

Global Dynamics of the Kummer–Schwarz Differential Equation

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Abstract. This paper studies the Kummer–Schwarz differential equation $2\dot{x}\ddot{x} - 3\ddot{x}^2 = 0$ which is of special interest due to its relationship with the Schwarzian derivative. This differential equation is transformed into a first order differential system in \mathbb{R}^3 , and we provide a complete description of its global dynamics adding the infinity.

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1. Introduction and Statements of Main Results

The Schwarzian derivative

$$\{x, t\} = \frac{\ddot{x}(t)}{\dot{x}(t)} - \frac{3}{2} \left(\frac{\ddot{x}(t)}{\dot{x}(t)} \right)^2, \quad (1.1)$$

plays an important role in the treatment of univalent functions; see details in [5] and references therein. Here, the dot denotes derivative with respect to the independent variable t . When the right hand in Eq. (1.1) is taken at zero, the resulting equation is the Kummer–Schwarz equation which is given by

$$2\dot{x}\ddot{x} - 3\ddot{x}^2 = 0, \quad (1.2)$$

and is of special interest due to its relationship to the Schwarzian derivative and its exceptional algebraic properties. This equation is also encountered in the study of geodesic curves in spaces of constant curvature; Lie lists the characteristic functions for its contact symmetries, see more results on this differential equation in [1, 4–6]. But up to now nobody has described its global qualitative dynamics. This will be the objective of this paper.

To know the global qualitative dynamics of a differential equation is important because, in particular it describes where born and where died all

their orbits, if the equation has periodic orbits or not, ... For more information about the qualitative dynamics see for instance [3]. Moreover, when it is possible it is interesting to describe the orbits which go or come from the infinity, because these orbits complete the qualitative information on the behavior of the orbits of the differential equation.

The Kummer–Schwarz equation of third order (1.2) can be written as the following rational differential system of first order

$$\dot{x} = y, \quad \dot{y} = z, \quad \dot{z} = \frac{3z^2}{2y}, \tag{1.3}$$

in \mathbb{R}^3 . Rescaling the time according to $d\tau/dt = 2y$, we obtain the equivalent polynomial differential system (outside the plane $y = 0$)

$$x' = 2y^2, \quad y' = 2yz, \quad z' = 3z^2. \tag{1.4}$$

Here, the prime denotes derivative with respect to the new independent variable τ . This differential system is called the *Kummer–Schwarz differential system in \mathbb{R}^3* .

We will study the flow of the polynomial differential system (1.4) in the phase space \mathbb{R}^3 ; of course, in order to describe the flow of the differential system (1.3) we must omit the plane $y = 0$.

We remark that if we know the qualitative behavior of the orbits $(x(\tau), y(\tau), z(\tau))$ of the differential system (1.4), then we know the qualitative behavior of the orbits $x(t)$ of the Kummer–Schwarz differential equation (1.2).

As we shall see later on, Goviender and Leach [4] determined two independent first integrals $H_i = H_i(x, y, z)$, $i = 1, 2$ for the differential system (1.4). These integrals play a main role in the study of the global dynamics of this system. Of course, the explicit curves $\{H_1(x, y, z) = h_1\} \cap \{H_2(x, y, z) = h_2\}$ with $h_1, h_2 \in \mathbb{R}$ contain the orbits of the differential system (1.4), but in general it remains a big work to do in order to describe the global qualitative behavior of all the orbits of the differential system. This work is the one that we do here.

1.1. Symmetries and Reduction of the Flow to the Quadrant $y \geq 0$ and $z \geq 0$

The differential system (1.4) is invariant under the following two symmetries:

$$S_1(x, y, z) = (x, -y, z), \quad \text{and} \quad S_2(x, y, z, \tau) = (-x, y, -z, -\tau).$$

The symmetry S_1 says that the flow of system (1.4) is symmetric with respect to the plane $y = 0$. Therefore, if $(x(\tau), y(\tau), z(\tau))$ is a solution of (1.4), then $(x(\tau), -y(\tau), z(\tau))$ is also solution of (1.4). The symmetry S_2 says that the flow of system (1.4) is symmetric with respect to the y -axis reversing the sense of the orbits. Therefore, if $(x(\tau), y(\tau), z(\tau))$ is a solution of (1.4), then $(-x(-\tau), y(\tau), -z(-\tau))$ is also solution of (1.4). Using both symmetries in order to describe the flow of system (1.4) in \mathbb{R}^3 it is enough to describe the flow of system (1.4) on the quadrant

$$Q = \{(x, y, z) \in \mathbb{R}^3 : y \geq 0, z \geq 0\}.$$

1.2. The Poincaré Compactification

It is known that a polynomial differential system in \mathbb{R}^3 can be extended to a unique analytic differential system on the closed ball B of radius 1 centered at the origin of \mathbb{R}^3 , called the *Poincaré ball*. More precisely, the whole space \mathbb{R}^3 is identified with the interior of B , and the infinity of \mathbb{R}^3 is identified with the boundary of B , i.e., with the two-dimensional sphere S^2 . For more details see [2] and [7], and the appendix of this paper. The known technique for making such an extension is called the Poincaré compactification and it allows to study the dynamics of a polynomial differential system in a neighborhood of infinity. Poincaré introduced this compactification for polynomial differential systems in \mathbb{R}^2 .

1.3. The Global Dynamics

Our main result is the description of the global dynamics of the Kummer–Schwarz differential system (1.4) on the compactified quadrant \bar{Q} of Q inside the Poincaré ball B , see Fig. 1. More precisely, we describe all the α - and ω -limit sets of all the orbits of the Kummer–Schwarz differential system (1.4). For the standard definitions of orbit, α - and ω -limit sets of an orbit, and of the Poincaré compactification, see for instance [3].

Theorem 1.1. *The following statements hold for the Kummer–Schwarz differential system (1.4).*

- (a) *On the quadrant \bar{Q} the equilibrium points are all the points of the x -axis, including its endpoints at infinity (X_- at the end of the negative x -axis and X_+ at the end of the positive x -axis), and additionally the endpoint Z_+ at infinity of the positive z -axis, see Fig. 2.*
- (b) *On the invariant boundary $y = 0$ of the quadrant \bar{Q} the orbits are the half-straight lines parallel to the z -axis having α -limit an equilibrium point of the x -axis and ω -limit the equilibrium point Z_+ , see Fig. 2.*

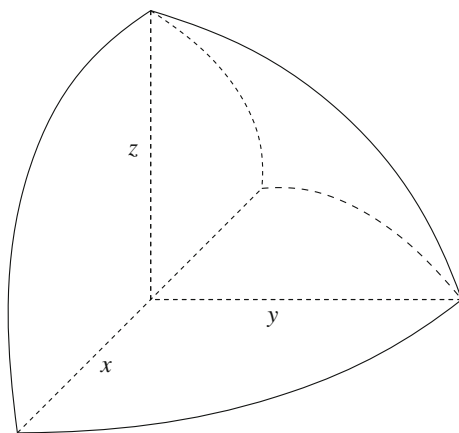


Figure 1. The quadrant \bar{Q} of the Poincaré ball B

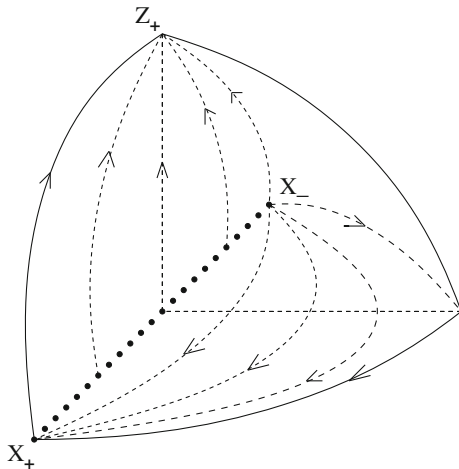


Figure 2. The flow on the invariant planes $y = 0$ and $z = 0$ restricted to the quadrant \bar{Q}

- (c) On the invariant boundary $z = 0$ of the quadrant \bar{Q} the orbits are the straight lines parallel to the x -axis having α -limit the equilibrium point X_- and ω -limit the equilibrium point X_+ , see Fig. 2.
- (d) On the infinity $\mathbb{S}^2 \cap \bar{Q}$ the flow is qualitatively the one described in Fig. 3, i.e., without taking into account the three equilibrium points Z_+ , X_- and X_+ at the infinity of the quadrant \bar{Q} all the other orbits have ω -limit at the equilibrium point Z_+ , and α -limit either at X_- , or at X_+ .
- (e) The explicit solution $(x(\tau), y(\tau), z(\tau))$ of the differential system (1.4) such that $(x(0), y(0), z(0)) = (x_0, y_0, z_0)$ is

$$\begin{aligned}
 x(\tau) &= x_0 + \frac{2y_0^2}{z_0} \left(\frac{1}{(1 - 3z_0\tau)^{1/3}} - 1 \right), \\
 y(\tau) &= \frac{y_0}{(1 - 3z_0\tau)^{2/3}}, \\
 z(\tau) &= \frac{z_0}{1 - 3z_0\tau}.
 \end{aligned}
 \tag{1.5}$$

- (f) Let γ be an orbit contained in the interior of the quadrant \bar{Q} . Then the α -limit of γ is the equilibrium point X_- and its ω -limit is the equilibrium point Z_+ .
- (g) The differential system (1.4) has two independent first integrals $H_1 = z^2/y^3$ and $H_2 = x - 2y^2/z$. The set $\{H_1 = h_1\} \cap \{H_2 = h_2\} \cap Q$ is an orbit γ with endpoints at X_- and Z_+ .

The proof of Theorem 1.1 is given in Sect. 2.

The two independent first integrals H_1 and H_2 of statement (g) of Theorem 1.1 are due to Goviender and Leach [4].

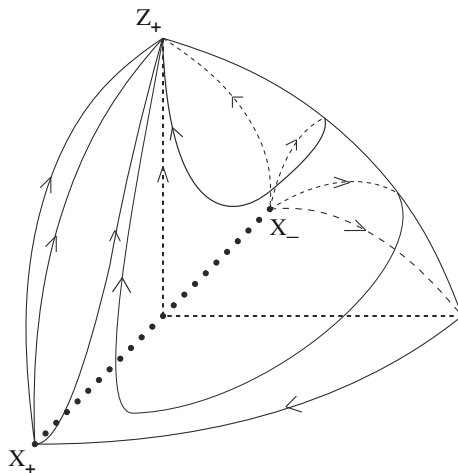


Figure 3. The flow on the infinity restricted to the quadrant \overline{Q}

2. Proof of Theorem 1.1

This section is dedicated to prove the different statements of Theorem 1.1.

Proof of statement (a) of Theorem 1.1. From the equations of the differential system (1.4) it follows immediately that the x -axis is filled with equilibrium points, because $x' = y' = z' = 0$ when $y = z = 0$. Now we shall study the equilibrium points at the infinity of the quadrant \overline{Q} using the Poincaré compactification of \mathbb{R}^3 described in the appendix.

We start studying the equilibrium points at the infinity located on the local chart U_1 , i.e., in $x > 0$ and its boundary at infinity. Thus the compactified differential system (1.4) in the local chart U_1 is given by:

$$\dot{z}_1 = -2z_1^3 + 2z_1z_2, \quad \dot{z}_2 = -2z_1^2z_2 + 3z_2^2, \quad \dot{z}_3 = -2z_1^2z_3. \tag{2.1}$$

At the infinity $z_3 = 0$ of U_1 , i.e., in the points of the sphere \mathbb{S}^2 system (2.1) reduces to

$$\dot{z}_1 = 2z_1(-z_1^2 + z_2), \quad \dot{z}_2 = z_2(-2z_1^2 + 3z_2). \tag{2.2}$$

So, the unique equilibrium point at the infinity of U_1 is the origin $(0, 0, 0)$ of U_1 . Its linear part has all its eigenvalues equal to zero. Therefore, we need to study it using the technique of blow ups, see for more details Chapter 3 of [3]. Then, we obtain that the local phase portrait of the equilibrium point $(0, 0)$ of the differential system (2.2) is qualitatively the one of Fig. 4. The equilibrium point $(0, 0, 0)$ of U_1 corresponds to the endpoint X_+ of the positive x -axis.

The local phase portrait at the equilibrium point $(0, 0, 0)$ of V_1 which corresponds to the endpoint X_- of the negative x -axis, is obtained doing symmetry with respect to the center of the sphere \mathbb{S}^2 and reversing the orientation of the orbits because the degree of the polynomial differential system (1.4) is 2.

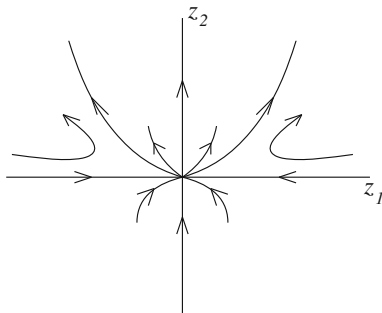


Figure 4. The local phase portrait of the equilibrium point $(0, 0)$ of the differential system (2.2)

The flow of system (1.4) in the local chart U_2 is given by the differential system

$$\dot{z}_1 = -2z_1z_2 + 2, \quad \dot{z}_2 = -2z_1^2z_2 + 3z_2^2, \quad \dot{z}_3 = -2z_2z_3. \tag{2.3}$$

So there are no equilibrium points at infinity in this local chart.

In the local chart U_3 the system (1.4) becomes

$$\dot{z}_1 = -3z_1 + 2z_2^2, \quad \dot{z}_2 = -z_2, \quad \dot{z}_3 = -3z_3. \tag{2.4}$$

At the infinity of U_3 the point $(0, 0, 0)$ is the unique equilibrium point, and its linear part has the eigenvalues -1 and -3 with multiplicity two. Therefore, by the Hartman Theorem this equilibrium point is a local attractor, and it corresponds to the endpoint Z_+ of the positive z -axis. \square

Proof of statement (b) of Theorem 1.1. From the differential system (1.4) it follows that $x' = 0$ and $y' = 0$ when $y = 0$, so the plane $y = 0$ and the straight lines $\{y = 0\} \cap \{x = \text{constant}\}$ are invariant by the flow of system (1.4). In other words, if an orbit of the differential system (1.4) has a point in $y = 0$, or in $\{y = 0\} \cap \{x = \text{constant}\}$, then the whole orbit is contained in that plane or straight line.

In short, on the invariant boundary $y = 0$ of the quadrant \bar{Q} the orbits are the half-straight lines parallel to the z -axis having α -limit an equilibrium point of the x -axis and ω -limit the equilibrium point Z_+ ; see for more details the proof of statement (a). \square

Proof of statement (c) of Theorem 1.1. From the differential system (1.4) we have that $y' = 0$ and $z' = 0$ when $z = 0$, so the plane $z = 0$ and the straight lines $\{z = 0\} \cap \{y = \text{constant}\}$ are invariant by the flow of system (1.4).

In summary, on the invariant boundary $z = 0$ of the quadrant \bar{Q} the orbits are the straight lines parallel to the x -axis having α -limit the equilibrium point X_- and ω -limit the equilibrium point X_+ , see again for more details the proof of statement (a). \square

Proof of statement (d) of Theorem 1.1. From the fact that the infinity \mathbb{S}^2 is invariant by the compactified flow of the polynomial differential system (1.4), and the local phase portraits at the three equilibrium points Z_+ , X_- and X_+

at the infinity of the quadrant \overline{Q} studied in the proof of statement (a), it follows that on the infinity $\mathbb{S}^2 \cap \overline{Q}$ the flow is qualitatively the one described in Fig. 3. □

Proof of statement (e) of Theorem 1.1. Let $(x(\tau), y(\tau), z(\tau))$ be the solution of the differential system (1.4) such that $(x(0), y(0), z(0)) = (x_0, y_0, z_0)$. From the differential equation $z' = 3z^2$ it follows easily that $z(\tau) = z_0/(1 - 3z_0\tau)$. Substituting $z = z(\tau)$ in $y' = 2yz$ we obtain that $y(\tau) = y_0/(1 - 3z_0\tau)^{2/3}$. Finally, substituting $y = y(\tau)$ in $x' = 2y^2$ and integrating we get that

$$x(\tau) = x_0 + \frac{2y_0^2}{z_0} \left(\frac{1}{(1 - 3z_0\tau)^{1/3}} - 1 \right).$$

□

Proof of statement (f) of Theorem 1.1. Since $x' = 2y^2 > 0$ and $z' = 3z^2 > 0$ in the interior of the quadrant \overline{Q} , for every orbit γ contained in the interior of the quadrant \overline{Q} we have that its α -limit has its x -coordinate equal to $-\infty$ and its z -coordinate equal to 0, and its ω -limit has its x -coordinate equal to $+\infty$ and its z -coordinate equal to $+\infty$. Taking into account either the solution of statement (e), or the phase portrait on the boundary of the quadrant \overline{Q} described in the statements (b), (c) and (d) we get that the α -limit of γ is the equilibrium point X_- and its ω -limit is the equilibrium point Z_+ . □

Proof of statement (g) of Theorem 1.1. Let $H_1 = z^2/y^3$ and $H_2 = x - 2y^2/z$. Then, since

$$\frac{dH_k}{d\tau} = \frac{\partial H_k}{\partial x} x' + \frac{\partial H_k}{\partial y} y' + \frac{\partial H_k}{\partial z} z' = 0,$$

for $k = 1, 2$, we obtain that H_1 and H_2 are first integrals of the differential system (1.4), i.e., they are constant on the solutions of system (1.4) where they are defined. Since their gradients are linearly independent except at the points of $z = 0$ and $y = 0$ which have zero Lebesgue measure, these two first integrals are independent.

It is not difficult to show that the set $\{H_1 = h_1\} \cap \{H_2 = h_2\} \cap Q$ has a unique component. Then, from either statement (e), or statement (f) it follows that this set is formed by an orbit γ with endpoints at X_- and Z_+ . □

3. The Appendix: Poincaré Compactification of \mathbb{R}^3

In \mathbb{R}^3 we consider the polynomial differential system

$$\dot{x} = P^1(x, y, z), \quad \dot{y} = P^2(x, y, z), \quad \dot{z} = P^3(x, y, z),$$

or equivalently its associated polynomial vector field $X = (P^1, P^2, P^3)$. The degree n of X is defined as $n = \max\{\deg(P^i) : i = 1, 2, 3\}$.

Let $\mathbb{S}^3 = \{y = (y_1, y_2, y_3, y_4) \in \mathbb{R}^4 : \|y\| = 1\}$ be the unit sphere in \mathbb{R}^4 , and

$$\mathbb{S}_+ = \{y \in \mathbb{S}^3 : y_4 > 0\} \quad \text{and} \quad \mathbb{S}_- = \{y \in \mathbb{S}^3 : y_4 < 0\}$$

be the northern and southern hemispheres, respectively. The tangent space to \mathbb{S}^3 at the point y is denoted by $T_y\mathbb{S}^3$. Then, the tangent hyperplane

$$T_{(0,0,0,1)}\mathbb{S}^3 = \{(x_1, x_2, x_3, 1) \in \mathbb{R}^4 : (x_1, x_2, x_3) \in \mathbb{R}^3\}$$

is identified with \mathbb{R}^3 .

We consider the central projections

$$f_+ : \mathbb{R}^3 = T_{(0,0,0,1)}\mathbb{S}^3 \rightarrow \mathbb{S}_+ \quad \text{and} \quad f_- : \mathbb{R}^3 = T_{(0,0,0,1)}\mathbb{S}^3 \rightarrow \mathbb{S}_-,$$

defined by

$$f_+(x) = \frac{1}{\Delta x}(x_1, x_2, x_3, 1) \quad \text{and} \quad f_-(x) = -\frac{1}{\Delta x}(x_1, x_2, x_3, 1),$$

where $\Delta x = \left(1 + \sum_{i=1}^3 x_i^2\right)^{1/2}$. Through these central projections, \mathbb{R}^3 can be identified with the northern and the southern hemispheres, respectively. The equator of \mathbb{S}^3 is $\mathbb{S}^2 = \{y \in \mathbb{S}^3 : y_4 = 0\}$. Clearly, \mathbb{S}^2 can be identified with the *infinity* of \mathbb{R}^3 .

The maps f_+ and f_- define two copies of X , one $Df_+ \circ X$ in the northern hemisphere and the other $Df_- \circ X$ in the southern one. Denote by \bar{X} the vector field on $\mathbb{S}^3 \setminus \mathbb{S}^2 = \mathbb{S}_+ \cup \mathbb{S}_-$ which restricted to \mathbb{S}_+ coincides with $Df_+ \circ X$ and restricted to \mathbb{S}_- coincides with $Df_- \circ X$.

In what follows we shall work with the orthogonal projection of the closed northern hemisphere to $y_4 = 0$. Note that this projection is a closed ball B of radius one, whose interior is diffeomorphic to \mathbb{R}^3 and whose boundary \mathbb{S}^2 corresponds to the infinity of \mathbb{R}^3 . We shall extend analytically the polynomial vector field \bar{X} to the boundary, in such a way that the flow on the boundary is invariant. This new vector field on B will be called the Poincaré compactification of X , and B will be called the *Poincaré ball*. Poincaré introduced this compactification for polynomial vector fields in \mathbb{R}^2 , and its extension to \mathbb{R}^m can be found in [7].

The expression for $\bar{X}(y)$ on $\mathbb{S}_+ \cup \mathbb{S}_-$ is

$$\bar{X}(y) = y_4 \begin{pmatrix} 1 - y_1^2 & -y_2y_1 & -y_3y_1 \\ -y_1y_2 & 1 - y_2^2 & -y_3y_2 \\ -y_1y_3 & -y_2y_3 & 1 - y_3^2 \\ -y_1y_4 & -y_2y_4 & -y_3y_4 \end{pmatrix} \begin{pmatrix} P^1 \\ P^2 \\ P^3 \end{pmatrix},$$

where $P^i = P^i(y_1/|y_4|, y_2/|y_4|, y_3/|y_4|)$. Written in this way $\bar{X}(y)$ is a vector field in \mathbb{R}^4 tangent to the sphere \mathbb{S}^3 .

Now we can extend analytically the vector field $\bar{X}(y)$ to the whole sphere \mathbb{S}^3 by

$$p(X)(y) = y_4^{n-1}\bar{X}(y);$$

this extended vector field $p(X)$ is called the *Poincaré compactification of X*.

As \mathbb{S}^3 is a differentiable manifold, to compute the expression for $p(X)$ we can consider the eight local charts $(U_i, F_i), (V_i, G_i)$ where $U_i = \{y \in \mathbb{S}^3 : y_i > 0\}$, and $V_i = \{y \in \mathbb{S}^3 : y_i < 0\}$ for $i = 1, 2, 3, 4$; the diffeomorphisms $F_i : U_i \rightarrow \mathbb{R}^3$ and $G_i : V_i \rightarrow \mathbb{R}^3$ for $i = 1, 2, 3, 4$ are the inverses of the central projections from the origin to the tangent planes at the points $(\pm 1, 0, 0, 0), (0, \pm 1, 0, 0)$,

$(0, 0, \pm 1, 0)$ and $(0, 0, 0, \pm 1)$, respectively. We now do the computations on U_1 . Suppose that the origin $(0, 0, 0, 0)$, the point $(y_1, y_2, y_3, y_4) \in \mathbb{S}^3$ and the point $(1, z_1, z_2, z_3)$ in the tangent plane to \mathbb{S}^3 at $(1, 0, 0, 0)$ are collinear, then we have

$$\frac{1}{y_1} = \frac{z_1}{y_2} = \frac{z_2}{y_3} = \frac{z_3}{y_4},$$

and consequently

$$F_1(y) = \left(\frac{y_2}{y_1}, \frac{y_3}{y_1}, \frac{y_4}{y_1} \right) = (z_1, z_2, z_3)$$

defines the coordinates on U_1 .

As

$$DF_1(y) = \begin{pmatrix} -y_2/y_1^2 & 1/y_1 & 0 & 0 \\ -y_3/y_1^2 & 0 & 1/y_1 & 0 \\ -y_4/y_1^2 & 0 & 0 & 1/y_1 \end{pmatrix}$$

and $y_4^{n-1} = \left(\frac{z_3}{\Delta z}\right)^{n-1}$, the analytical field $p(X)$ in U_1 becomes

$$\frac{z_3^n}{(\Delta z)^{n-1}} (-z_1 P^1 + P^2, -z_2 P^1 + P^3, -z_3 P^1), \tag{3.1}$$

where $P^i = P^i(1/z_3, z_1/z_3, z_2/z_3)$.

In a similar way we can deduce the expressions of $p(X)$ in U_2 and U_3 . These are

$$\frac{z_3^n}{(\Delta z)^{n-1}} (-z_1 P^2 + P^1, -z_2 P^2 + P^3, -z_3 P^2), \tag{3.2}$$

where $P^i = P^i(z_1/z_3, 1/z_3, z_2/z_3)$ in U_2 , and

$$\frac{z_3^n}{(\Delta z)^{n-1}} (-z_1 P^3 + P^1, -z_2 P^3 + P^2, -z_3 P^3), \tag{3.3}$$

where $P^i = P^i(z_1/z_3, z_2/z_3, 1/z_3)$ in U_3 .

The expression for $p(X)$ in U_4 is $z_3^{n+1}(P^1, P^2, P^3)$ where the component $P^i = P^i(z_1, z_2, z_3)$.

The expression for $p(X)$ in the local chart V_i is the same as in U_i multiplied by $(-1)^{n-1}$.

When we shall work with the expression of the compactified vector field $p(X)$ in the local charts we omit the factor $1/(\Delta z)^{n-1}$. We can do that through a rescaling of the time.

We remark that all the points on the sphere at infinity in the coordinates of any local chart have $z_3 = 0$.

The orthogonal projection of the closed north hemisphere of \mathbb{S}^3 on the hyperplane $y_4 = 0$ is a closed ball B of radius 1 centered at the origin of coordinates, whose interior is diffeomorphic to $\mathbb{R}^3 = T_{(0,0,0,1)}\mathbb{S}^3$ and whose boundary \mathbb{S}^2 is the infinity of \mathbb{R}^3 . This closed ball B is the *Poincaré ball*.

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