (n\varrho+k-j)_j]} \\
&\quad \times \sum_{i=0}^{j-1} \frac{c^{\varrho}(k-i,k,n)}{(n\varrho)_{k-i}} \sum_{l=k-j}^{k-i} c_{k-i-l}^{(k-i)} \varrho^l \frac{S(l,k-j)}{(n-k+j)!}.
\end{aligned}
$$

From here we get easily the Eqs. (3.6). In particular, for $j = 1$ we get

$$
c^{\varrho}(k-1,k,n) = \frac{c_1^{(k)} + c_0^{(k)} \varrho \frac{k(k-1)}{2}}{(n-k+1)\varrho - (n\varrho+k-1)} = -\frac{k}{2} \tag{3.11}
$$

because $c_0^{(k)} = 1$ and $c_1^{(k)} = 1+2+\cdots+(k-1) = \frac{k(k-1)}{2}$ and $S(k-1,k-1) = 1$, $S(k,k-1) = \frac{k(k-1)}{2}$. □

Theorem 3.2. *The dual functional* $\mu_{\varrho,k}^{(n)} \in span\{f \to F_{n,j}^{\varrho}(f); j = 0,1,\ldots,n\}$ *defined on* $C[0,1]$ *satisfies*

$$
\mu_{\varrho,k}^{(n)}(p_{\varrho,i}^{(n)}) = \delta_{i,k}; \qquad i,k = 0,1,\ldots,n,
$$

and is given by

$$
\mu_{\varrho,k}^{(n)}(f) = \sum_{j=0}^{n} v^{\varrho}(j,k,n) F_{n,j}^{\varrho}(f); \qquad k = 0,1,\ldots,n, \tag{3.12}
$$

where the $(n+1)\times(n+1)$ *matrix of coefficients* $V := [v^{\varrho}(j,k,n)]_{j,k=0}^{n}$ *is the inverse of* $P := [F_{n,j}^{\varrho}(p_{\varrho,i}^{(n)})]_{i,j=0}^{n}$.

Proof. The biorthogonality condition $\mu_{\varrho,k}^{(n)}(p_{\varrho,i}^{(n)}) = \delta_{i,k}$ follows easily from (3.2) and (3.3). Using (3.12) it can be written as

$$\sum_{j=0}^{n} F_{n,j}^{\varrho}(p_{\varrho,i}^{(n)})v^{\varrho}(j,k,n)=\delta_{i,k},$$

i.e., $PV=I$, and so $V=P^{-1}$. □

Theorem 3.3. *The eigenfunctions and the dual functionals satisfy the equations*

$$p_{\varrho,k}^{(n)}(x)=(-1)^k p_{\varrho,k}^{(n)}(1-x),\quad \mu_{\varrho,k}^{(n)}(f)=(-1)^k(f\circ R), \tag{3.13}$$

where $R(x)=1-x$ *is reflection about the point* $\frac{1}{2}$. *The eigenfunctions of degree* ≥ 2 *can be factored as follows:*

$$\begin{aligned} p_{\varrho,2j}^{(n)}(x)&=x(x-1)q(x-1/2),\\ p_{\varrho,2j+1}^{(n)}&=x(x-1/2)(x-1)q(x-1/2),\quad j=1,2,\ldots. \end{aligned} \tag{3.14}$$

In each case q *is an even monic polynomial.*

Proof. From (1.1) it follows that

$$U_n^{\varrho}(f\circ R)=(U_n^{\varrho}f)\circ R, \tag{3.15}$$

so that

$$U_n^{\varrho}(p_{\varrho,k}^{(n)}\circ R)=(U_n^{\varrho}p_{\varrho,k}^{(n)})\circ R=\lambda_{\varrho,k}^{(n)}(p_{\varrho,k}^{(n)}\circ R),$$

and $p_{\varrho,k}^{(n)}\circ R$ is a $\lambda_{\varrho,k}^{(n)}$-eigenfunction. For $k=0,1$ the property (3.13) of $p_{\varrho,k}^{(n)}$ is obvious, and for $k\geq 2$ the eigenfunction $p_{\varrho,k}^{(n)}\circ R$ must be a scalar multiple of $p_{\varrho,k}^{(n)}$ (the eigenspace is 1-dimensional). By equating the coefficients of x^k yields

$$p_{\varrho,k}^{(n)}=(-1)^k p_{\varrho,k}^{(n)}\circ R. \tag{3.16}$$

So $p_{\varrho,k}^{(n)}$ is even (odd) about the point 1/2 when k is even (odd). In particular, the zeros of $p_{\varrho,k}^{(n)}$ are symmetric about 1/2. Moreover, (3.15) implies that

$$\begin{aligned} \lambda_{\varrho,k}^{(n)}p_{\varrho,k}^{(n)}\mu_{\varrho,k}^{(n)}(f\circ R)&=\lambda_{\varrho,k}^{(n)}(p_{\varrho,k}^{(n)}\circ R)\mu_{\varrho,k}^{(n)}(f)\\ &=\lambda_{\varrho,k}^{(n)}(-1)^k p_{\varrho,k}^{(n)}\mu_{\varrho,k}^{(n)}(f), \end{aligned}$$

and equating the coefficients of $p_{\varrho,k}^{(n)}$ in the preceding relation we get

$$\mu_{\varrho,k}^{(n)}(f)=(-1)^k\mu_{\varrho,k}^{(n)}(f\circ R). \tag{3.17}$$

Taking $j=k$ in (3.6) and using $S(m,0)=0, m\geq 1$, we obtain $c^{\varrho}(0,k,n)=0$, $k\geq 2$.

Thus, for $k\geq 2, x=0$ is a zero of $p_{\varrho,k}^{(n)}$, and by the symmetry property so is $x=1$. Further, when k is odd the symmetry property of the zeros implies that $x=1/2$ must be a zero of $p_{\varrho,k}^{(n)}$, which proves (3.14). This completes the proof. □

4. Asymptotics of the Eigenfunctions

We show that for each $\varrho > 0$ and $k \geq 0$ the sequence $(p_{\varrho,k}^{(n)})_{n\geq 1}$ is convergent.

Theorem 4.1. *For $0 \leq j \leq k$,*

$$\lim_{n\to\infty} c^{\varrho}(j,k,n) = c^*(j,k), \tag{4.1}$$

where

$$c^*(0,1) = -\frac{1}{2}, c^*(j,k) := \prod_{i=1}^{k-j} \frac{(k+1-i)(k-i)}{i(i-2k+1)}, \quad (j,k) \neq (0,1); \tag{4.2}$$

the empty product is interpreted as 1. This means that $p_{\varrho,k}^{(n)}$ converges uniformly on $[0,1]$ to $p_k^* \in \Pi_k$ as $n \to \infty$, where

$$p_k^*(x) := \sum_{j=0}^{k} c^*(j,k)x^j = x^k - \frac{k}{2}x^{k-1} + \frac{k(k-1)(k-2)}{4(2k-3)}x^{k-2} - \cdots. \tag{4.3}$$

Proof. Noticing that $p_{\varrho,0}^{(n)}(x) = 1 = p_0^*(x), p_{\varrho,1}^{(n)}(x) = x - 1/2 = p_1^*(x)$, it is sufficient to prove the result for $k \geq 2$. This will be done using induction on j in order to prove that $\lim_{n\to\infty} c^{\varrho}(k-j,k,n)$ exists and is given by (4.2). Since $c^{\varrho}(k,k,n) = 1$, this result holds for $j = 0$. Suppose it is true for $\lim_{n\to\infty} c^{\varrho}(k-i,k,n), i = 0,\ldots,j-1$, where $0 < j \leq k$. Since for all $j > 0$,

$$\varrho^{k-j}[\varrho^j(n-k+1)_j - (n\varrho+k-j)_j] = -\varrho^{k-1}(\varrho+1)\frac{j(2k-j-1)}{2}n^{j-1} + \text{lower order powers of } n,$$

taking the limit as $n \to \infty$ on both sides of the last equality in (3.6) and using the induction hypothesis gives

$$\lim_{n\to\infty} c^{\varrho}(k-j,k,n) = -\frac{\varrho^k}{\varrho^{k-1}(\varrho+1)\frac{j(2k-j-1)}{2}\varrho^{k-j+1}} \times \Big[c^*(k-j+1,k)c_1^{(k-j+1)}\varrho^{k-j}S(k-j,k-j) + c^*(k-j+1,k)c_0^{(k-j+1)}\varrho^{k-j+1}S(k-j+1,k-j)\Big].$$

But $c_0^{(k-j+1)} = 1, c_1^{(k-j+1)} = 1+2+\cdots+(k-j) = \frac{1}{2}(k-j)(k-j+1)$, $S(k-j,k-j) = 1$ and $S(k-j+1,k-j) = \binom{k-j+1}{2} = \frac{1}{2}(k-j)(k-j+1)$; so we get

$$\begin{aligned}\lim_{n\to\infty} c^{\varrho}(k-j,k,n) &= \frac{2\varrho}{(\varrho+1)\varrho^{k-j+1}j(j-2k+1)}\\ &\quad\times\left[\frac{(k-j)(k-j+1)}{2}c^*(k-j+1,k)\varrho^{k-j}\right.\\ &\qquad\left.+\frac{(k-j)(k-j+1)}{2}c^*(k-j+1,k)\varrho^{k-j+1}\right]\\ &= \frac{2\varrho^{k-j}(\varrho+1)(k-j)(k-j+1)}{2\varrho^{k-j}(\varrho+1)j(j-2k+1)}c^*(k-j+1,k)\\ &= \frac{(k-j)(k-j+1)}{j(j-2k+1)}c^*(k-j+1,k)\\ &= \frac{(k-j)(k-j+1)}{j(j-2k+1)}\prod_{i=1}^{j-1}\frac{(k-i)(k-i+1)}{i(i-2k+1)}\\ &= \prod_{i=1}^{j}\frac{(k-i)(k-i+1)}{i(i-2k+1)},\end{aligned}$$

which completes the induction. □

5. The Structure of the Dual Functionals

In the first part of this section we provide a recurrence relation for calculating the coefficients $v^{\varrho}(j,k,n)$ of the dual functional $\mu_{\varrho,k}^{(n)}$, i.e.,

$$\mu_{\varrho,k}^{(n)}(f) = \sum_{j=0}^{n} v^{\varrho}(j,k,n)F_{n,j}^{\varrho}(f), \quad k=0,1,\ldots,n.$$

Let $n\geq 1$ be fixed. For each $j\in\{0,1,\ldots,n\}$ there exists a unique polynomial $l_{\varrho,j}^{(n)}$ of degree $\leq n$ satisfying

$$F_{n,i}^{\varrho}(l_{\varrho,j}^{(n)}) = \delta_{i,j}. \tag{5.1}$$

Its coefficients can be determined from a system of linear equations with non-zero determinant. Indeed, consider the positive linear functionals $F_{n,i}^{\varrho}\colon C[0,1]\to\mathbb{R}$, $\varrho>0$, and search the polynomials $l_{\varrho,j}^{(n)}\in\Pi_n$ of the form $l_{\varrho,j}^{(n)} = c_{j0}e_0+c_{j1}e_1+\cdots+c_{jn}e_n$ so that $F_{n,i}^{\varrho}(l_{\varrho,j}^{(n)})=\delta_{i,j}$. For a fixed j we have $F_{n,i}^{\varrho}(l_{\varrho,j}^{(n)}) = c_{j0}F_{n,i}^{\varrho}(e_0)+c_{j1}F_{n,i}^{\varrho}(e_1)+\cdots+c_{jn}F_{n,i}^{\varrho}(e_n)=\delta_{i,j}$ which can be written as a system of linear equations:

$$\begin{cases} c_{j0}F_{n,0}^{\varrho}(e_0)+c_{j1}F_{n,0}^{\varrho}(e_1)+\cdots+c_{jn}F_{n,0}^{\varrho}(e_n)=\delta_{0,j}\\ c_{j0}F_{n,1}^{\varrho}(e_0)+c_{j1}F_{n,1}^{\varrho}(e_1)+\cdots+c_{jn}F_{n,1}^{\varrho}(e_n)=\delta_{1,j}\\ \cdots\\ c_{j0}F_{n,n}^{\varrho}(e_0)+c_{j1}F_{n,n}^{\varrho}(e_1)+\cdots+c_{jn}F_{n,n}^{\varrho}(e_n)=\delta_{n,j}. \end{cases}$$

We claim that

$$A := \begin{vmatrix} F^{\varrho}_{n,0}(e_0) & F^{\varrho}_{n,0}(e_1) & \cdots & F^{\varrho}_{n,0}(e_n) \\ F^{\varrho}_{n,1}(e_0) & F^{\varrho}_{n,1}(e_1) & \cdots & F^{\varrho}_{n,1}(e_n) \\ \cdots & \cdots & \cdots & \cdots \\ F^{\varrho}_{n,n}(e_0) & F^{\varrho}_{n,n}(e_1) & \cdots & F^{\varrho}_{n,n}(e_n) \end{vmatrix} \neq 0. \tag{5.2}$$

We have seen that $F^{\varrho}_{n,i}(e_m) = \dfrac{(i\varrho)_m}{(n\varrho)_m}$, so the determinant becomes

$$A = \begin{vmatrix} 1 & \dfrac{(0\varrho)_1}{(n\varrho)_1} & \dfrac{(0\varrho)_2}{(n\varrho)_2} & \cdots & \dfrac{(0\varrho)_n}{(n\varrho)_n} \\ 1 & \dfrac{(1\varrho)_1}{(n\varrho)_1} & \dfrac{(1\varrho)_2}{(n\varrho)_2} & \cdots & \dfrac{(1\varrho)_n}{(n\varrho)_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \dfrac{(n\varrho)_1}{(n\varrho)_1} & \dfrac{(n\varrho)_2}{(n\varrho)_2} & \cdots & \dfrac{(n\varrho)_n}{(n\varrho)_n} \end{vmatrix}$$

Elementary manipulations of the determinant yield that

$$\begin{aligned} A &= \frac{1}{(n\varrho)_1(n\varrho)_2\cdots(n\varrho)_n} \begin{vmatrix} 1 & 0\cdot\varrho & (0\cdot\varrho)^2 & \cdots & (0\cdot\varrho)^n \\ 1 & \varrho & (\varrho)^2 & \cdots & (\varrho)^n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & n\varrho & (n\varrho)^2 & \cdots & (n\varrho)^n \end{vmatrix} \\ &= \frac{1}{(n\varrho)_1(n\varrho)_2\cdots(n\varrho)_n} \prod_{0\le i<j\le n} (j\cdot\varrho - i\cdot\varrho) \neq 0, \end{aligned}$$

which means that $l^{(n)}_{\varrho,j}$ is uniquely determined. We have by (1.1),

$$U^{\varrho}_n(l^{(n)}_{\varrho,j}) = \sum_{k=0}^{n} F^{\varrho}_{n,k}(l^{(n)}_{\varrho,j}) p_{n,k}$$

and by (3.3) and (3.12),

$$U^{\varrho}_n(l^{(n)}_{\varrho,j}) = \sum_{k=0}^{n} \lambda^{(n)}_{\varrho,k} p^{(n)}_{\varrho,k} \sum_{i=0}^{n} v^{\varrho}(i,k,n) F^{\varrho}_{n,i}(l^{(n)}_{\varrho,j}).$$

By using (5.1) and (3.10) we get successively

$$\begin{aligned} p_{n,j}(x) &= \sum_{k=0}^{n} \lambda^{(n)}_{\varrho,k} p^{(n)}_{\varrho,k}(x) v^{\varrho}(j,k,n), \quad j = 0,1,\ldots,n, \\ \binom{n}{j} x^j (1-x)^{n-j} &= \sum_{k=0}^{n} \lambda^{(n)}_{\varrho,k} \sum_{s=0}^{k} c^{\varrho}(s,k,n) x^s v^{\varrho}(j,k,n), \\ \binom{n}{j} x^j \sum_{i=0}^{n-j} (-1)^i \binom{n-j}{i} x^i &= \sum_{s=0}^{n} \sum_{l=s}^{n} \lambda^{(n)}_{\varrho,l} c^{\varrho}(s,l,n) v^{\varrho}(j,l,n) x^s. \end{aligned}$$

For $i = n - j - k$, equating the coefficients of x^{n-k} we get

$$(-1)^{n-j-k}\binom{n}{j}\binom{n-j}{k} = \sum_{l=n-k}^{n} \lambda_{\varrho,l}^{(n)} c^{\varrho}(n-k,l,n) v^{\varrho}(j,l,n).$$

Setting now $s = n - l$, we get

$$(-1)^{n-j-k}\binom{n}{j}\binom{n-j}{k} = \sum_{s=0}^{k-1} \lambda_{\varrho,n-s}^{(n)} v^{\varrho}(j,n-s,n) c^{\varrho}(n-k,n-s,n) + \lambda_{\varrho,n-k}^{(n)} v^{\varrho}(j,n-k,n).$$

For $k = 0$ this reduces to

$$v^{\varrho}(j,n,n) = (-1)^{n-j}\frac{(n\varrho)_n}{\varrho^n j!(n-j)!}, \tag{5.3}$$

while for $k = 1, \ldots, n$ we get

$$v^{\varrho}(j,n-k,n) = \frac{(-1)^{n-j-k}(n\varrho)_{n-k}}{\varrho^{n-k} j!(n-j-k)!} - \sum_{s=0}^{k-1} \frac{k!}{s!}\varrho^{k-s}\frac{(n\varrho)_{n-k}}{(n\varrho)_{n-s}} v^{\varrho}(j,n-s,n) c^{\varrho}(n-k,n-s,n). \tag{5.4}$$

Now (5.3) and (5.4) constitute the required recurrence.

In the sequel we shall study the limits of the dual functionals, acting on polynomials, as $n \to \infty$. Consider the linear functionals $\mu_k^*: C[0,1] \to \mathbb{R}$ defined by

$$\mu_0^*(f) := \frac{f(0)+f(1)}{2}, \quad \mu_1^*(f) := f(1)-f(0),$$
$$\mu_k^*(f) := \frac{1}{2}\binom{2k}{k}\left((-1)^k f(0)+f(1) - k\int_0^1 f(x) P_{k-2}^{(1,1)}(2x-1)\,\mathrm{d}x\right), \quad k \geq 2,$$

where $(P_j^{(1,1)}(x))_{j\geq 0}$ are the Jacobi polynomials, orthogonal with respect to the weight $(1-t)(1+t)$ on the interval $[-1,1]$.

These functionals were introduced in [4], where it was proved that they are limits of the dual functionals in the setting of Bernstein operators. We shall obtain a similar result for the operators U_n^{ϱ}.

Theorem 5.1. *Let $k \geq 0$ and $\varrho > 0$ be fixed. For every $f \in \Pi$,*

$$\lim_{n\to\infty} \mu_{\varrho,k}^{(n)}(f) = \mu_k^*(f). \tag{5.5}$$

Proof. First we prove that for each $j \geq 0$,

$$\lim_{n\to\infty} \mu_{\varrho,j}^{(n)}(f) = \mu_j^*(f), \quad f \in \Pi_j. \tag{5.6}$$

So, let $f \in \Pi_j$. Because U_n^ϱ is degree reducing and $\lim_{n\to\infty} U_n^\varrho f = f$ (see [7,8]), we have

$$U_n^\varrho f = \sum_{i=0}^{j} \lambda_{\varrho,i}^{(n)} p_{\varrho,i}^{(n)} \mu_{\varrho,i}^{(n)}(f) \to f = \sum_{i=0}^{j} p_i^* \mu_i^*(f), \ n \to \infty.$$

The last equality is a consequence of (4.18) in [4]. Since the above convergence takes place in the finite-dimensional space Π_j, we may consider the coefficients of x^j in order to obtain

$$\lambda_{\varrho,j}^{(n)} \mu_{\varrho,j}^{(n)}(f) \to \mu_j^*(f).$$

Together with $\lambda_{\varrho,j}^{(n)} \to 1$, this leads to (5.6).

We shall prove by induction on $r \geq 0$ that

$$\lim_{n\to\infty} \mu_{\varrho,k}^{(n)}(f) = \mu_k^*(f), \quad \text{for all } k \geq 0, \ \ f \in \Pi_{k+r}, \tag{5.7}$$

and this will complete the proof of (5.5).

For $r = 0$, (5.7) is a consequence of (5.6). Suppose that (5.7) is also true for $1, \ldots, r-1$, and let $f \in \Pi_{k+r}$. As before, we have

$$U_n^\varrho f = \sum_{i=0}^{k+r} \lambda_{\varrho,i}^{(n)} p_{\varrho,i}^{(n)} \mu_{\varrho,i}^{(n)}(f) \to f = \sum_{i=0}^{k+r} p_i^* \mu_i^*(f).$$

By considering the coefficients of x^k as $n \to \infty$ we get

$$\begin{aligned} \lambda_{\varrho,k}^{(n)} \mu_{\varrho,k}^{(n)}(f) &+ \sum_{i=1}^{r} \lambda_{\varrho,k+i}^{(n)} c^\varrho(k, k+i, n) \mu_{\varrho,k+i}^{(n)}(f) \\ &\to \mu_k^*(f) + \sum_{i=1}^{r} c^*(k, k+i) \mu_{k+i}^*(f). \end{aligned} \tag{5.8}$$

We know that for all $i = 1, \ldots, r$,

$$\lambda_{\varrho,k+i}^{(n)} \to 1, c^\varrho(k, k+i, n) \to c^*(k, k+i).$$

By the induction hypothesis, $\mu_{\varrho,k+i}^{(n)}(f) \to \mu_{k+i}^*(f), \ i = 1, \ldots, r$. Now (5.8) implies

$$\lambda_{\varrho,k}^{(n)} \mu_{\varrho,k}^{(n)}(f) \to \mu_k^*(f),$$

and so $\mu_{\varrho,k}^{(n)}(f) \to \mu_k^*(f)$. This concludes the induction. □

6. Applications to Iterates of U_n^ϱ

By Theorem 3.1,

$$(U_n^\varrho)^i f = \sum_{k=0}^{n} (\lambda_{\varrho,k}^{(n)})^i p_{\varrho,k}^{(n)} \mu_{\varrho,k}^{(n)}(f), \quad f \in C[0,1], \ i = 1, 2, \ldots. \tag{6.1}$$

The linear function $B_1(f;x) = f(0)(1-x)+f(1)x$ is the uniform limit of the overiterated operator images $(U_n^\varrho)^i f$, as $i \to \infty$ according to Remark 3.2 in [8]. More generally we have

Corollary 6.1. *Suppose $(g_j)_{j\geq 1}$ is a sequence of polynomials with $g_j(0) = 0$ and*

$$\lim_{j\to\infty} g_j(\lambda_{\varrho,k}^{(n)}) = G(\varrho, k, n), \quad k = 0, 1, \ldots, n.$$

Then

$$\lim_{j\to\infty} (g_j(U_n^\varrho))f = \sum_{k=0}^{n} G(\varrho, k, n) p_{\varrho,k}^{(n)} \mu_{\varrho,k}^{(n)}(f), \tag{6.2}$$

the convergence being uniform.

Proof. By using (6.1) we get

$$(g_j(U_n^\varrho))f = \sum_{k=0}^{n} g_j(\lambda_{\varrho,k}^{(n)}) p_{\varrho,k}^{(n)} \mu_{\varrho,k}^{(n)}(f), \quad f \in C[0,1],\ j = 1, 2, \ldots. \tag{6.3}$$

Letting $j \to \infty$ yields (6.2). □

Lemma 6.2. *Suppose that j_n is a sequence of positive integers with*

$$\lim_{n\to\infty} \frac{j_n}{n} = t, \tag{6.4}$$

then

$$\lim_{n\to\infty} (\lambda_{\varrho,k}^{(n)})^{j_n} = e^{-\frac{k(k-1)}{2}\left(\frac{1}{\varrho}+1\right)t}, \ \textit{for all}\, k, \quad 0 \leq t < \infty, \tag{6.5}$$

and

$$\lim_{n\to\infty} (\lambda_{\varrho,k}^{(n)})^{j_n} = 0, \ \textit{for all}\, k \geq 2, \quad t = \infty. \tag{6.6}$$

Proof. Let

$$\begin{aligned} y_n &= (\lambda_{\varrho,k}^{(n)})^{j_n - nt} \\ &= \left[\left(1 + \frac{1}{n\varrho}\right)^{-1} \cdots \left(1 + \frac{k-1}{n\varrho}\right)^{-1} \left(1 - \frac{1}{n}\right) \cdots \left(1 - \frac{k-1}{n}\right)\right]^{j_n - nt}. \end{aligned}$$

Then

$$\begin{aligned} \log y_n = (j_n - nt) &\left[\log\left(1 + \frac{1}{n\varrho}\right)^{-1} + \cdots + \log\left(1 + \frac{k-1}{n\varrho}\right)^{-1} \right.\\ &\left. + \log\left(1 - \frac{1}{n}\right) + \cdots + \log\left(1 - \frac{k-1}{n}\right)\right] \\ = \left(\frac{j_n}{n} - t\right) &\left(-\frac{k(k-1)}{2}\frac{\varrho+1}{\varrho} + O\left(\frac{1}{n}\right)\right) \to 0, \quad n \to \infty. \end{aligned}$$

Therefore

$$\lim_{n\to\infty} (\lambda_{\varrho,k}^{(n)})^{j_n - nt} = \lim_{n\to\infty} y_n = 1. \tag{6.7}$$

But

$$\lim_{n\to\infty}(\lambda_{\varrho,k}^{(n)})^{nt} = \lim_{n\to\infty}\left\{\left(1+\frac{1}{n\varrho}\right)^{-nt}\cdots\left(1+\frac{k-1}{n\varrho}\right)^{-nt}\right.$$
$$\left.\times\left(1-\frac{1}{n}\right)^{nt}\cdots\left(1-\frac{k-1}{n}\right)^{nt}\right\}$$
$$= e^{-\frac{k(k-1)}{2}\left(\frac{1}{\varrho}+1\right)t}. \tag{6.8}$$

Combining (6.7) and (6.8) gives (6.5). For $t \to \infty$ we obtain (6.6). □

Corollary 6.3. *Suppose that*

$$\lim_{n\to\infty}\frac{j_n}{n} = t.$$

Then for $0 \le t < \infty$,

$$\lim_{n\to\infty}(U_n^\varrho)^{j_n} f = \sum_{k=0}^{s} e^{-\frac{k(k-1)}{2}\left(\frac{1}{\varrho}+1\right)t} p_k^* \mu_k^*(f)$$
$$= \sum_{k=0}^{\infty} e^{-\frac{k(k-1)}{2}\left(\frac{1}{\varrho}+1\right)t} p_k^* \mu_k^*(f), \quad \textit{for all } f \in \Pi_s, \tag{6.9}$$

and for $t = \infty$,

$$\lim_{n\to\infty}(U_n^\varrho)^{j_n} f = B_1 f = \sum_{k=0}^{1} p_k^* \mu_k^*(f), \quad \textit{for all } f \in \Pi. \tag{6.10}$$

The convergence in (6.9) *and* (6.10) *is uniform.*

Proof. Suppose that $f \in \Pi_s$. Since U_n^ϱ is degree reducing, (6.1) gives

$$(U_n^\varrho)^{j_n} f = \sum_{k=0}^{s} (\lambda_{\varrho,k}^{(n)})^{j_n} p_{\varrho,k}^{(n)} \mu_{\varrho,k}^{(n)}(f), \quad n \ge s.$$

Take the limit as $n \to \infty$ in the above and use Lemma 6.2, Theorem 4.1 and Theorem 5.1 to obtain (6.10) and the first equality in (6.9). The second equality in (6.9) follows from (4.19) in [4]. □

Remark 6.4. (1) For $\varrho \to \infty$, each result of this article has a corresponding one in [4], concerning the Bernstein operators B_n.

(2) For $\varrho = 1$ we cover some results concerning the eigenstructure of the genuine Bernstein–Durrmeyer operators, scattered in the literature; see, e.g., [5,6,9] and the references therein.

(3) The Markov semigroup approximated by suitable iterates of B_n was deeply investigated; see [1, Section 6], [2,4,11,12], and the references therein. The semigroup approximated by iterates of U_n was studied in [5,6]. The uniform asymptotic relation given in Theorem 5.2 in [8]

$$\lim_{n\to\infty} n(U_n^\varrho f(x) - f(x)) = \frac{\varrho+1}{2\varrho} x(1-x) f''(x), \qquad f \in C^2[0,1],$$

opens the way for studying the Markov semigroup approximated by iterates of U_n^ϱ, but this will be the subject of a forthcoming paper.

References

[1] Altomare, F., Campiti, M.: Korovkin-Type Approximation Theory and its Applications. Walter de Gruyter, Berlin (1994)

[2] Altomare, F., Leonessa, V., Raşa, I.: On Bernstein-Schnabl operators on the unit interval. Zeit. Anal. Anwend. **27**, 353–379 (2008)

[3] Comtet, L.: Advanced Combinatorics—the Art of Finite and Infinite Expansions. Reidel, Dordrecht (1974)

[4] Cooper, S., Waldron, S.: The eigenstructure of the Bernstein operator. J. Approx. Theory **105**(1), 133–165 (2000)

[5] Gonska, H., Kacsó, D., Raşa, I.: On genuine Bernstein–Durrmeyer operators. Results Math. **50**(3–4), 213–225 (2007)

[6] Gonska, H., Kacsó, D., Raşa, I.: The genuine Bernstein–Durrmeyer operators revisited. Results Math. **62**(3–4), 295–310 (2012)

[7] Gonska, H., Păltănea, R.: Simultaneous approximation by a class of Bernstein–Durrmeyer operators preserving linear functions. Czechoslovak Math. J. **60**(3), 783–799 (2010)

[8] Gonska, H., Păltănea, R.: Quantitative convergence theorems for a class of Bernstein–Durrmeyer operators preserving linear functions. Ukrainian Math. J. **62**(7), 1061–1072 (2010)

[9] Goodman, T.N.T., Sharma, A.: A Bernstein-type operator on the simplex. Math. Balkanica (N.S.) **5**(2), 129–145 (1991)

[10] Păltănea, R.: A class of Durrmeyer type operators preserving linear functions. Ann. Tiberiu Popoviciu Sem. Funct. Equat. Approxim. Convex. (Cluj-Napoca) **5**, 109–117 (2007)

[11] Raşa, I.: Estimates for the semigroup associated with Bernstein operators. Rev. Anal. Numér. Théor. Approx. **33**(2), 243–245 (2004)

[12] Raşa, I.: Estimates for the semigroup associated with Bernstein-Schnabl operators. Carpathian J. Math. **20**(1), 157–162 (2012)

Heiner Gonska and Elena-Dorina Stănilă
Faculty of Mathematics
University of Duisburg-Essen
Forsthausweg 2
47057 Duisburg
Germany
e-mail: heiner.gonska@uni-due.de

Elena-Dorina Stănilă
e-mail: elena.stanila@stud.uni-due.de

Ioan Raşa
Department of Mathematics
Technical University of Cluj-Napoca
Str. Memorandumului nr. 28
400114 Cluj-Napoca
Romania
e-mail: Ioan.Rasa@math.utcluj.ro

Received: January 29, 2013.
Revised: July 1, 2013.
Accepted: September 2, 2013.