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Properties (t) and (gt) for Bounded Linear Operators

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Abstract. In this paper we introduce and study the properties (t) and (gt), which extend properties (w) and (gw). We establish for a bounded linear operator defined on a Banach space several sufficient and necessary conditions for which property (t) and property (gt) hold. We also relate these properties with Weyl's type theorems. We show that if T is a bounded linear operator acting on a Banach space $\mathscr{X}$, then property (gt) holds for T if and only if property (gw) holds for T and $\sigma(T) = \sigma_a(T)$. Analogously, we show that property (t) holds for T if and only if property (w) holds for T and $\sigma(T) = \sigma_a(T)$. We also study the properties (t) and (gt) for the operators satisfying the single valued extension property. Moreover, these properties are also studied in the framework of polaroid operators.

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1. Introduction and Preliminary

Throughout this paper, $\mathscr{X}$ denotes an infinite-dimensional complex Banach space, $\mathscr{L}(\mathscr{X})$ the algebra of all bounded linear operators on $\mathscr{X}$. For $T \in \mathscr{L}(\mathscr{X})$, let T^*, $\ker(T)$, $\Re(T)$, $\sigma(T)$, $\sigma_a(T)$ and $\sigma_s(T)$ denote the *adjoint*, the *null space*, the *range*, the *spectrum*, the *approximate point spectrum* and the *surjectivity spectrum* of T respectively. Let $\alpha(T)$ and $\beta(T)$ be the *nullity* and the *deficiency* of T defined by $\alpha(T) = \dim\ker(T)$ and $\beta(T) = \operatorname{co}\dim\Re(\mathrm{T})$. Let $\mathrm{SF}_+(\mathscr{X}) = \{T \in \mathscr{L}(\mathscr{X}) : \alpha(T) < \infty \text{ and } \Re(T) \text{ is closed}\}$ and $\mathrm{SF}_-(\mathscr{X}) = \{T \in \mathscr{L}(\mathscr{X}) : \beta(T) < \infty\}$ denote the semigroup of upper semi-Fredholm and lower semi-Fredholm operators on $\mathscr{X}$ respectively. An operator $T \in \mathscr{L}(\mathscr{X})$ is said to be *semi-Fredholm* if T is either upper semi-Fredholm or lower semi-Fredholm. If both $\alpha(T)$ and $\beta(T)$ are finite, then T is called *Fredholm operator*. If T is semi-Fredholm operator then *index* of T is defined by $\operatorname{ind}(T) = \alpha(T) - \beta(T)$.

A bounded linear operator T acting on a Banach space $\mathscr{X}$ is *Weyl* if it is Fredholm of index zero and *Browder* if T is Fredholm of finite ascent and descent. Let $\mathbb{C}$ denote the set of complex numbers and let $\sigma(T)$

denote the spectrum of T. The *Weyl spectrum* $\sigma_w(T)$ and *Browder spectrum* $\sigma_b(T)$ of T are defined by $\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\}$ and $\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Browder }\}$ respectively. For $T \in \mathscr{L}(\mathscr{X})$, $\mathrm{SF}_+^-(\mathscr{X}) = \{\mathrm{T} \in \mathrm{SF}_+(\mathscr{X}) : \mathrm{ind}(T) \leq 0\}$. Then the *upper Weyl spectrum* of T is defined by $\sigma_{\mathrm{SF}_+^-}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin \mathrm{SF}_+^-(\mathscr{X})\}$. Let $\Delta(T) = \sigma(T)\backslash\sigma_w(T)$ and $\Delta_a(T) = \sigma_a(T)\backslash\sigma_{\mathrm{SF}_+^-}(T)$. Following Coburn [18], we say that *Weyl's theorem* holds for $T \in \mathscr{L}(\mathscr{X})$ if $\Delta(T) = E^0(T)$, where $E^0(T) = \{\lambda \in \mathrm{iso}\,\sigma(T) : 0 < \alpha(T - \lambda) < \infty\}$. Here and elsewhere in this paper, for $K \subset \mathbb{C}$, iso$K$ is the set of isolated points of K.

According to Rakočević [24], an operator $T \in \mathscr{L}(\mathscr{X})$ is said to satisfy *a-Weyl's theorem* if $\sigma_a(T)\backslash\sigma_{\mathrm{SF}_+^-}(T) = E_a^0(T)$, where

$$E_a^0(T) = \{\lambda \in \mathrm{iso}\,\sigma_{\mathrm{a}}(T) : 0 < \alpha(T - \lambda) < \infty\}.$$

It is known from [24] that an operator satisfying a- Weyl's theorem satisfies Weyl's theorem, but the converse does not hold in general.

For $T \in \mathscr{L}(\mathscr{X})$ and a non negative integer n define $T_{[n]}$ to be the restriction T to $\Re(T^n)$ viewed as a map from $\Re(T^n)$ to $\Re(T^n)$(in particular $T_{[0]} = T$). If for some integer n the range space $\Re(T^n)$ is closed and $T_{[n]}$ is an upper (resp., lower) semi-Fredholm operator, then T is called *upper* (resp., *lower*) *semi-B-Fredholm operator*. In this case index of T is defined as the index of semi-B-Fredholm operator $T_{[n]}$. Moreover, if $T_{[n]}$ is a Fredholm operator then T is called a *B-Fredholm operator*. An operator T is said to be *B-Weyl operator* if it is a B-Fredholm operator of index zero. Let $\sigma_{\mathrm{BW}}(T) = \{\lambda \in \mathbb{C} : T - \lambda \ \text{ is not } B\text{-Weyl}\}$.

Recall that the *ascent*, $\mathrm{a}(T)$, of an operator $T \in \mathscr{L}(\mathscr{X})$ is the smallest non negative integer p such that $\ker(T^p) = \ker(T^{p+1})$ and if such integer does not exist we put $\mathrm{a}(T) = \infty$. Analogously the *descent*, $\mathrm{des}(T)$, of an operator $T \in \mathscr{L}(\mathscr{X})$ is the smallest non negative integer q such that $\Re(T^q) = \Re(T^{q+1})$ and if such integer does not exist we put $\mathrm{d}(T) = \infty$.

According to Berkani [13], an operator $T \in \mathscr{L}(\mathscr{X})$ is said to be *Drazin invertible* if it has finite ascent and descent. The *Drazin spectrum* of T is defined by $\sigma_{\mathrm{D}}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Drazin invertible}\}$. Define the set $\mathrm{LD}(\mathscr{X}) = \{\mathrm{T} \in \mathscr{L}(\mathscr{X}) : \mathrm{a(T)} < \infty \text{ and } \Re(\mathrm{T}^{\mathrm{a(T)}+1}) \text{ is closed}\}$ and $\sigma_{\mathrm{LD}}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin \mathrm{LD}(\mathscr{X})\}$. Following [14], an operator $T \in \mathscr{L}(\mathscr{X})$ is said to be *left Drazin invertible* if $T \in \mathrm{LD}(\mathscr{X})$. We say that $\lambda \in \sigma_a(T)$ is a left pole of T if $T - \lambda \in \mathrm{LD}(\mathscr{X})$, and that $\lambda \in \sigma_a(T)$ is a left pole of T of finite rank if λ if λ is a left pole of T and $\alpha(T - \lambda) < \infty$ [14, Definition 2.6]. Let $\pi_a(T)$ denotes the set of all left poles of T and let $\pi_a^0(T)$ denotes the set of all left poles of finite rank. It follows from [14, Theorem 2.8] that if $T \in \mathscr{L}(\mathscr{X})$ is left Drazin invertible, then T is upper semi-B-Fredholm of index less than or equal to 0.

We say that *Browder's theorem* holds for $T \in \mathscr{L}(\mathscr{X})$ if $\Delta(T) = \pi^0(T)$, where $\pi^0(T)$ is the set of all poles of T of finite rank and that *a-Browder's theorem* holds for T if $\Delta_a(T) = \pi_a^0(T)$. Let $\Delta^g(T) = \sigma(T)\backslash\sigma_{\mathrm{BW}}(T)$. Following [13], we say that *generalized Weyl's theorem* holds for $T \in \mathscr{L}(\mathscr{X})$ if $\Delta^g(T) = E(T)$, $E(T)$ is the set of all eigenvalues of T which are isolated in

$\sigma(T)$, and that *generalized Browder's theorem* holds for T if $\Delta^g(T) = \pi(T)$, where $\pi(T)$ is the set of poles of T. It is proved in [10, Theorem 2.1] that generalized Browder's theorem is equivalent to Browder's theorem.

Let $\mathrm{SBF}_+^-(\mathscr{X})$ denote the class of all *upper semi-B-Fredholm* operators such that $\mathrm{ind}(T) \leq 0$. The *upper B-Weyl spectrum* $\sigma_{\mathrm{SBF}_+^-}(T)$ of T is defined by $\sigma_{\mathrm{SBF}_+^-}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin \mathrm{SBF}_+^-(\mathrm{X})\}$. Let $\Delta_a^g(T) = \sigma_a(T)\backslash\sigma_{\mathrm{SBF}_+^-}(T)$. We say that $T \in \mathscr{L}(\mathscr{X})$ satisfies *generalized a-Weyl's theorem*, if $\sigma_a(T)\backslash\sigma_{\mathrm{SBF}_+^-}(T) = E_a(T)$, where $E_a(T)$ is the set of all eigenvalues of T which are isolated in $\sigma_a(T)$ and that $T \in \mathscr{L}(\mathscr{X})$ satisfies *generalized a-Browder's theorem* if $\Delta_a^g(T) = \pi_a(T)$ [14, Definition 2.13]. It is proved in [10, Theorem 2.2] that generalized a-Browder's theorem is equivalent to a-Browder's theorem.

Following [23], we say that $T \in \mathscr{L}(\mathscr{X})$ satisfies *property* (w) if $\Delta_a(T) = E^0(T)$. The property (w) has been studied in [1,5,23]. In Theorem 2.8 of [5], it is shown that property (w) implies Weyl's theorem, but the converse is not true in general. We say that $T \in \mathscr{L}(\mathscr{X})$ satisfies *property* (gw) if $\Delta_a^g(T) = E(T)$. Property (gw) has been introduced and studied in [11]. Property (gw) extends property (w) to the context of B-Fredholm theory, and it is proved in [11] that an operator possessing property (gw) satisfies property (w) but the converse is not true in general. According to [16], an operator $T \in \mathscr{L}(\mathscr{X})$ is said to possess *property* (gb) if $\Delta_a^g(T) = \pi(T)$, and is said to possess *property* (b) if $\Delta_a(T) = \pi^0(T)$. It is shown in Theorem 2.3 of [16] that an operator possessing property (gb) satisfies property (b) but the converse is not true in general. Following [8], we say an operator $T \in \mathscr{L}(\mathscr{X})$ is said to be satisfies *property* (R) if $\pi_a^0(T) = E^0(T)$. In Theorem 2.4 of [8], it is shown that T satisfies property (w) if and only if T satisfies a-Browder's theorem and T satisfies property (R).

The single valued extension property plays an important role in local spectral theory, see the recent monograph of Laursen and Neumann [21] and Aiena [1]. In this article we shall consider the following local version of this property, which has been studied in recent papers, [5,20] and previously by Finch [19]. Following [19] we say that $T \in \mathscr{L}(\mathscr{X})$ has the *single-valued extension property* (SVEP) at point $\lambda \in \mathbb{C}$ if for every open neighborhood U_λ of λ, the only analytic function $f : U_\lambda \longrightarrow \mathscr{X}$ which satisfies the equation $(T - \mu)f(\mu) = 0$ is the constant function $f \equiv 0$. It is well-known that $T \in \mathscr{L}(\mathscr{X})$ has SVEP at every point of the resolvent $\rho(T) := \mathbb{C}\backslash\sigma(T)$. Moreover, from the identity Theorem for analytic function it easily follows that $T \in \mathscr{L}(\mathscr{X})$ has SVEP at every point of the boundary $\partial\sigma(T)$ of the spectrum. In particular, T has SVEP at every isolated point of $\sigma(T)$. In [20, Proposition 1.8], Laursen proved that if T is of finite ascent, then T has SVEP.

Theorem 1.1. [2, Theorem 1.3] *If $T \in \mathrm{SF}_\pm(\mathbb{X})$ the following statements are equivalent:*

(i) *T has SVEP at λ_0;*
(ii) *$a(T - \lambda_0 I) < \infty$;*

(iii) *$\sigma_a(T)$ does not cluster at λ_0;*
(iv) *$H_0(T-\lambda_0 I)$ is finite dimensional.*

By duality we have

Theorem 1.2. *If $T \in \mathrm{SF}_{\pm}(\mathbb{X})$ the following statements are equivalent:*
(i) *T^* has SVEP at λ_0;*
(ii) *$d(T-\lambda_0 I) < \infty$;*
(iii) *$\sigma_s(T)$ does not cluster at λ_0.*

In this paper we shall consider properties which are related to Weyl type theorem for bounded linear operators $T \in \mathscr{L}(\mathscr{X})$, defined on a complex Banach space $\mathscr{X}$. These properties, that we call *property* (t), means that the isolated points of the spectrum $\sigma(T)$ of T which are eigenvalues of finite multiplicity are exactly those points λ of the spectrum for which $T-\lambda$ is an upper semi-Fredholm with index less than or equal to 0 (see Definition 2.1) and we call *property* (gt), means that the isolated points of the spectrum $\sigma(T)$ of T which are eigenvalues are exactly those points λ of the spectrum for which $T-\lambda$ is an upper semi-B-Fredholm with index less than or equal to 0 (see Definition 2.1). Properties (t) and (gt) are related to a variant of Weyl type theorems. We shall characterize properties (t) and (gt) in several ways and we shall also describe the relationships of it with the other variants of Weyl type theorems. Our main tool is localized version of the SVEP. Also, we consider the properties (t) and (gt) in the frame of polaroid type operators.

2. Properties (t) and (gt)

Let $\Delta_+(T) = \sigma(T) \backslash \sigma_{\mathrm{SF}_+^-}(T)$ and $\Delta_+^g(T) = \sigma(T) \backslash \sigma_{\mathrm{SBF}_+^-}(T)$.

Definition 2.1. *Let $T \in \mathscr{L}(\mathscr{X})$. We say that T satisfies*
(i) *property (t) if $\Delta_+(T) = E^0(T)$.*
(ii) *property (gt) if $\Delta_+^g(T) = E(T)$.*

Theorem 2.2. *Let $T \in \mathscr{L}(\mathscr{X})$. If T satisfies property (gt), then T satisfies property (t).*

Proof. Suppose that T satisfies property (gt), then $\Delta_+^g(T) = E(T)$. If $\lambda \in \Delta_+(T)$, then $\lambda \in \Delta_+^g(T) = E(T)$. Since $\lambda \in \mathrm{iso}\sigma(T)$ and $T-\lambda$ is semi-Fredholm, then $\alpha(T-\lambda) < \infty$. So $\lambda \in E^0(T)$ and $\Delta_+(T) \subseteq E^0(T)$. To show the opposite inclusion, let $\lambda \in E^0(T)$ be arbitrary. Then λ is an eigenvalue isolated in $\sigma(T)$. Since T satisfies property (gt), it follows that $\lambda \in \Delta_+^g(T)$ and $T-\lambda$ is a semi-B-Fredholm operator. As $\alpha(T-\lambda)$ is finite, then it follows from Lemma 2.2 of [11] that $T-\lambda$ is semi-Fredholm of index less than or equal to 0. Hence $\lambda \in \Delta_+(T)$. Therefore, $\Delta_+(T) = E^0(T)$, i.e., T satisfies property (t). □

The converse of the Theorem 2.2 is not true in general as shown by the following example.

Example 2.3. *Let Q be defined for each $x = \{\xi_i\} \in \ell^1(\mathbb{N})$ by*

$$Q(\xi_1, \xi_2, \cdots) = (0, \alpha_1\xi_2, \alpha_2\xi_3, \cdots, \alpha_{k-1}\xi_k, \cdots),$$

where $\{\alpha_i\}$ is a sequence of complex numbers such that $0 < |\alpha_i| \leq 1$ and $\sum_{i=1}^{\infty} |\alpha_i| < \infty$.

Define T on $\mathscr{X} = \ell^1(\mathbb{N}) \oplus \ell^1(\mathbb{N})$ by $T = Q \oplus 0$. Then $\sigma(T) = \sigma_a(T) = \{0\}, E(T) = \{0\}, E^0(T) = \emptyset$. It follows from Example 3.12 of [14] that $\Re(T^n)$ is not closed for all $n \in \mathbb{N}$. This implies that $\sigma_{\mathrm{SF}_+^-}(T) = \sigma_{\mathrm{SBF}_+^-}(T) = \{0\}$. We then have $\Delta_+^g(T) = \emptyset \neq E(T) = \{0\}$ and $\Delta_+(T) = E^0(T)$. Hence T satisfies property (t), but T does not satisfy property (gt).

Theorem 2.4. *Let $T \in \mathscr{L}(\mathscr{X})$. Then the following assertions hold.*

(i) *If T satisfies property (t), then T satisfies property (w).*
(ii) *If T satisfies property (gt), then T satisfies property (gw).*

Proof. (i) Assume that T satisfies property (t), then $\Delta_+(T) = E^0(T)$. Let $\lambda \in \Delta_a(T)$. Then $\lambda \in \Delta_+(T) = E^0(T)$ and so $\Delta_a(T) \subseteq E^0(T)$. Conversely, let $\lambda \in E^0(T)$, then since $\lambda \in \mathrm{iso}\sigma(T)$ and $0 < \alpha(T - \lambda) < \infty$ we know that T and T^* have SVEP at λ. From the equality $\sigma(T)\backslash\sigma_{\mathrm{SF}_+^-}(T) = E^0(T)$ we see that $\lambda \notin \sigma_{\mathrm{SF}_+^-}(T)$ and hence $T - \lambda \in \mathrm{SF}_+(\mathscr{X})$. The SVEP for T and T^* at λ by Remark 1.2 of [5] implies that $a(T - \lambda) = d(T - \lambda) < \infty$. From Theorem 3.4 of [1] we then obtain that $\alpha(T - \lambda) = \beta(T - \lambda) < \infty$, so $\lambda \in \pi^0(T) \subseteq \pi_a^0(T) \subseteq \Delta_a(T)$. Therefore, $\Delta_a(T) = E^0(T)$, i.e., T satisfies property (w).
(ii) Assume that T satisfies property (gt), then $\Delta_+^g(T) = E(T)$. Let $\lambda \in \Delta_a^g(T)$. Then $\lambda \in \Delta_+^g(T) = E(T)$ and so $\Delta_a^g(T) \subseteq E(T)$. Conversely, let $\lambda \in E(T)$, then since $\lambda \in \mathrm{iso}\sigma(T)$ and $0 < \alpha(T - \lambda)$ we know that T and T^* have SVEP at λ. From the equality $\sigma(T)\backslash\sigma_{\mathrm{SBF}_+^-}(T) = E(T)$ we see that $\lambda \notin \sigma_{\mathrm{SBF}_+^-}(T)$ and hence $T - \lambda \in \mathrm{SBF}_+(\mathscr{X})$ The SVEP for T and T^* at λ by Remark 1.2 of [5] implies that $a(T - \lambda) = d(T - \lambda) < \infty$. Hence $\lambda \in \pi(T) \subseteq \pi_a(T) \subseteq \Delta_a^g(T)$. Therefore, $\Delta_a^g(T) = E(T)$, i.e., T satisfies property (gw). □

The following example shows the converse of the previous theorem does not hold in general.

Example 2.5. *Consider the operator $T = R \oplus S$ that defined on $\mathscr{X} = \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$, where R is the right unilateral shift operator and $S(x_1, x_2, \cdots) = (x_2/2, x_3/3, \cdots)$. Then $\sigma(T) = D(0, 1)$, where $D(0, 1)$ is the unit disc of $\mathbb{C}$. Hence, $\mathrm{iso}\sigma(T) = \emptyset$ and so, $E^0(T) = E(T) = \emptyset$. Moreover, $\sigma_a(T) = \sigma_{\mathrm{SF}_+^-}(T) = \sigma_{\mathrm{SBF}_+^-}(T) = C(0, 1) \cup \{0\}$, where $C(0, 1)$ is the unit circle of $\mathbb{C}$. Since $\Delta_a(T) = \emptyset = E^0(T)$ and $\Delta_a^g(T) = E(T)$, then T satisfies both property (w) and property (gw). On the other hand, since $\Delta_+(T) \neq E^0(T)$ and $\Delta_+^g(T) \neq E(T)$, then T does not satisfy property (t) nor the property (gt).*

Theorem 2.6. *Let $T \in \mathscr{L}(\mathscr{X})$. Then the following are equivalent.*

(i) *T satisfies property (gt) if and only if T satisfies property (gw) and $\sigma(T) = \sigma_a(T)$.*

(ii) *T satisfies property (t) if and only if T satisfies property (w) and $\sigma(T) = \sigma_a(T)$*

Proof. (i) If T satisfies property (gt), then T satisfies property (gw) by Theorem 2.4, i.e., $\Delta_+^g(T) = \Delta_a^g(T) = E(T)$ and so $\sigma(T) = \sigma_a(T)$. Conversely, assume that T satisfies property (gw) and $\sigma(T) = \sigma_a(T)$. Then

$$E(T) = \sigma_a(T)\backslash\sigma_{\mathrm{SBF}_+^-}(T) = \sigma(T)\backslash\sigma_{\mathrm{SBF}_+^-}(T).$$

Hence T satisfies property (gt).

(ii) If T satisfies property (t), then T satisfies property (w) by Theorem 2.4, i.e., $\Delta_+(T) = \Delta_a(T) = E^0(T)$ and so $\sigma(T) = \sigma_a(T)$. Conversely, assume that T satisfies property (w) and $\sigma(T) = \sigma_a(T)$. Then

$$E^0(T) = \sigma_a(T)\backslash\sigma_{\mathrm{SF}_+^-}(T) = \sigma(T)\backslash\sigma_{\mathrm{SF}_+^-}(T).$$

Hence T satisfies property (t). □

As a consequence of Theorem 2.4 and [5, Theorem 2.7, Theorem 2.8], we have

Proposition 2.7. *Suppose that $T \in \mathscr{L}(\mathscr{X})$ satisfies property (t), then*

(i) *T satisfies a-Browder's theorem and $\pi_a^0(T) = E^0(T)$.*

(ii) *T satisfies Weyl's theorem.*

Also, as a consequence of Theorem 2.4, [11, Theorem 2.4] and [11, Theorem 2.6], we have

Proposition 2.8. *Suppose that $T \in \mathscr{L}(\mathscr{X})$ satisfies property (gt), then*

(i) *T satisfies generalized Weyl's theorem.*

(ii) *T satisfies generalized a-Browder's theorem and $\pi_a(T) = E(T)$.*

Theorem 2.9. *Let $T \in \mathscr{L}(\mathscr{X})$. Then the following are equivalent.*

(i) *T satisfies property (gt);*

(ii) *T satisfies property (t) and $E(T) = \pi_a(T)$.*

Proof. (i)⇒(ii) Assume that T satisfies property (gt), then T satisfies property (t) by Theorem 2.2. If $\lambda \in E(T)$, then $\lambda \in \mathrm{iso}\sigma(T)$ and since T satisfies property (gt), then $T - \lambda$ is semi-B-Fredholm operator with $\mathrm{ind}(T - \lambda) \leq 0$. From Theorem 4.2 of [12] we deduce that $\lambda \in \pi_a(T)$. By Proposition 2.8, T satisfies generalized a-Browder's theorem, then $\pi_a(T) = \Delta_a^g(T) \subseteq \Delta_+^g(T) = E(T)$. Therefore, $E(T) = \pi_a(T)$.

(ii)⇒(i) Assume that T satisfies property (t) and $E(T) = \pi_a(T)$. Then T satisfies property (w) and $\sigma(T) = \sigma_a(T)$ and hence by Theorem 2.7 of [5] it then follows that T satisfies a-Browder theorem. As we know from Theorem 2.2 of [10] that a-Browder's theorem is equivalent to generalized a-Browder's theorem. Hence we have $\pi_a(T) = \Delta_a^g(T) = \Delta_+^g$. But $E(T) = \pi_a(T)$. Hence T satisfies property (gt). □

Theorem 2.10. *Let $T \in \mathscr{L}(\mathscr{X})$. Then the following are equivalent.*

(i) *T satisfies property (gt) if and only if T satisfies property generalized Weyl's theorem and $\sigma_{\mathrm{BW}}(T) = \sigma_{\mathrm{SBF}_+^-}(T)$.*

(ii) *T satisfies property (t) if and only if T satisfies Weyl's theorem and $\sigma_w(T) = \sigma_{\mathrm{SF}_+^-}(T)$*

Proof. (i) If T satisfies property (gt), then T satisfies generalized Weyl's theorem and $\sigma_{\mathrm{BW}}(T) = \sigma_{\mathrm{SBF}_+^-}(T)$ by Proposition 2.8. Conversely, if T generalized Weyl's theorem and $\sigma_{\mathrm{BW}}(T) = \sigma_{\mathrm{SBF}_+^-}(T)$. Then

$$E(T) = \sigma(T) \backslash \sigma_{\mathrm{BW}}(T) = \sigma(T) \backslash \sigma_{\mathrm{SBF}_+^-}(T).$$

Hence $\Delta_+^g(T) = E(T)$, i.e., T satisfies property (gt).
(ii) If T satisfies property (t), then T satisfies Weyl's theorem and $\sigma_w(T) = \sigma_{\mathrm{SF}_+^-}(T)$ by Proposition 2.7. Conversely, if T Weyl's theorem and $\sigma_w(T) = \sigma_{\mathrm{SF}_+^-}(T)$. Then

$$E^0(T) = \sigma(T) \backslash \sigma_w(T) = \sigma(T) \backslash \sigma_{\mathrm{SF}_+^-}(T).$$

Hence $\Delta_+(T) = E(T)$, i.e., T satisfies property (t). □

The following example shows generalized a-Weyl's theorem and generalized Weyl's theorem do not imply property (gt). It shows also that a-Weyl's theorem and Weyl's theorem do not imply property (t).

Example 2.11. *Let R be the unilateral right shift operator defined on the Hilbert space $\ell^2(\mathbb{N})$. Define T On the Banach space $\mathscr{X} = \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ by $T = 0 \oplus R$. Then $\sigma(T) = \sigma_w(T) = \sigma_{\mathrm{BW}}(T) = D(0,1)$ is the closed disc in $\mathbb{C}$. Hence $\mathrm{iso}\sigma(T) = \emptyset$ and so $E^0(T) = E(T) = \emptyset$. So, $\Delta^g(T) = E(T)$ and $\Delta(T) = E^0(T)$, i.e., T satisfies generalized Weyl's theorem and Weyl's theorem. Moreover, $\sigma_a(T) = \sigma_{\mathrm{SF}_+^-}(T) = C(0,1) \cup \{0\}, \sigma_{\mathrm{SBF}_+^-}(T) = C(0,1)$, where $C(0,1)$ is the unit circle in $\mathbb{C}$. So, $E_a^0(T) = \{0\}$ and $E^a(T) = \{0\}$. Hence $\Delta_a^g(T) = E_a(T)$ and $\Delta_a(T) = E_a^0(T)$, i.e., T satisfies generalized a-Weyl's theorem and a-Weyl's theorem. But T does not satisfy property (gt) or property (t), since $\Delta_+^g(T) \neq E(T)$ and $\Delta_+(T) \neq E^0(T)$.*

The next result shows that the equivalence of property (R), property (t), property (b), property (w), Weyl's theorem and a-Weyl's theorem is true whenever we assume that T^* has SVEP at the points $\lambda \notin \sigma_{\mathrm{SF}_+^-}(T)$.

Theorem 2.12. *Let $T \in \mathscr{L}(\mathscr{X})$. If T^* has SVEP at every $\lambda \notin \sigma_{\mathrm{SF}_+^-}(T)$. Then property ($w$), property ($b$), property ($R$), property ($t$), Weyl's theorem and a-Weyl's theorem are equivalent for T.*

Proof. We conclude from Theorem 2.10 and Theorem 2.19 of [8] that

$$\sigma(T) = \sigma_a(T), \sigma_w(T) = \sigma_b(T) = \sigma_{\mathrm{SF}_+^-}(T) = \sigma_{\mathrm{ub}}(T),$$

and

$$\pi^0(T) = E^0(T), \pi_a^0(T) = E_a^0(T), E^0(T) = \pi_a^0(T).$$

Hence

$$\pi^0(T) = \Delta(T) = \Delta_+(T) = E^0(T) = \Delta_a(T) = E_a^0(T)$$
$$= \sigma_a(T)\backslash\sigma_{\mathrm{ub}}(T) = \pi_a^0(T).$$

Therefore, then property (w), property (b), property (R), property (t), Weyl's theorem and a-Weyl's theorem are equivalent for T. □

Theorem 2.13. *Let $T \in \mathscr{L}(\mathscr{X})$. If T has SVEP at every $\lambda \notin \sigma_{\mathrm{SF}_-^+}(T)$. Then property (w), property (b), property (R), property (t), Weyl's theorem and a-Weyl's theorem are equivalent for T^*.*

Proof. We conclude from Theorem 2.10 and Theorem 2.20 of [8] that

$$\sigma(T^*) = \sigma(T) = \sigma_s(T) = \sigma_a(T^*), \sigma_w(T^*) = \sigma_w(T)$$
$$\sigma_w(T^*) = \sigma_b(T) = \sigma_{\mathrm{SF}_-^+}(T) = \sigma_{\mathrm{lb}}(T) = \sigma_{\mathrm{ub}}(T^*),$$

and

$$\pi^0(T^*) = E^0(T^*), \pi_a^0(T^*) = E_a^0(T^*), E^0(T^*) = \pi_a^0(T^*).$$

Hence

$$\pi^0(T^*) = \Delta(T^*) = \Delta_+(T^*) = E^0(T^*) = \Delta_a(T^*) = E_a^0(T^*)$$
$$= \sigma_a(T^*)\backslash\sigma_{\mathrm{ub}}(T^*) = \pi_a^0(T^*).$$

Therefore, then property (w), property (b), property (R), property (t), Weyl's theorem and a-Weyl's theorem are equivalent for T^*. □

Theorem 2.14. *Suppose that T^* has SVEP at every $\lambda \notin \sigma_{\mathrm{SBF}_+^-}(T)$. Then the following assertions are equivalent:*

(i) $E(T) = \pi(T)$;
(ii) $E_a(T) = \pi_a(T)$;
(iii) $E(T) = \pi_a(T)$.

Consequently, property (gw), property (gb), property (gt), generalized a-Weyl's theorem and generalized Weyl's theorem are equivalent for T.

Proof. Suppose that T^* has SVEP at every $\lambda \notin \sigma_{\mathrm{SBF}_+^-}(T)$. We prove first the equality $\sigma_{\mathrm{SBF}_+^-}(T) = \sigma_{\mathrm{BW}}(T)$. If $\lambda \notin \sigma_{\mathrm{SBF}_+^-}(T)$ then $T - \lambda$ is an upper semi-B-Fredholm operator and $\mathrm{ind}(T - \lambda) \leq 0$. As T^* has SVEP, then it follows from Corollary 2.8 of [15] that $T - \lambda$ is a B-Weyl operator and so $\lambda \notin \sigma_{\mathrm{BW}}(T)$. Therefore, $\sigma_{\mathrm{SBF}_+^-}(T) \subseteq \sigma_{\mathrm{BW}}(T)$. Since the other inclusion is always verified, we have the equality. Now we prove that $\sigma_{\mathrm{D}}(T) = \sigma_{\mathrm{BW}}(T)$. Since $\sigma_{\mathrm{SBF}_+^-}(T) \subseteq \sigma_{\mathrm{SF}_+^-}(T)$ is always verified, then T^* has SVEP at every $\lambda \notin \sigma_{\mathrm{SF}_+^-}(T)$. This implies that T satisfies Browder's theorem. As we know from Theorem 2.1 of [10] that Browder's theorem is equivalent to generalized Browder's theorem, we have $\sigma_{\mathrm{BW}}(T) = \sigma_{\mathrm{D}}(T)$. On the other hand, as T^* has

SVEP at every $\lambda \notin \sigma_{\mathrm{SF}_+^-}(T)$, then $\sigma(T) = \sigma_a(T)$. From this we deduce that $E(T) = E_a(T)$ and

$$\pi_a(T) = \sigma_a(T) \backslash \sigma_{\mathrm{SBF}_+^-}(T) = \sigma(T) \backslash \sigma_{\mathrm{D}}(T) = \pi(T),$$

from which the equivalence of (i), (ii) and (iii) easily follows. To show the last statement observed that the SVEP of T^* at the points $\lambda \notin \sigma_{\mathrm{SBF}_+^-}(T)$ entails that generalized a-Browder's theorem (and hence generalized Browder's theorem) holds for T, see [17, Corollary 2.7]. Therefore,

$$\pi(T) = \Delta^g(T) = E(T) = \Delta_+^g(T) = \Delta_a^g(T) = E_a(T).$$

That is, property (gt), property (gb), property (gw), generalized a-Weyl's theorem and generalized Weyl's theorem are equivalent for T. □

Dually, we have

Theorem 2.15. *Suppose that T has SVEP at every $\lambda \notin \sigma_{\mathrm{SBF}_-^+}(T)$. Then the following assertions are equivalent:*

(i) $E(T^*) = \pi(T^*)$;
(ii) $E_a(T^*) = \pi_a(T^*)$;
(iii) $E(T^*) = \pi_a(T^*)$.

Consequently, property (gb), property (gw), generalized a-Weyl's theorem and generalized Weyl's theorem are equivalent for T^*.

Proof. Suppose that T has SVEP at every $\lambda \notin \sigma_{\mathrm{SBF}_-^+}(T)$. We prove first the equality $\sigma_{\mathrm{SBF}_+^-}(T^*) = \sigma_{\mathrm{BW}}(T^*)$. If $\lambda \notin \sigma_{\mathrm{SBF}_-^+}(T)$ then $T - \lambda$ is a lower semi-B-Fredholm operator and $\mathrm{ind}(T - \lambda) \geq 0$. As T has SVEP, then it follows from Theorem 2.5 of [15] that $T - \lambda$ is a B-Weyl operator and so $\lambda \notin \sigma_{\mathrm{BW}}(T)$. As $\sigma_{\mathrm{BW}}(T) = \sigma_{\mathrm{BW}}(T^*)$. Then $\lambda \notin \sigma_{\mathrm{BW}}(T^*)$. So $\sigma_{\mathrm{BW}}(T^*) \subseteq \sigma_{\mathrm{SBF}_-^+}(T)$. As $\sigma_{\mathrm{SBF}_-^+}(T) = \sigma_{\mathrm{SBF}_+^-}(T^*)$, then $\sigma_{\mathrm{BW}}(T^*) \subseteq \sigma_{\mathrm{SBF}_+^-}(T^*)$. Since the other inclusion is always verified, it then follows that $\sigma_{\mathrm{BW}}(T^*) = \sigma_{\mathrm{SBF}_+^-}(T^*)$. Now we show that $\sigma_{\mathrm{BW}}(T^*) = \sigma_{\mathrm{D}}(T^*)$. Since we have always $\sigma_{\mathrm{SBF}_-^+}(T) \subseteq \sigma_{\mathrm{SF}_-^+}(T)$, then T has SVEP at every $\lambda \in \sigma_{\mathrm{SF}_-^+}(T)$. Hence T^* satisfies generalized Browder's theorem. So $\sigma_{\mathrm{D}}(T^*) = \sigma_{\mathrm{BW}}(T^*)$. Finally, we have $\sigma_{\mathrm{BW}}(T^*) = \sigma_{\mathrm{SBF}_+^-}(T^*) = \sigma_{\mathrm{D}}(T^*)$ and $\sigma(T^*) = \sigma_a(T^*)$, from which we obtain $E(T^*) = E_a(T^*)$ and $\pi(T^*) = \pi_a(T^*)$. The SVEP at every $\lambda \in \sigma_{\mathrm{SBF}_-^+}(T)$ ensure by Corollary 2.7 of [17] that generalized a-Browder's theorem holds for T^*. Hence

$$\pi(T^*) = \Delta^g(T^*) = E(T^*) = \Delta_+^g(T^*) = \Delta_a^g(T^*) = E_a(T^*).$$

That is, property (gb), property (gt), property (gw), generalized a-Weyl's theorem and generalized Weyl's theorem are equivalent for T^*. □

3. Properties (t) and (gt) for polaroid type operators

In this section we study the properties (t) and (gt) for classes of operators for which the isolated points of the spectrum are poles of the resolvent.

Definition 3.1. *([7, Definition 2.5]) A bounded operator $T \in \mathscr{L}(\mathscr{X})$ is said to be left polaroid if every isolated point of $\sigma_a(T)$ is a left pole of the resolvent of T. $T \in \mathscr{L}(\mathscr{X})$ is said to be right polaroid if every isolated point of $\sigma_s(T)$ is a right pole of the resolvent of T. $T \in \mathscr{L}(\mathscr{X})$ is said to be polaroid if every isolated point of $\sigma(T)$ is a pole of the resolvent of T.*

It is well known that λ is a pole of the resolvent of T if and only if λ is a pole of the resolvent of T^*. Since $\sigma(T) = \sigma(T^*)$ we then have

$$T \text{ is polaroid if and only if } T^* \text{ is polaroid.} \tag{3.1}$$

Definition 3.2. *([7, Definition 2.9]) A bounded operator $T \in \mathscr{L}(\mathscr{X})$ is said to be a-polaroid if every $\lambda \in \text{iso}\sigma_a(T)$ is a pole of the resolvent of T.*

Since a pole is always a left pole we have

$$T \text{ is a-polaroid} \Longrightarrow T \text{ is left polaroid,} \tag{3.2}$$

but the converse of implication (3.2) is not true in general, see [7, Example 2.10]. Moreover, it follows from [9, Theorem 2.5] that

$$\text{if } T \text{ is either left or right polaroid} \Longrightarrow T \text{ is polaroid.} \tag{3.3}$$

However, we note that the implication (3.3) cannot be reversed, see [7, Example 2.7].

Since $\text{iso}\sigma(T) \subseteq \sigma_{\mathrm{a}}(T)$ for every $T \in \mathscr{L}(\mathscr{X})$ and the boundary of $\sigma(T)$ is contained in $\sigma_a(T)$, from which we easily obtain:

$$\text{if } T \text{ is } a\text{-polaroid} \Longrightarrow T \text{ is polaroid,} \tag{3.4}$$

while in general the converse is not true.
From Corollary 2.45 of [1] we know that if T^* has SVEP, then $\sigma(T) = \sigma_a(T)$. Therefore, if T^* has SVEP then

$$T \text{ is } a\text{-polaroid if and only if } T \text{ is polaroid.} \tag{3.5}$$

If T has SVEP, we know that $\sigma(T^*) = \sigma_a(T^*)$. Therefore, if T has SVEP, then

$$T^* \ a\text{-polaroid} \Leftrightarrow T^* \text{ polaroid} \Leftrightarrow T \text{ polaroid.} \tag{3.6}$$

Theorem 3.3. *Suppose that T is a-polaroid and $\sigma(T) = \sigma_a(T)$. Then the following assertions hold.*

(i) *T satisfies a-Weyl's theorem if and only if T satisfies property (t).*
(ii) *T satisfies generalized a-Weyl's theorem if and only if T satisfies property (gt).*

Proof. (i) Note first that if T is a-polaroid then $E_a^0(T) = \pi^0(T)$. In fact, if $\lambda \in E_a^0(T)$ then λ is isolated in $\sigma_a(T)$ and hence $a(T-\lambda) = d(T-\lambda) < \infty$. Moreover, $\alpha(T-\lambda) < \infty$, so by Theorem 3.4 of [1] it follows that $\beta(T-\lambda)$ is also finite, thus $\lambda \in \pi^0(T)$. This shows $E_a^0(T) \subseteq \pi^0(T)$. Since the other inclusion is always verified, we have $E_a^0(T) = \pi^0(T)$.
Now, if T satisfies a-Weyl's theorem then $\Delta_a(T) = E_a^0(T) = \pi^0(T)$, and since Weyl's theorem holds for T by Theorem 2.4 of [5] that $\pi^0(T) = E^0(T)$. Therefore,

$$E^0(T) = E_a^0(T) = \Delta_a(T) = \sigma_a(T)\setminus\sigma_{\mathrm{SF}_+^-}(T) = \sigma(T)\setminus\sigma_{\mathrm{SF}_+^-}(T).$$

That is, T satisfies property (t).

Conversely, if T satisfies property (t) then $\Delta_+(T) = E^0(T)$. Since by Proposition 2.7 T satisfies Weyl's theorem we also have by Theorem 2.4 of [5] that $E^0(T) = \pi^0(T) = E_a^0(T)$. Hence

$$E_a^0(T) = E^0(T) = \Delta_+(T) = \sigma(T)\setminus\sigma_{\mathrm{SF}_+^-}(T) = \sigma_a(T)\setminus\sigma_{\mathrm{SF}_+^-}(T).$$

That is, T satisfies a-Weyl's theorem.
(ii) Note first that if T is a-polaroid then $E_a(T) = \pi(T)$. In fact, if $\lambda \in E_a(T)$ then λ is isolated in $\sigma_a(T)$ and hence $a(T-\lambda) = d(T-\lambda) < \infty$. Therefore, $\lambda \in \pi(T)$. This shows $E_a(T) \subseteq \pi(T)$. Since we have always $\pi(T) \subseteq E_a(T)$, and so $E_a(T) = \pi(T)$.
Now, if T satisfies generalized a-Weyl's theorem then $\Delta_a^g(T) = E_a(T) = \pi(T)$, and since generalized Weyl's theorem holds for T by Corollary 2.1 of [10] that $\pi(T) = E(T)$. Therefore,

$$E(T) = E_a(T) = \Delta_a^g(T) = \sigma_a(T)\setminus\sigma_{\mathrm{SBF}_+^-}(T) = \sigma(T)\setminus\sigma_{\mathrm{SBF}_+^-}(T).$$

That is, T satisfies property (gt).

Conversely, if T satisfies property (gt) then $\Delta_+^g(T) = E(T)$. Since by Proposition 2.8 T satisfies generalized Weyl's theorem we also have by Corollary 2.1 of [10] that $E(T) = \pi(T) = E_a(T)$. Hence

$$E_a(T) = E(T) = \Delta_+^g(T) = \sigma(T)\setminus\sigma_{\mathrm{SBF}_+^-}(T) = \sigma_a(T)\setminus\sigma_{\mathrm{SBF}_+^-}(T).$$

That is, T satisfies generalized a-Weyl's theorem. □

Theorem 3.4. *Suppose that $T \in \mathscr{L}(\mathscr{X})$ is polaroid.*

(i) *If T^* has SVEP, then property (t) holds for T.*
(ii) *If T has SVEP, then property (t) holds for T^*.*

Proof. (i) Since T^* has SVEP, then it follows from [1, Corollary 2.45] that $\sigma(T) = \sigma_a(T), \sigma_{\mathrm{ub}}(T) = \sigma_{\mathrm{SF}_+^-}(T)$, see also [3, Theorem 1.5]. Since T is polaroid and T^* has SVEP, then by equivalence 3.5, we have T is a-polaroid and so $\pi^0(T) = \pi_a^0(T) = E^0(T) = E_a^0(T)$, see proof of Theorem 3.3. Therefore,

$$E^0(T) = \pi_a^0(T) = \sigma_a(T)\setminus\sigma_{\mathrm{ub}}(T) = \sigma(T)\setminus\sigma_{\mathrm{SF}_+^-}(T).$$

That is, T satisfies property (t).
(ii) The SVEP for T implies by Corollary 2.45 of [1] that $\sigma(T^*) = \sigma(T) = \sigma_a(T^*)$ and $\sigma_{\mathrm{lb}}(T) = \sigma_{\mathrm{ub}}(T^*) = \sigma_{\mathrm{SF}_-^+}(T) = \sigma_{\mathrm{SF}_+^-}(T^*)$. Since T^* is polaroid

by equivalence 3.1 and T has SVEP, then T^* is a-polaroid and so $\pi^0(T^*) = \pi_a^0(T^*) = E^0(T^*) = E_a^0(T^*)$. Therefore,

$$E^0(T^*) = \pi_a^0(T^*) = \sigma_a(T^*) \backslash \sigma_{\mathrm{ub}}(T^*) = \sigma(T^*) \backslash \sigma_{\mathrm{SF}_+^-}(T^*).$$

That is, T^* satisfies property (t). □

Theorem 3.5. *Suppose that $T \in \mathscr{L}(\mathscr{X})$ is polaroid.*

(i) *If T^* has SVEP, then property (gt) holds for T.*
(ii) *If T has SVEP, then property (gt) holds for T^*.*

Proof. (i) Since T^* has SVEP, then it follows from [1, Corollary 2.45] that $\sigma(T) = \sigma_a(T)$ and from the proof of Theorem 2.14 we then have $\sigma_{\mathrm{LD}}(T) = \sigma_{\mathrm{SBF}_+^-}(T)$. Since T is polaroid and T^* has SVEP, then by equivalence 3.5, we have T is a-polaroid and so $\pi(T) = \pi_a(T) = E(T) = E_a(T)$, see proof of Theorem 3.3. Therefore,

$$E(T) = \pi_a(T) = \sigma_a(T) \backslash \sigma_{\mathrm{LD}}(T) = \sigma(T) \backslash \sigma_{\mathrm{SBF}_+^-}(T).$$

That is, T satisfies property (gt).
(ii) The SVEP for T implies by Corollary 2.45 of [1] that $\sigma(T^*) = \sigma(T) = \sigma_a(T^*)$ and from the proof of Theorem 2.15 we then have $\sigma_{\mathrm{LD}}(T^*) = \sigma_{\mathrm{SBF}_+^-}(T^*)$. Since T^* is polaroid by equivalence 3.1 and T has SVEP, then T^* is a-polaroid and so $\pi(T^*) = \pi_a(T^*) = E(T^*) = E_a(T^*)$. Therefore,

$$E(T^*) = \pi_a(T^*) = \sigma_a(T^*) \backslash \sigma_{\mathrm{LD}}(T^*) = \sigma(T^*) \backslash \sigma_{\mathrm{SBF}_+^-}(T^*).$$

That is, T^* satisfies property (gt). □

Let $H_{\mathrm{nc}}(\sigma(T))$ denote the set of all analytic functions, defined on an open neighborhood of $\sigma(T)$, such that f is non-constant on each of the components of its domain. Define, by the classical calculus, $f(T)$ for every $f \in H_{\mathrm{nc}}(\sigma(T))$.

Theorem 3.6. *Suppose that T is polaroid and $f \in H_{\mathrm{nc}}(\sigma(T))$.*

(i) *If T^* has SVEP, then property (t) holds for $f(T)$, or equivalently property (w), property (R), property (b), Weyl's theorem, a-Weyl's theorem hold for $f(T)$.*
(ii) *If T has SVEP, then property (t) holds for $f(T^*)$, or equivalently property (w), property (R), property (b), Weyl's theorem, a-Weyl's theorem hold for $f(T^*)$.*

Proof. (i) It follows from Lemma 3.11 of [7] that $f(T)$ is polaroid. By Theorem 2.40 of [1], we have $f(T^*)$ has SVEP, hence from equivalence 3.5 we conclude that $f(T)$ is a-polaroid and hence it then follows by Theorem 3.4 that property (t) holds for $f(T)$ and this by Theorem 2.12 is equivalent to saying that property (w), property (R), property (b), Weyl's theorem, a-Weyl's theorem hold for $f(T)$.
(ii) From the equivalence 3.1 we know that T^* is polaroid and hence by Lemma 3.11 of [7] that $f(T^*)$ is polaroid. By Theorem 2.40 of [1], we have $f(T)$ has SVEP, hence from equivalence 3.6 we conclude that $f(T^*)$ is a-polaroid and hence it then follows by Theorem 3.4 that property (t) holds

for $f(T^*)$ and this by Theorem 2.13 is equivalent to saying that property (w), property (R), property (b), Weyl's theorem, a-Weyl's theorem hold for $f(T^*)$. $\square$

Theorem 3.7. *Suppose that T is polaroid and $f \in H_{\mathrm{nc}}(\sigma(T))$.*

(i) *If T^* has SVEP, then property (gt) holds for $f(T)$, or equivalently property (gw), property (gb), generalized Weyl's theorem, generalized a-Weyl's theorem hold for $f(T)$.*

(ii) *If T has SVEP, then property (gt) holds for $f(T^*)$, or equivalently property (gw), property (gb), generalized Weyl's theorem, generalized a-Weyl's theorem hold for $f(T^*)$.*

Proof. (i) It follows from Lemma 3.11 of [7] that $f(T)$ is polaroid. By Theorem 2.40 of [1], we have $f(T^*)$ has SVEP, hence from equivalence 3.5 we conclude that $f(T)$ is a-polaroid and hence it then follows by Theorem 3.5 that property (gt) holds for $f(T)$ and this by Theorem 2.12 is equivalent to saying that property (gw), property (gb), generalized Weyl's theorem, generalized a-Weyl's theorem hold for $f(T)$.
(ii) From the equivalence 3.1 we know that T^* is polaroid and hence by Lemma 3.11 of [7] that $f(T^*)$ is polaroid. By Theorem 2.40 of [1], we have $f(T)$ has SVEP, hence from equivalence 3.6 we conclude that $f(T^*)$ is a-polaroid and hence it then follows by Theorem 3.5 that property (gt) holds for $f(T^*)$ and this by Theorem 2.13 is equivalent to saying that property (gw), property (gb), generalized Weyl's theorem, generalized a-Weyl's theorem hold for $f(T^*)$. $\square$

An important T-invariant subspace in local spectral theory is the quasi-nilpotent part of T, defined as

$$H_0(T) := \left\{ x \in \mathscr{X} : \lim_{n\to\infty} \|T^n x\|^{\frac{1}{n}} = 0 \right\}.$$

we have also

$$H_0(T-\lambda)\,\text{closed} \Rightarrow T\,\text{has SVEP at}\,\lambda. \tag{3.7}$$

In the case of Hilbert space operators the SVEP for the dual T^* and the SVEP for the Hilbert adjoint T' are equivalent. Consequently, for Hilbert space operators in the statements of Theorem 3.6, T^* may be replaced by T'.

The condition of being polaroid may be characterized by means of the quasi-nilpotent part: T is polaroid if and only if there exists $p := p(T-\lambda) \in \mathbb{N}$ such that:

$$H_0(T-\lambda) = \ker(T-\lambda)^p \text{ for all } \lambda \in \operatorname{iso}\sigma(T), \tag{3.8}$$

see [4]. The class of polaroid operators having SVEP is very large. An interesting class of polaroid operators is given by the $H(p)$-operators, where an operator $T \in \mathscr{L}(\mathscr{X})$ is said to belong to the class $H(p)$ if there exists a natural $p := p(\lambda)$ such that:

$$H_0(T-\lambda) = \ker(T-\lambda)^p \text{ for all } \lambda \in \mathbb{C}. \tag{3.9}$$

From the implication 3.7 it is obvious that every operator T which belongs to the class $H(p)$ has SVEP. Moreover, from (3.8) we also see that every $H(p)$ operator T is polaroid. The class $H(p)$ has been introduced by Oudghiri in [22]. Property $H(p)$ is satisfied by every generalized scalar operator (see [21] for definition and properties of this class), and in particular the property $H(p)$ is satisfied by algebraically w-hyponormal [27], algebraically $wF(p,r,q)$ operators [26], algebraically (p,k)-quasihyponormal operators [28] or quasi-class (A,k) operators [25].

Theorem 3.8. *Suppose that $T \in \mathscr{L}(\mathscr{X})$ and $f \in H_{nc}(\sigma(T))$. Then the following assertions hold:*

- (i) *If T^* has SVEP and T is left polaroid, then property (t) holds for $f(T)$, or equivalently, property (w), Weyl's theorem and a-Weyl's theorem hold for $f(T)$.*
- (ii) *If T has SVEP and T is right polaroid, then property (t) holds for $f(T^*)$, or equivalently, property (w), Weyl's theorem and a-Weyl's theorem hold for $f(T^*)$.*

Proof. (i) Let $\lambda \in \text{iso}\sigma(T)$. Since T^* has SVEP, it then follows by Corollary 2.45 of [1] that $\sigma(T) = \sigma_a(T)$, so $\lambda \in \text{iso}\sigma_{\rm a}(\rm T)$ and hence a left pole for T. In particular, $T-\lambda$ is left Drazin invertible and hence $a(T-\lambda) < \infty$. By [6, Theorem 2.5] we know that $\lambda - T$ is semi-B-Fredholm, i.e., there exists a natural number $n \in \mathbb{N}$ such that $\Re(T-\lambda)^n$ is closed and the restriction $T-\lambda|_{\Re(T-\lambda)^n}$ is semi-Fredholm, in particular $T-\lambda$ is quasi-Fredholm. The SVEP for T^* implies that $d(T-\lambda) < \infty$. Hence λ is a pole of the resolvent of T. This proves that T is polaroid. By Lemma 3.11 of [7], $f(T)$ is polaroid and since by [1, Theorem 2.40], $f(T^*)$ has SVEP, the assertion follows now from part (i) of Theorem 3.6.

(ii) Suppose that T has SVEP and T is right polaroid. The SVEP of T entails $\sigma_s(T) = \sigma(T)$, see Corollary 2.5 of [1]. Let $\lambda \in \text{iso}\sigma(T)$. Then $\lambda \in \sigma_s(T)$ and hence is a right pole of T. Therefore, $d(T-\lambda) < \infty$. On the other hand, since $T-\lambda$ is right Drazin invertible, then $T-\lambda$ is semi B-Fredholm, in particular $T-\lambda$ is quasi-Fredholm. The SVEP for T at λ entails that $a(t-\lambda) < \infty$. Consequently, λ is a pole of the resolvent of T and hence T is polaroid. By Lemma 3.11 of [7], $f(T^*)$ is polaroid and since $f(T)$ has SVEP, the assertion follows now from part (ii) of Theorem 3.6. □

Similar to the proof of Theorem 3.8, we can prove the following result.

Theorem 3.9. *Suppose that $T \in \mathscr{L}(\mathscr{X})$ and $f \in H_{\rm nc}(\sigma(T))$. Then the following assertions hold:*

- (i) *If T^* has SVEP and T is left polaroid, then property (gt) holds for $f(T)$, or equivalently, property (gw), generalized Weyl's theorem and generalized a-Weyl's theorem hold for $f(T)$.*
- (ii) *If T has SVEP and T is right polaroid, then property (gt) holds for $f(T^*)$, or equivalently, property (gw), generalized Weyl's theorem and generalized a-Weyl's theorem hold for $f(T^*)$.*

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