Mediterr. J. Math. 11 (2014), 463–475
DOI 10.1007/s00009-013-0310-0
0378-620X/14/020463-13
published online June 14, 2013
© Springer Basel 2013

Mediterranean Journal of Mathematics

Oscillation Theorems for Fourth-Order Half-Linear Delay Dynamic Equations with Damping

Ravi P. Agarwal, Martin Bohner, Tongxing Li, and Chenghui Zhang

Abstract. This article is concerned with oscillatory behavior of a class of fourth-order half-linear delay dynamic equations with damping on a time scale. Some new oscillation criteria are established.

Mathematics Subject Classification (2010). 34K11, 34N05, 39A10.

Keywords. Oscillatory solution, delay equation, dynamic equation with damping, fourth-order equation, time scale.

1. Introduction

Consider the oscillatory behavior of a fourth-order half-linear delay dynamic equation with damping

$$(r(x^{\Delta^3})^\gamma)^\Delta(t) + p(t)(x^{\Delta^3})^\gamma(t) + q(t)x^\gamma(\tau(t)) = 0 \tag{1.1}$$

on a time scale $\mathbb{T}$ unbounded above; here $\gamma > 0$ is the ratio of positive odd integers, r, p, and q are positive real-valued rd-continuous functions defined on $\mathbb{T}$, $r(t) - \mu(t)p(t) \neq 0, \tau \in \mathrm{C_{rd}}(\mathbb{T}, \mathbb{T}), \tau(t) \leq t$, and $\tau(t) \to \infty$ as $t \to \infty$.

Since we are interested in oscillation, we assume throughout this paper that the given time scale $\mathbb{T}$ is unbounded above and is a time scale interval of the form $[t_0, \infty)_\mathbb{T} := [t_0, \infty) \cap \mathbb{T}$ with $t_0 \in \mathbb{T}$. On any time scale we define the forward and backward jump operators by $\sigma(t) := \inf\{s \in \mathbb{T} | s > t\}$ and $\rho(t) := \sup\{s \in \mathbb{T} | s < t\}$, where $\inf \emptyset := \sup \mathbb{T}$ and $\sup \emptyset := \inf \mathbb{T}$, $\emptyset$ denotes the empty set. A point $t \in \mathbb{T}$ is said to be left-dense if $\rho(t) = t$ and $t > \inf \mathbb{T}$, right-dense if $\sigma(t) = t$ and $t < \sup \mathbb{T}$, left-scattered if $\rho(t) < t$, and right-scattered if $\sigma(t) > t$. The graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) := \sigma(t) - t$, and for any function $f : \mathbb{T} \to \mathbb{R}$ the notation $f^\sigma(t) := f(\sigma(t))$. For further discussion on time scales, we refer the reader to [5,6,13].

By a solution of (1.1) we mean a nontrivial real-valued function $x \in \mathrm{C^3_{rd}}[T_x, \infty)_\mathbb{T}$, $T_x \in [t_0, \infty)_\mathbb{T}$ which has the property $r(x^{\Delta^3})^\gamma \in \mathrm{C^1_{rd}}[T_x, \infty)_\mathbb{T}$

and satisfies (1.1) on $[T_x, \infty)_{\mathbb{T}}$. The solutions vanishing in some neighbourhood of infinity will be excluded from our consideration. A solution x of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is nonoscillatory. Equation (1.1) is called oscillatory if all its solutions are oscillatory.

Recently, there has been much activity concerning oscillatory and asymptotic behavior of different classes of dynamic equations on time scales. We refer the reader to the books [2,5,6,17], the papers [1,3,4,7–12,14–16,18–20], and the references cited therein. Saker et al. [18] considered a second-order dynamic equation with damping

$$(rx^{\Delta})^{\Delta}(t) + p(t)x^{\Delta^{\sigma}}(t) + q(t)f(x^{\sigma}(t)) = 0.$$

Erbe et al. [7] studied a class of second-order delay dynamic equations with a nonlinear damping

$$(r(x^{\Delta})^{\gamma})^{\Delta}(t) + p(t)(x^{\Delta^{\sigma}}(t))^{\gamma} + q(t)f(x(\tau(t))) = 0.$$

Erbe et al. [8] investigated a third-order dynamic equation

$$x^{\Delta^3}(t) + q(t)x(t) = 0.$$

Agarwal et al. [1], Hassan [12], and Li et al. [15] considered a third-order nonlinear delay dynamic equation

$$\left(a((rx^{\Delta})^{\Delta})^{\gamma}\right)^{\Delta}(t) + f(t, x(\tau(t))) = 0.$$

Grace et al. [9] studied a fourth-order dynamic equation

$$x^{\Delta^4}(t) + q(t)x^{\gamma}(\sigma(t)) = 0.$$

Monotone and oscillatory behavior of solutions of a fourth-order dynamic equation

$$(a(x^{\Delta^2})^{\alpha})^{\Delta^2}(t) + q(t)x^{\beta}(\sigma(t)) = 0$$

with the property that

$$\frac{x(t)}{\int_{t_0}^{t}\int_{t_0}^{s} a^{-1/\alpha}(\tau)\Delta\tau\Delta s} \to 0 \quad \text{as} \quad t \to \infty$$

were established by Grace et al. [10]. Grace et al. [11] considered a fourth-order dynamic equation

$$x^{\Delta^4}(t) + q(t)x^{\gamma}(t) = 0.$$

Li et al. [16] studied oscillation of unbounded solutions to a fourth-order delay dynamic equation

$$(rx^{\Delta^3})^{\Delta}(t) + q(t)x(\tau(t)) = 0$$

under the assumption $\int_{t_0}^{\infty} r^{-1}(t)\Delta t < \infty$ and obtained some comparison theorems. Zhang et al. [20] investigated a fourth-order dynamic equation

$$(rx^{\Delta^3})^{\Delta}(t) + q(t)f(x(\sigma(t))) = 0$$

in the case where $\int_{t_0}^{\infty} r^{-1}(t)\Delta t < \infty$.

So far, there are few results on oscillatory behavior of (1.1). Hence the aim of this paper is to give some oscillation criteria for this equation.

2. Main Results

In what follows, all functional inequalities are assumed to hold eventually, that is, they are satisfied for all sufficiently large t. We begin with the following lemma.

Lemma 2.1 (See [5, Theorem 2.33]). *Assume $1+\mu(t)p(t)\neq 0$ and fix $t_0\in\mathbb{T}$. Then $e_p(\cdot,t_0)$ is a solution of the initial value problem*

$$y^\Delta = p(t)y, \quad y(t_0)=1.$$

Lemma 2.2. *Assume x is an eventually positive solution of (1.1). If*

$$\int_{t_0}^{\infty}\left(\frac{e_{-p/r}(t,t_0)}{r(t)}\right)^{1/\gamma}\Delta t=\infty, \tag{2.1}$$

then there are only the following two possible cases for $t\in[t_1,\infty)_\mathbb{T}$, where $t_1\in[t_0,\infty)_\mathbb{T}$ sufficiently large:

(1) $x(t)>0, x^\Delta(t)>0, x^{\Delta\Delta}(t)>0, x^{\Delta^3}(t)>0, (r(x^{\Delta^3})^\gamma)^\Delta(t)<0$;
(2) $x(t)>0, x^\Delta(t)>0, x^{\Delta\Delta}(t)<0, x^{\Delta^3}(t)>0, (r(x^{\Delta^3})^\gamma)^\Delta(t)<0$.

Proof. Let x be an eventually positive solution of (1.1). Then there exists a $t_1\in[t_0,\infty)_\mathbb{T}$ such that $x(t)>0$ and $x(\tau(t))>0$ for $t\in[t_1,\infty)_\mathbb{T}$. From (1.1), we have

$$(r(x^{\Delta^3})^\gamma)^\Delta(t)+p(t)(x^{\Delta^3})^\gamma(t)=-q(t)x^\gamma(\tau(t))<0 \tag{2.2}$$

for $t\in[t_1,\infty)_\mathbb{T}$. Hence, we obtain by (2.2) and Lemma 2.1 that

$$\begin{aligned}\left(\frac{r(x^{\Delta^3})^\gamma}{e_{-p/r}(\cdot,t_0)}\right)^\Delta &= \frac{(r(x^{\Delta^3})^\gamma)^\Delta e_{-p/r}(\cdot,t_0)-r(x^{\Delta^3})^\gamma e^\Delta_{-p/r}(\cdot,t_0)}{e_{-p/r}(\cdot,t_0)e^\sigma_{-p/r}(\cdot,t_0)}\\ &=\frac{(r(x^{\Delta^3})^\gamma)^\Delta+p(x^{\Delta^3})^\gamma}{e^\sigma_{-p/r}(\cdot,t_0)}<0.\end{aligned}$$

Thus, $r(x^{\Delta^3})^\gamma/e_{-p/r}(\cdot,t_0)$ is decreasing. Then $x^\Delta, x^{\Delta\Delta}$, and x^{Δ^3} are of constant sign eventually. We claim that $x^{\Delta^3}>0$. If not, there exist a constant $M>0$ and $t_2\in[t_1,\infty)_\mathbb{T}$ such that

$$\frac{r(t)(x^{\Delta^3})^\gamma(t)}{e_{-p/r}(t,t_0)}\leq -M<0, \quad t\in[t_2,\infty)_\mathbb{T}.$$

That is,

$$x^{\Delta^3}(t)\leq -M^{1/\gamma}\left(\frac{e_{-p/r}(t,t_0)}{r(t)}\right)^{1/\gamma}, \quad t\in[t_2,\infty)_\mathbb{T}.$$

Integrating this from t_2 to t, we get

$$x^{\Delta\Delta}(t) \le x^{\Delta\Delta}(t_2) - M^{1/\gamma} \int_{t_2}^{t} \left(\frac{e_{-p/r}(s,t_0)}{r(s)} \right)^{1/\gamma} \Delta s.$$

Hence, we have $\lim_{t\to\infty} x^{\Delta\Delta}(t) = -\infty$, and then $\lim_{t\to\infty} x(t) = -\infty$, which is a contradiction. If

$$x^{\Delta\Delta} > 0,$$

then $x^{\Delta} > 0$ due to $x^{\Delta^3} > 0$. If

$$x^{\Delta\Delta} < 0,$$

then $x^{\Delta} > 0$ by $x > 0$. The proof is complete. □

Lemma 2.3. *Assume x is a solution of* (1.1) *which satisfies case* (1) *of Lemma 2.2. Then*

$$x^{\Delta\Delta}(t) \ge \left(r^{1/\gamma}(t) \int_{t_1}^{t} \frac{\Delta s}{r^{1/\gamma}(s)} \right) x^{\Delta^3}(t). \tag{2.3}$$

If there exist a function $\phi \in \mathrm{C}^1_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}}, (0,\infty))$ and a $t_2 \in [t_1,\infty)_{\mathbb{T}}$ such that

$$\frac{\phi(t)}{r^{1/\gamma}(t) \int_{t_1}^{t} \frac{\Delta s}{r^{1/\gamma}(s)}} - \phi^{\Delta}(t) \le 0, \quad t \in [t_2,\infty)_{\mathbb{T}}, \tag{2.4}$$

then $x^{\Delta\Delta}/\phi$ is a nonincreasing function on $t \in [t_2,\infty)_{\mathbb{T}}$, and

$$x^{\Delta}(t) \ge \left(\frac{1}{\phi(t)} \int_{t_2}^{t} \phi(s)\Delta s \right) x^{\Delta\Delta}(t) \quad \textit{for} \quad t \in [t_2,\infty)_{\mathbb{T}}. \tag{2.5}$$

Further, if there exist a function $\varphi \in \mathrm{C}^1_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}}, (0,\infty))$ and a $t_3 \in [t_2,\infty)_{\mathbb{T}}$ such that

$$\frac{\varphi(t)}{\frac{1}{\phi(t)} \int_{t_2}^{t} \phi(s)\Delta s} - \varphi^{\Delta}(t) \le 0, \quad t \in [t_3,\infty)_{\mathbb{T}}, \tag{2.6}$$

then x^{Δ}/φ is a nonincreasing function on $t \in [t_3,\infty)_{\mathbb{T}}$, and

$$x(t) \ge \left(\frac{1}{\varphi(t)} \int_{t_3}^{t} \varphi(s)\Delta s \right) x^{\Delta}(t) \quad \textit{for} \quad t \in [t_3,\infty)_{\mathbb{T}}. \tag{2.7}$$

Proof. From $x^{\Delta\Delta} > 0$, $x^{\Delta^3} > 0$, and $(r(x^{\Delta^3})^{\gamma})^{\Delta} < 0$, we have

$$x^{\Delta\Delta}(t) = x^{\Delta\Delta}(t_1) + \int_{t_1}^{t} \frac{(r(x^{\Delta^3})^{\gamma})^{1/\gamma}(s)}{r^{1/\gamma}(s)} \Delta s \ge \left(r^{1/\gamma}(t) \int_{t_1}^{t} \frac{\Delta s}{r^{1/\gamma}(s)} \right) x^{\Delta^3}(t).$$

Thus,

$$\left(\frac{x^{\Delta\Delta}}{\phi}\right)^{\Delta}(t)=\frac{x^{\Delta^3}(t)\phi(t)-x^{\Delta\Delta}(t)\phi^{\Delta}(t)}{\phi(t)\phi^{\sigma}(t)}\leq\frac{x^{\Delta\Delta}(t)}{\phi(t)\phi^{\sigma}(t)}\left(\frac{\phi(t)}{r^{1/\gamma}(t)\int_{t_1}^{t}\frac{\Delta s}{r^{1/\gamma}(s)}}-\phi^{\Delta}(t)\right)\leq 0.$$

Therefore, $x^{\Delta\Delta}/\phi$ is a nonincreasing function on $[t_2,\infty)_{\mathbb{T}}$. Then, we obtain

$$x^{\Delta}(t)=x^{\Delta}(t_2)+\int_{t_2}^{t}\frac{x^{\Delta\Delta}(s)}{\phi(s)}\phi(s)\Delta s\geq\left(\frac{1}{\phi(t)}\int_{t_2}^{t}\phi(s)\Delta s\right)x^{\Delta\Delta}(t).$$

Hence,

$$\left(\frac{x^{\Delta}}{\varphi}\right)^{\Delta}(t)=\frac{x^{\Delta\Delta}(t)\varphi(t)-x^{\Delta}(t)\varphi^{\Delta}(t)}{\varphi(t)\varphi^{\sigma}(t)}\leq\frac{x^{\Delta}(t)}{\varphi(t)\varphi^{\sigma}(t)}\left(\frac{\varphi(t)}{\frac{1}{\phi(t)}\int_{t_2}^{t}\phi(s)\Delta s}-\varphi^{\Delta}(t)\right)\leq 0.$$

Thus, x^{Δ}/φ is a nonincreasing function on $[t_3,\infty)_{\mathbb{T}}$. So we have

$$x(t)=x(t_3)+\int_{t_3}^{t}\frac{x^{\Delta}(s)}{\varphi(s)}\varphi(s)\Delta s\geq\left(\frac{1}{\varphi(t)}\int_{t_3}^{t}\varphi(s)\Delta s\right)x^{\Delta}(t).$$

This completes the proof. □

Remark 2.4. The functions ϕ and φ are existent, e.g., by letting $\phi(t):=\int_{t_1}^{t}r^{-1/\gamma}(s)\Delta s$ and $\varphi(t):=\int_{t_2}^{t}\phi(s)\Delta s$.

In the following, we use the notation $(d(t))_+:=\max\{0,d(t)\}$.

Theorem 2.5. *Let* (2.1) *hold and* $\gamma\geq 1$. *Assume that there exists a positive function* $\alpha\in \mathrm{C}^1_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ *such that for all sufficiently large* $t_1\in[t_0,\infty)_{\mathbb{T}}$, *for some* $t_2\in[t_1,\infty)_{\mathbb{T}}$, $t_3\in[t_2,\infty)_{\mathbb{T}}$, *and* $t_4\in[t_3,\infty)_{\mathbb{T}}$,

$$\limsup_{t\to\infty}\int_{t_4}^{t}\left[\alpha^{\sigma}(s)q(s)f(s,t_2,t_3)-\frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}\frac{D^{\gamma+1}(s)}{C^{\gamma}(s)}\right]\Delta s=\infty, \tag{2.8}$$

where ϕ *and* φ *are defined as in Lemma 2.3, and*

$$f(s,t_2,t_3):=\left(\frac{1}{\varphi(\tau(s))\phi(\sigma(s))}\int_{t_3}^{\tau(s)}\varphi(u)\Delta u\int_{t_2}^{\tau(s)}\phi(u)\Delta u\right)^{\gamma},$$

$$C(t):=\gamma\left(\frac{\phi(t)}{\phi(\sigma(t))}\right)^{\gamma}\frac{\alpha^{\sigma}(t)}{\alpha^{(\gamma+1)/\gamma}(t)r^{1/\gamma}(t)},$$

$$D(t):=\left[\frac{\alpha^{\Delta}(t)}{\alpha(t)}-\left(\frac{\phi(t)}{\phi(\sigma(t))}\right)^{\gamma}\frac{\alpha^{\sigma}(t)p(t)}{\alpha(t)r(t)}\right]_+.$$

If there exist positive functions $\beta,\varsigma\in \mathrm{C}^1_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ *such that*

$$\frac{\varsigma(t)}{t-t_1}-\varsigma^{\Delta}(t)\leq 0 \quad \textit{for} \quad t \quad \textit{large enough}, \tag{2.9}$$

and

$$\limsup_{t\to\infty}\int_{t_2}^{t}\left\{\beta^{\sigma}(\xi)\frac{\varsigma(\xi)}{\varsigma(\sigma(\xi))}g(\xi)-\frac{\varsigma(\sigma(\xi))((\beta^{\Delta}(\xi))_{+})^{2}}{4\beta^{\sigma}(\xi)\varsigma(\xi)}\right\}\Delta\xi=\infty, \tag{2.10}$$

where

$$g(\xi):=\int_{\xi}^{\infty}\left[\frac{1}{r(s)}\int_{s}^{\infty}q(v)\left(\frac{\varsigma(\tau(v))}{\varsigma(v)}\right)^{\gamma}\Delta v\right]^{1/\gamma}\Delta s,$$

then (1.1) *is oscillatory.*

Proof. Suppose that (1.1) has a nonoscillatory solution x on $[t_0,\infty)_{\mathbb{T}}$. We may assume without loss of generality that there exists a $t_1\in[t_0,\infty)_{\mathbb{T}}$ such that $x(t)>0$ and $x(\tau(t))>0$ for $t\in[t_1,\infty)_{\mathbb{T}}$. Proceeding as in the proof of Lemma 2.2, we get (2.2) and then x satisfies either (1) or (2).

Assume (1). Define the function ω by

$$\omega(t):=\alpha(t)\frac{r(t)(x^{\Delta^3})^{\gamma}(t)}{(x^{\Delta\Delta})^{\gamma}(t)},\quad t\in[t_1,\infty)_{\mathbb{T}}. \tag{2.11}$$

Then $\omega(t)>0$ for $t\in[t_1,\infty)_{\mathbb{T}}$ and

$$\omega^{\Delta}(t)=\alpha^{\Delta}(t)\frac{r(t)(x^{\Delta^3})^{\gamma}(t)}{(x^{\Delta\Delta})^{\gamma}(t)}+\alpha^{\sigma}(t)\left(\frac{r(x^{\Delta^3})^{\gamma}}{(x^{\Delta\Delta})^{\gamma}}\right)^{\Delta}(t),$$

which implies that

$$\begin{aligned}\omega^{\Delta}(t)=&\ \alpha^{\Delta}(t)\frac{r(t)(x^{\Delta^3})^{\gamma}(t)}{(x^{\Delta\Delta})^{\gamma}(t)}+\alpha^{\sigma}(t)\frac{(r(x^{\Delta^3})^{\gamma})^{\Delta}(t)}{(x^{\Delta\Delta})^{\gamma}(\sigma(t))}\\&-\alpha^{\sigma}(t)\frac{r(t)(x^{\Delta^3})^{\gamma}(t)((x^{\Delta\Delta})^{\gamma})^{\Delta}(t)}{(x^{\Delta\Delta})^{\gamma}(t)(x^{\Delta\Delta})^{\gamma}(\sigma(t))}.\end{aligned} \tag{2.12}$$

By virtue of Pötzsche chain rule [5, Theorem 1.90], we have

$$\begin{aligned}((x^{\Delta\Delta})^{\gamma})^{\Delta}(t)&=\gamma x^{\Delta^3}(t)\int_{0}^{1}\left[hx^{\Delta\Delta}(\sigma(t))+(1-h)x^{\Delta\Delta}(t)\right]^{\gamma-1}\mathrm{d}h\\&\geq\gamma(x^{\Delta\Delta})^{\gamma-1}(t)x^{\Delta^3}(t),\quad\gamma\geq1.\end{aligned} \tag{2.13}$$

Substituting (2.13) into (2.12), we find

$$\begin{aligned}\omega^{\Delta}(t)\leq&\ \alpha^{\Delta}(t)\frac{r(t)(x^{\Delta^3})^{\gamma}(t)}{(x^{\Delta\Delta})^{\gamma}(t)}+\alpha^{\sigma}(t)\frac{(r(x^{\Delta^3})^{\gamma})^{\Delta}(t)}{(x^{\Delta\Delta})^{\gamma}(\sigma(t))}\\&-\gamma\alpha^{\sigma}(t)\frac{r(t)(x^{\Delta^3})^{\gamma+1}(t)}{(x^{\Delta\Delta})^{\gamma+1}(t)}\left(\frac{x^{\Delta\Delta}(t)}{x^{\Delta\Delta}(\sigma(t))}\right)^{\gamma}.\end{aligned}$$

From (2.2), (2.11), and the above inequality, we obtain

$$\omega^{\Delta}(t) \leq \frac{\alpha^{\Delta}(t)}{\alpha(t)}\omega(t) - \frac{\alpha^{\sigma}(t)p(t)}{\alpha(t)r(t)}\frac{(x^{\Delta\Delta})^{\gamma}(t)}{(x^{\Delta\Delta})^{\gamma}(\sigma(t))}\omega(t)$$
$$-\alpha^{\sigma}(t)q(t)\frac{x^{\gamma}(\tau(t))}{(x^{\Delta\Delta})^{\gamma}(\sigma(t))}$$
$$-\gamma\alpha^{\sigma}(t)\frac{\omega^{(\gamma+1)/\gamma}(t)}{\alpha^{(\gamma+1)/\gamma}(t)r^{1/\gamma}(t)}\left(\frac{x^{\Delta\Delta}(t)}{x^{\Delta\Delta}(\sigma(t))}\right)^{\gamma}. \qquad (2.14)$$

In view of (2.5), (2.7), and the fact that $x^{\Delta\Delta}/\phi$ is a nonincreasing function, we have

$$\frac{x^{\gamma}(\tau(t))}{(x^{\Delta\Delta})^{\gamma}(\sigma(t))} = \left(\frac{x(\tau(t))}{x^{\Delta\Delta}(\tau(t))}\frac{x^{\Delta\Delta}(\tau(t))}{x^{\Delta\Delta}(\sigma(t))}\right)^{\gamma}$$
$$\geq \left(\frac{1}{\varphi(\tau(t))}\int_{t_3}^{\tau(t)}\varphi(s)\Delta s\right)^{\gamma}\left(\frac{1}{\phi(\tau(t))}\int_{t_2}^{\tau(t)}\phi(s)\Delta s\right)^{\gamma}$$
$$\times\left(\frac{\phi(\tau(t))}{\phi(\sigma(t))}\right)^{\gamma}$$
$$= \left(\frac{1}{\varphi(\tau(t))\phi(\sigma(t))}\int_{t_3}^{\tau(t)}\varphi(s)\Delta s\int_{t_2}^{\tau(t)}\phi(s)\Delta s\right)^{\gamma} \qquad (2.15)$$

and

$$\left(\frac{x^{\Delta\Delta}(t)}{x^{\Delta\Delta}(\sigma(t))}\right)^{\gamma} \geq \left(\frac{\phi(t)}{\phi(\sigma(t))}\right)^{\gamma}. \qquad (2.16)$$

Substituting (2.15) and (2.16) into (2.14), we get

$$\omega^{\Delta}(t) \leq -\alpha^{\sigma}(t)q(t)\left(\frac{1}{\varphi(\tau(t))\phi(\sigma(t))}\int_{t_3}^{\tau(t)}\varphi(s)\Delta s\int_{t_2}^{\tau(t)}\phi(s)\Delta s\right)^{\gamma}$$
$$+\left[\frac{\alpha^{\Delta}(t)}{\alpha(t)} - \left(\frac{\phi(t)}{\phi(\sigma(t))}\right)^{\gamma}\frac{\alpha^{\sigma}(t)p(t)}{\alpha(t)r(t)}\right]_{+}\omega(t)$$
$$-\gamma\alpha^{\sigma}(t)\left(\frac{\phi(t)}{\phi(\sigma(t))}\right)^{\gamma}\frac{\omega^{(\gamma+1)/\gamma}(t)}{\alpha^{(\gamma+1)/\gamma}(t)r^{1/\gamma}(t)}.$$

Set

$$A := C(t), \quad B := D(t), \quad \text{and} \quad y := \omega(t).$$

Using the inequality

$$By - Ay^{(\gamma+1)/\gamma} \leq \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}\frac{B^{\gamma+1}}{A^{\gamma}}, \quad A > 0,$$

we get

$$\omega^{\Delta}(t) \leq -\alpha^{\sigma}(t)q(t)f(t,t_2,t_3) + \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}\frac{D^{\gamma+1}(t)}{C^{\gamma}(t)}.$$

Integrating the above inequality from t_4 ($t_4 \in [t_3, \infty)_{\mathbb{T}}$) to t, we obtain

$$\int_{t_4}^{t} \left[\alpha^{\sigma}(s)q(s)f(s,t_2,t_3) - \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}\frac{D^{\gamma+1}(s)}{C^{\gamma}(s)}\right]\Delta s \le \omega(t_4) - \omega(t) \le \omega(t_4),$$

which contradicts (2.8).

Assume (2). Define the function ν by

$$\nu(t) := \beta(t)\frac{x^{\Delta}(t)}{x(t)}, \quad t \in [t_1, \infty)_{\mathbb{T}}. \tag{2.17}$$

Then $\nu(t) > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$ and

$$\nu^{\Delta}(t) = \beta^{\Delta}(t)\frac{x^{\Delta}(t)}{x(t)} + \beta^{\sigma}(t)\frac{x^{\Delta\Delta}(t)}{x(\sigma(t))} - \beta^{\sigma}(t)\frac{x^{\Delta}(t)x^{\Delta}(t)}{x(t)x(\sigma(t))}. \tag{2.18}$$

Hence by (2.17) and (2.18), we have

$$\nu^{\Delta}(t) = \frac{\beta^{\Delta}(t)}{\beta(t)}\nu(t) + \beta^{\sigma}(t)\frac{x^{\Delta\Delta}(t)}{x(\sigma(t))} - \frac{\beta^{\sigma}(t)}{\beta^2(t)}\frac{x(t)}{x(\sigma(t))}\nu^2(t). \tag{2.19}$$

Since $x > 0, x^{\Delta} > 0$, and $x^{\Delta\Delta} < 0$, we obtain

$$x(t) = x(t_1) + \int_{t_1}^{t} x^{\Delta}(s)\Delta s \ge (t - t_1)x^{\Delta}(t).$$

Thus,

$$\left(\frac{x}{\varsigma}\right)^{\Delta}(t) = \frac{x^{\Delta}(t)\varsigma(t) - x(t)\varsigma^{\Delta}(t)}{\varsigma(t)\varsigma^{\sigma}(t)} \le \frac{x(t)}{\varsigma(t)\varsigma^{\sigma}(t)}\left(\frac{\varsigma(t)}{t-t_1} - \varsigma^{\Delta}(t)\right) \le 0.$$

Hence x/ς is nonincreasing eventually, and so

$$\frac{x(t)}{x(\sigma(t))} \ge \frac{\varsigma(t)}{\varsigma(\sigma(t))}, \quad \frac{x(\tau(t))}{x(t)} \ge \frac{\varsigma(\tau(t))}{\varsigma(t)}. \tag{2.20}$$

Hence by (2.19) and (2.20), we see that

$$\nu^{\Delta}(t) \le \frac{\beta^{\Delta}(t)}{\beta(t)}\nu(t) + \beta^{\sigma}(t)\frac{x^{\Delta\Delta}(t)}{x(\sigma(t))} - \frac{\varsigma(t)}{\varsigma(\sigma(t))}\frac{\beta^{\sigma}(t)}{\beta^2(t)}\nu^2(t). \tag{2.21}$$

On the other hand, we get by (1.1) that

$$r(z)(x^{\Delta^3})^{\gamma}(z) - r(t)(x^{\Delta^3})^{\gamma}(t) + \int_{t}^{z} q(s)x^{\gamma}(\tau(s))\Delta s \le 0.$$

It follows from $x^{\Delta} > 0$ and (2.20) that

$$r(z)(x^{\Delta^3})^{\gamma}(z) - r(t)(x^{\Delta^3})^{\gamma}(t) + x^{\gamma}(t)\int_{t}^{z} q(s)\left(\frac{\varsigma(\tau(s))}{\varsigma(s)}\right)^{\gamma}\Delta s \le 0.$$

Letting $z \to \infty$ in the above inequality, we obtain

$$-r(t)(x^{\Delta^3})^{\gamma}(t) + x^{\gamma}(t)\int_{t}^{\infty} q(s)\left(\frac{\varsigma(\tau(s))}{\varsigma(s)}\right)^{\gamma}\Delta s \le 0$$

due to $\lim_{z\to\infty} r(z)(x^{\Delta^3})^\gamma(z) = l_1 \geq 0$. That is,

$$x^{\Delta^3}(t) \geq x(t)\left[\frac{1}{r(t)}\int_t^\infty q(s)\left(\frac{\varsigma(\tau(s))}{\varsigma(s)}\right)^\gamma \Delta s\right]^{1/\gamma}.$$

Therefore,

$$-x^{\Delta\Delta}(z) + x^{\Delta\Delta}(t) + x(t)\int_t^z \left[\frac{1}{r(s)}\int_s^\infty q(v)\left(\frac{\varsigma(\tau(v))}{\varsigma(v)}\right)^\gamma \Delta v\right]^{1/\gamma} \Delta s \leq 0.$$

Letting $z \to \infty$ in the last inequality and using $\lim_{z\to\infty}(-x^{\Delta\Delta}(z)) = l_2 \geq 0$, we have

$$x^{\Delta\Delta}(t) + x(t)\int_t^\infty \left[\frac{1}{r(s)}\int_s^\infty q(v)\left(\frac{\varsigma(\tau(v))}{\varsigma(v)}\right)^\gamma \Delta v\right]^{1/\gamma} \Delta s \leq 0.$$

Thus, we get by (2.20) that

$$\begin{aligned}\frac{x^{\Delta\Delta}(t)}{x(\sigma(t))} &\leq -\frac{x(t)}{x(\sigma(t))}\int_t^\infty \left[\frac{1}{r(s)}\int_s^\infty q(v)\left(\frac{\varsigma(\tau(v))}{\varsigma(v)}\right)^\gamma \Delta v\right]^{1/\gamma} \Delta s \\ &\leq -\frac{\varsigma(t)}{\varsigma(\sigma(t))}\int_t^\infty \left[\frac{1}{r(s)}\int_s^\infty q(v)\left(\frac{\varsigma(\tau(v))}{\varsigma(v)}\right)^\gamma \Delta v\right]^{1/\gamma} \Delta s. \quad (2.22)\end{aligned}$$

Substituting (2.22) into (2.21), we obtain

$$\begin{aligned}\nu^\Delta(t) \leq &-\beta^\sigma(t)\frac{\varsigma(t)}{\varsigma(\sigma(t))}\int_t^\infty \left[\frac{1}{r(s)}\int_s^\infty q(v)\left(\frac{\varsigma(\tau(v))}{\varsigma(v)}\right)^\gamma \Delta v\right]^{1/\gamma} \Delta s \\ &+\frac{(\beta^\Delta(t))_+}{\beta(t)}\nu(t) - \frac{\varsigma(t)}{\varsigma(\sigma(t))}\frac{\beta^\sigma(t)}{\beta^2(t)}\nu^2(t),\end{aligned}$$

which yields

$$\begin{aligned}\nu^\Delta(t) \leq &-\beta^\sigma(t)\frac{\varsigma(t)}{\varsigma(\sigma(t))}\int_t^\infty \left[\frac{1}{r(s)}\int_s^\infty q(v)\left(\frac{\varsigma(\tau(v))}{\varsigma(v)}\right)^\gamma \Delta v\right]^{1/\gamma} \Delta s \\ &+\frac{\varsigma(\sigma(t))((\beta^\Delta(t))_+)^2}{4\beta^\sigma(t)\varsigma(t)}.\end{aligned}$$

Integrating the latter inequality from t_2 ($t_2 \in [t_1, \infty)_{\mathbb{T}}$) to t, we have

$$\int_{t_2}^{t} \left\{ \beta^\sigma(\xi) \frac{\varsigma(\xi)}{\varsigma(\sigma(\xi))} \int_{\xi}^{\infty} \left[\frac{1}{r(s)} \int_{s}^{\infty} q(v) \left(\frac{\varsigma(\tau(v))}{\varsigma(v)} \right)^\gamma \Delta v \right]^{1/\gamma} \Delta s \right.$$
$$\left. - \frac{\varsigma(\sigma(\xi))((\beta^\Delta(\xi))_+)^2}{4\beta^\sigma(\xi)\varsigma(\xi)} \right\} \Delta\xi \le \nu(t_2) - \nu(t) \le \nu(t_2),$$

which contradicts (2.10). The proof is complete. □

Remark 2.6. The function ς is existent, e.g., by letting $\varsigma(t) := t - t_1$.

Now we establish an oscillation result for (1.1) in the case where $\gamma \le 1$.

Theorem 2.7. *Let (2.1) hold and $\gamma \le 1$. Assume that there exists a positive function $\alpha \in C^1_{\mathrm{rd}}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ such that for all sufficiently large $t_1 \in [t_0, \infty)_{\mathbb{T}}$, for some $t_2 \in [t_1, \infty)_{\mathbb{T}}$, $t_3 \in [t_2, \infty)_{\mathbb{T}}$, and $t_4 \in [t_3, \infty)_{\mathbb{T}}$,*

$$\limsup_{t\to\infty} \int_{t_4}^{t} \left[\alpha^\sigma(s) q(s) f(s, t_2, t_3) - \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}} \frac{D^{\gamma+1}(s)}{E^\gamma(s)} \right] \Delta s = \infty, \tag{2.23}$$

where ϕ and φ are defined as in Lemma 2.3, f and D are as in Theorem 2.5, and

$$E(t) := \gamma \frac{\phi(t)}{\phi(\sigma(t))} \frac{\alpha^\sigma(t)}{\alpha^{(\gamma+1)/\gamma}(t) r^{1/\gamma}(t)}.$$

If there exist positive functions $\beta, \varsigma \in C^1_{\mathrm{rd}}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ such that (2.9) and (2.10) hold, then (1.1) is oscillatory.

Proof. Suppose that (1.1) has a nonoscillatory solution x on $[t_0, \infty)_{\mathbb{T}}$. We may assume without loss of generality that there exists a $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t) > 0$ and $x(\tau(t)) > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$. Proceeding as in the proof of Lemma 2.3, we get (2.2) and then x satisfies either (1) or (2).

Assume (1). Define the function ω by (2.11). Then we obtain (2.12). From Pötzsche chain rule [5, Theorem 1.90], we see that

$$((x^{\Delta\Delta})^\gamma)^\Delta(t) = \gamma x^{\Delta^3}(t) \int_0^1 \left[h x^{\Delta\Delta}(\sigma(t)) + (1-h) x^{\Delta\Delta}(t) \right]^{\gamma-1} \mathrm{d}h$$
$$\ge \gamma (x^{\Delta\Delta})^{\gamma-1}(\sigma(t)) x^{\Delta^3}(t), \quad \gamma \le 1. \tag{2.24}$$

It follows from (2.12) and (2.24) that

$$\omega^\Delta(t) \le \alpha^\Delta(t) \frac{r(t)(x^{\Delta^3})^\gamma(t)}{(x^{\Delta\Delta})^\gamma(t)} + \alpha^\sigma(t) \frac{(r(x^{\Delta^3})^\gamma)^\Delta(t)}{(x^{\Delta\Delta})^\gamma(\sigma(t))}$$
$$- \gamma \alpha^\sigma(t) \frac{r(t)(x^{\Delta^3})^{\gamma+1}(t)}{(x^{\Delta\Delta})^{\gamma+1}(t)} \frac{x^{\Delta\Delta}(t)}{x^{\Delta\Delta}(\sigma(t))}.$$

The rest of the proof is similar to that of Theorem 2.5, and hence is omitted.

Assume (2). The remainder of the proof is the same as that of Theorem 2.5. This completes the proof. □

3. Examples and Discussions

Below, we present two examples to show applications of the main results.

Example 3.1. Consider a fourth-order differential equation with damping

$$\left(\frac{1}{t}x'''(t)\right)' + \frac{1}{t^3}x'''(t) + \frac{\lambda}{t^5}x\left(\frac{t}{2}\right) = 0, \quad t \geq 1. \tag{3.1}$$

Here, $\gamma = 1, \lambda > 0$ is a constant, $r(t) = 1/t, p(t) = 1/t^3, q(t) = \lambda/t^5$, and $\tau(t) = t/2$. Set $\phi(t) = \int_{t_1}^t s\mathrm{d}s = (t^2 - {t_1}^2)/2, \varphi(t) = \int_{t_2}^t (s^2 - {t_1}^2)/2\mathrm{d}s = (t^3 - {t_2}^3)/6 - {t_1}^2(t - t_2)/2, \varsigma(t) = t - t_1, \alpha(t) = t^2$, and $\beta(t) = t$. Then (2.1) holds,

$$\frac{t^2}{3} \leq \phi(t) \leq \frac{t^2}{2}, \quad \frac{t^3}{7} \leq \varphi(t) \leq \frac{t^3}{6} \quad \text{for} \quad t \quad \text{large enough},$$

$$f(s, t_2, t_3) = \frac{1}{\varphi(\tau(s))\phi(\sigma(s))} \int_{t_3}^{\tau(s)} \varphi(u)\Delta u \int_{t_2}^{\tau(s)} \phi(u)\Delta u \geq \frac{s^2}{672},$$

and

$$C(t) = \frac{1}{t}, \quad D(t) = \frac{2t - 1}{t^2}.$$

Thus, (2.8) holds if $\lambda > 672$ and (2.10) holds if $\lambda > 6$. Therefore by Theorem 2.5, we see that equation (3.1) is oscillatory if $\lambda > 672$.

Example 3.2. Consider a fourth-order delay dynamic equation with damping

$$\left(\frac{1}{t}x^{\Delta^3}(t)\right)^{\Delta} + \frac{1}{2t^2}x^{\Delta^3}(t) + x\left(\frac{t}{2}\right) = 0, \tag{3.2}$$

where $t \in \mathbb{T} := \overline{2^{\mathbb{Z}}} := \{2^k : k \in \mathbb{Z}\} \cup \{0\}$. Let $\gamma = 1, r(t) = 1/t, p(t) = 1/(2t^2)$, and $q(t) = 1$. Then it is not difficult to verify that condition (2.1) is satisfied. On the other hand, letting $\phi(t) = \int_{t_1}^t s\Delta s, \varphi(t) = \int_{t_2}^t \phi(s)\Delta s, \varsigma(t) = t - t_1, \alpha(t) = 1$, and $\beta(t) = 1$, then all assumptions of Theorem 2.7 hold. Hence, equation (3.2) is oscillatory.

Remark 3.3. One can define the function

$$\omega(t) := \alpha(t)\frac{r(t)(x^{\Delta^3})^{\gamma}(t)}{(x^{\Delta})^{\gamma}(t)}$$

or

$$\omega(t) := \alpha(t)\frac{r(t)(x^{\Delta^3})^{\gamma}(t)}{x^{\gamma}(t)}$$

in the proof of case (1) in Theorem 2.5 or Theorem 2.7, and then obtain other different classes of criteria for oscillation of (1.1).

Remark 3.4. From the method given in this paper, one can obtain some Philos-type oscillation criteria for (1.1). The details are left to the reader.

Acknowledgments

This research is supported by NNSF of P. R. China (Grant Nos. 61034007, 51277116, and 50977054).

References

[1] Agarwal, R.P., Bohner, M., Tang, S., Li, T., Zhang, C.: Oscillation and asymptotic behavior of third-order nonlinear retarded dynamic equations. Appl. Math. Comput. **219**, 3600–3609 (2012)

[2] Agarwal, R.P., Grace, S.R., O'Regan, D.: Oscillation Theory for Second Order Dynamic Equations, Series in Mathematical Analysis and Applications, vol. 5. Taylor and Francis Ltd, London (2003)

[3] Agarwal, R.P., O'Regan, D., Saker, S.H.: Oscillation of second order nonlinear neutral delay dynamic equations on time scales. J. Math. Anal. Appl. **300**, 203–217 (2004)

[4] Akın-Bohner, E., Bohner, M., Saker, S.H.: Oscillation criteria for a certain class of second order Emden–Fowler dynamic equations. Electron. Trans. Numer. Anal. **27**, 1–12 (2007)

[5] Bohner, M., Peterson, A.: Dynamic Equations on Time Scales: An Introduction with Applications. Birkhäuser, Boston (2001)

[6] Bohner, M., Peterson, A.: Advances in Dynamic Equations on Time Scales. Birkhäuser, Boston (2003)

[7] Erbe, L., Hassan, T.S., Peterson, A.: Oscillation criteria for nonlinear damped dynamic equations on time scales. Appl. Math. Comput. **203**, 343–357 (2008)

[8] Erbe, L., Peterson, A., Saker, S.H.: Hille and Nehari type criteria for third-order dynamic equations. J. Math. Anal. Appl. **329**, 112–131 (2007)

[9] Grace, S.R., Agarwal, R.P., Pinelas, S.: On the oscillation of fourth order superlinear dynamic equations on time scales. Dynam. Syst. Appl. **20**, 45–54 (2011)

[10] Grace, S.R., Agarwal, R.P., Sae-jie, W.: Monotone and oscillatory behavior of certain fourth order nonlinear dynamic equations. Dynam. Syst. Appl. **19**, 25–32 (2010)

[11] Grace, S.R., Bohner, M., Sun, S.: Oscillation of fourth-order dynamic equations. Hacet. J. Math. Stat. **39**, 545–553 (2010)

[12] Hassan, T.S.: Oscillation of third order nonlinear delay dynamic equations on time scales. Math. Comput. Modelling **49**, 1573–1586 (2009)

[13] Hilger, S.: Analysis on measure chains–a unified approach to continuous and discrete calculus. Results Math. **18**, 18–56 (1990)

[14] Li T., Agarwal R.P., Bohner, M.: Some oscillation results for second-order neutral dynamic equations. Hacet. J. Math. Stat. **41**, 715–721 (2012)

[15] Li, T., Han, Z., Sun, S., Zhao, Y.: Oscillation results for third order nonlinear delay dynamic equations on time scales. Bull. Malays. Math. Sci. Soc. **34**, 639–648 (2011)

[16] Li, T., Thandapani, E., Tang, S.: Oscillation theorems for fourth-order delay dynamic equations on time scales. Bull. Math. Anal. Appl. **3**, 190–199 (2011)

[17] Saker S.H.: Oscillation Theory of Dynamic Equations on Time Scales, Second and Third Orders. Lambert Academic Publisher (2010)

[18] Saker, S.H., Agarwal, R.P., O'Regan, D.: Oscillation of second-order damped dynamic equations on time scales. J. Math. Anal. Appl. **330**, 1317–1337 (2007)

[19] Zhang, B., Zhu, S.: Oscillation of second order nonlinear delay dynamic equations on time scales. Comput. Math. Appl. **49**, 599–609 (2005)

[20] Zhang, C., Li, T., Agarwal, R.P., Bohner, M.: Oscillation results for fourth-order nonlinear dynamic equations. Appl. Math. Lett. **25**, 2058–2065 (2012)

Ravi P. Agarwal
Department of Mathematics
Texas A&M University
700 University Blvd.
Kingsville, TX 78363-8202, USA
e-mail: agarwal@tamuk.edu

Martin Bohner
Department of Mathematics and Statistics
Missouri S&T, Rolla, MO 65409-0020, USA
e-mail: bohner@mst.edu

Tongxing Li and Chenghui Zhang
School of Control Science and Engineering
Shandong University, Jinan, Shandong 250061, P. R. China
e-mail: litongx2007@163.com

Chenghui Zhang
e-mail: zchui@sdu.edu.cn

Received: December 15, 2012.
Accepted: February 6, 2013.