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Sampling in Reproducing Kernel Banach Spaces

Antonio G. García* and Alberto Portal

Abstract. It is well-known the close relationship between reproducing kernel Hilbert spaces and sampling theory. The concept of reproducing kernel Hilbert space has been recently generalized to the case of Banach spaces. In this paper, some sampling results are proven in this new setting of reproducing kernel Banach spaces.

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1. Introduction

A reproducing kernel Hilbert space (RKHS in short) is a Hilbert space $\mathcal{H}$ of functions defined on a fixed set Ω such that for each $t \in \Omega$ the evaluation functional at t, i.e., $\mathcal{E}_t(f) := f(t),\ \ f \in \mathcal{H}$, is continuous on $\mathcal{H}$. The Riesz representation theorem gives a unique function $k : \Omega \times \Omega \longrightarrow \mathbb{C}$ such that

- $\{k(\cdot, t) : t \in \Omega\} \subset \mathcal{H}$, and
- $f(t) = \langle f, k(\cdot, t)\rangle_{\mathcal{H}}\,,\ \ t \in \Omega\,,\ \ f \in \mathcal{H}\,.$

The function k is called the *reproducing kernel* of $\mathcal{H}$. For the theory of RKHS see, for instance, [4, 24] and references therein; notice that the story of the reproducing kernel property goes back to a paper of Zaremba [26].

Concerning sampling results in a separable RKHS $\mathcal{H}$, let us assume that there exists a sequence $\{t_j\}_{j\in\mathbb{I}} \subset \Omega$ of sampling points, where $\mathbb{I}$ denotes an indexing set contained in $\mathbb{Z}$, such that the sequence $\{k(\cdot, t_j)\}_{j\in\mathbb{I}}$ is a frame for $\mathcal{H}$. In particular, the frame concept includes Riesz and orthonormal bases.

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*Corresponding author.

The well established frame theory (see, for instance, [5]) says that the frame operator $\mathcal{S}:\mathcal{H}\longrightarrow\mathcal{H}$, defined by

$$\mathcal{S}f := \sum_{j\in\mathbb{I}} \langle f, k(\cdot,t_j)\rangle_{\mathcal{H}}\, k(\cdot,t_j)\,, \quad f\in\mathcal{H}\,,$$

is bounded, self-adjoint, positive, and invertible. Applying the inverse operator $\mathcal{S}^{-1}$ to both sides of the above expression we obtain, for each $f\in\mathcal{H}$, the sampling expansion

$$f(t)=\sum_{j\in\mathbb{I}} f(t_j)\mathcal{S}^{-1}[k(\cdot,t_j)](t)\,, \quad t\in\Omega\,.$$

The convergence of the series above is absolute, and uniform on any subset of Ω where the function $t\mapsto k(t,t)$ is bounded. Unless for some Riesz and orthonormal bases examples the reconstruction funtions $\mathcal{S}^{-1}[k(\cdot,t_j)]$ are not avalaible, in general, in closed form. When $\mathcal{H}=PW_\pi$ is the classical Paley-Wiener space of square-integrable functions on $\mathbb{R}$ whose Fourier transforms are supported on $[-\pi,\pi]$, and $t_j=j$, $j\in\mathbb{Z}$, the reproducing kernel is the sine cardinal function $k(t,s)=\operatorname{sinc}(t-s)$, and the sampling expansion above becomes the celebrated Whittaker-Shannon-Kotel'nikov sampling series

$$f(t)=\sum_{j\in\mathbb{Z}} f(j)\frac{\sin\pi(t-j)}{\pi(t-j)}\,, \qquad t\in\mathbb{R}\,. \tag{1.1}$$

For sampling in RKHS see, among others, [9, 22] and references therein.

The generalization of the concept of reproducing kernel in Banach spaces has been proposed by several authors (see, for instance, [16, 28, 29]). The aim in this work is to give some sampling results in the light of these new reproducing kernel Banach space theories. Sampling in Banach spaces is not a new topic in the mathematical literature: see, for instance, [17] for sampling in Bernstein and Paley-Wiener spaces, and also [1, 2, 3, 13, 16] for sampling in L^p shift-invariant spaces. Next, we briefly introduce the reproducing kernel Banach space concept as it appears in [28].

1.1. Reproducing Kernel Banach Spaces

A normed vector space $\mathcal{B}$ is called a Banach space of functions on Ω if it is a Banach space whose elements are functions on Ω, and for each $f\in\mathcal{B}$, its norm $\|f\|_{\mathcal{B}}$ vanishes if and only if f, as a function, vanishes everywhere on Ω. Thus, the Lebesgue space $L^p[0,1]$, $1\le p\le\infty$, is not a Banach space of functions as it consists of equivalent classes of functions with respect to the Lebesgue measure.

Having in mind the definition of RKHS, one could define a reproducing kernel Banach space (RKBS in short) as a Banach space of functions on Ω such that the point evaluations are continuous linear functionals. If such a definition was adopted, then the Banach space $\big(C[0,1],\|\cdot\|_\infty\big)$ of continuous functions on $[0,1]$ equipped with the maximum norm satisfies the definition. However, since for each $f\in C[0,1]$, $f(t)=\mathcal{E}_t(f)$, $t\in[0,1]$, the reproducing kernel for $C[0,1]$ would have to be the delta distribution, which is not a function that can be evaluated. This example suggests that there should exists a way of identifying the elements in the dual of an RKBS with functions. Recall that two normed vector spaces are isometric if there is a bijective linear norm-preserving mapping between them, and we say that each one is an identication of each other. We would like the dual space $\mathcal{B}^*$ of an RKBS $\mathcal{B}$ on Ω to be isometric to a Banach space of functions on Ω. In addition to this requirement, later on we will find it very convenient to go from a Banach space to its dual. For this reason, we would like a RKBS $\mathcal{B}$ to be reflexive in the sense that $(\mathcal{B}^*)^* = \mathcal{B}$. Thus, we give the following definition [28]:

Definition 1.1. A reproducing kernel Banach space on Ω is a reflexive Banach space $\mathcal{B}$ of functions on Ω for which $\mathcal{B}^*$ is isometric to a Banach space $\widetilde{\mathcal{B}}$ of functions on Ω and the point evaluation is continuous on both $\mathcal{B}$ and $\widetilde{\mathcal{B}}$.

As pointed out in [28], the identification $\widetilde{\mathcal{B}}$ of $\mathcal{B}^*$ is not unique; we will refer to the dual space $\mathcal{B}^*$ of a RKBS $\mathcal{B}$ as its chosen identification. It has been proved in [28] that there exists a reproducing kernel for an RKBS as defined above. To this end, we introduce the bilinear form on $\mathcal{B} \times \mathcal{B}^*$ by setting

$$\big(u, v^*\big)_{\mathcal{B}} := v^*(u)\,, \quad u \in \mathcal{B},\ v^* \in \mathcal{B}^*\,.$$

Notice that as $\mathcal{B}$ is a reflexive Banach space then for any bounded linear functional T on $\mathcal{B}^*$ there exists a unique $u \in \mathcal{B}$ such that $T(v^*) = \big(u, v^*\big)_{\mathcal{B}}$ for each $v^* \in \mathcal{B}^*$. The following result holds [28, Theorem 2]:

Theorem 1.2. *Suppose that $\mathcal{B}$ is an RKBS on Ω. Then there exists a unique function $k : \Omega \times \Omega \longrightarrow \mathbb{C}$ such that the following statements hold:*

(a) *For every $t \in \Omega$, $k(\cdot, t) \in \mathcal{B}^*$ and $f(t) = \big(f, k(\cdot, t)\big)_{\mathcal{B}}$ for all $f \in \mathcal{B}$.*
(b) *For every $t \in \Omega$, $k(t, \cdot) \in \mathcal{B}$ and $f^*(t) = \big(k(t, \cdot), f^*\big)_{\mathcal{B}}$ for all $f^* \in \mathcal{B}^*$.*
(c) *The linear span of $\{k(t, \cdot) : t \in \Omega\}$ is dense in $\mathcal{B}$.*
(d) *The linear span of $\{k(\cdot, t) : t \in \Omega\}$ is dense in $\mathcal{B}^*$.*
(e) *For all $t, s \in \Omega$, $k(t, s) = \big(k(t, \cdot), k(\cdot, s)\big)_{\mathcal{B}}$.*

The function k in Theorem 1.2 is the reproducing kernel for the RKBS $\mathcal{B}$. This reproducing kernel is unique. However, as showed in [28], different RKBS may have the same reproducing kernel: For $1 < p < \infty$, the Paley-Wiener classes

$$\mathcal{B}_p := \big\{f \in C(\mathbb{R}) \mid \operatorname{supp} \widehat{f} \subset [-1/2, 1/2] \text{ and } \widehat{f} \in L^p[-1/2, 1/2]\big\}$$

with norm $\|f\|_{\mathcal{B}_p} := \|\widehat{f}\|_{L^p[-1/2,1/2]}$ are RKBS (not isomorphic), and they all have the sinc function $k(t, s) = \operatorname{sinc}(t-s)$ as the reproducing kernel. In other words, although we have at hand a reproducing kernel k, we can not determine the norm on $\mathcal{B}$. Also, the reproducing kernel for a general RKBS can be an arbitrary function on $\Omega \times \Omega$ which, in particular, might be non-symmetric or non-positive definite [28, Proposition 5]. In order to have the reproducing kernel of a RKBS the desired properties of those of RKHS, we impose certain structures on RKBS, which in some sense are substitutes of the inner product for RKHS. For this purpose, we shall adopt the semi-inner-product introduced by Lumer [21] (see also [14]). A semi-inner-product possesses some but not all properties of an inner product. Some Hilbert space arguments and results become available in the presence of a semi-inner-product. Next, we briefly introduce the notion of semi-inner-product RKBS as in [28]:

1.2. Semi-Inner-Product Reproducing Kernel Banach Spaces

Let $\mathcal{B}$ be a Banach space. A semi-inner-product on $\mathcal{B}$ is a function

$$[\cdot,\cdot]: \quad \mathcal{B}\times\mathcal{B} \quad \longrightarrow \quad \mathbb{C}\,,$$

such that (see, for instance, [14, 21]), for all $x_1, x_2, x_3 \in \mathcal{B}$ and $\alpha \in \mathbb{C}$:

1. $[x_1 + x_2, x_3] = [x_1, x_3] + [x_2, x_3]\,,$
2. $[\alpha x_1, x_2] = \alpha[x_1, x_2]$ and $[x_1, \alpha x_2] = \overline{\alpha}[x_1, x_2]\,,$
3. $[x_1, x_1] > 0$ for all $x_1 \neq 0\,,$
4. $|[x_1, x_2]|^2 \leq [x_1, x_1][x_2, x_2]\,.$

The difference between a semi-inner-product and an inner product is the conjugate symmetry and, as a consequence, a semi-inner-product is not additive in the second variable. Every normed vector space $\mathcal{B}$ has a semi-inner-product that induces its norm [14, 21], i.e., $\|x\|_{\mathcal{B}} = [x, x]^{1/2}$ for each $x \in \mathcal{B}$. In general, a semi-inner-product for a normed vector space may not be unique; however, if the space $\mathcal{B}$ is uniformly Fréchet differentiable we obtain the uniqueness of the semi-inner-product (see [28] for the details). Recall that the space $\mathcal{B}$ is uniformly Fréchet differentiable for all $x, y \in \mathcal{B}$ with $x \neq 0$,

$$\lim_{\substack{t\in\mathbb{R}\\ t\to 0}} \frac{\big\|x + ty\big\|_{\mathcal{B}} - \big\|x\big\|_{\mathcal{B}}}{t}$$

exists and the limit is uniform on $\mathcal{S}(\mathcal{B}) \times \mathcal{S}(\mathcal{B})$ where $\mathcal{S}(\mathcal{B})$ denotes the unit sphere $\mathcal{S}(\mathcal{B}) := \big\{x \in \mathcal{B} : \big\|x\big\|_{\mathcal{B}} = 1\big\}$ in $\mathcal{B}$.

Assuming also that the Banach space is uniformly convex we obtain a Riesz representation theorem [14]: For each $f \in \mathcal{B}^*$ there exists a unique $x \in \mathcal{B}$ such that $f = x^*$. In other words,

$$f(y) = [y, x]_{\mathcal{B}} \quad \text{for all } y \in \mathcal{B}\,.$$

Moreover, $\|f\|_{\mathcal{B}^*} = \|x\|_{\mathcal{B}}$. Recall that $\mathcal{B}$ is uniformly convex if for all $\varepsilon > 0$ there exists $\delta > 0$ such that $\big\|x + y\big\|_{\mathcal{B}} \leq 2 - \delta$ for all $x, y \in \mathcal{S}(\mathcal{B})$ with $\big\|x - y\big\|_{\mathcal{B}} \geq \varepsilon$.

Notice that if $\mathcal{B}$ is uniformly convex then it is reflexive (see [23, p. 410]) and strictly convex, i.e., for every $x, y \in \mathcal{B}$ with $x \neq y$ and $\big\|x\big\| = \big\|y\big\| = 1$, we have that $\big\|x + y\big\| < 2$.

For $1 < p < \infty$, the classical $L^p(I)$, where I denotes any interval on $\mathbb{R}$, and $\ell^p(\mathbb{N})$ spaces are uniformly convex and uniformly Fréchet differentiable Banach spaces. Their semi-inner-product are given, respectively, by

$$[f, g]_p := \big\|g\big\|_p^{2-p} \int_I f(t)\overline{g(t)}|g(t)|^{p-2}\,dt\,,$$

and

$$[x,y]_p := \|y\|_p^{2-p} \sum_{n=1}^{\infty} x_n \overline{y_n} |y_n|^{p-2} .$$

Following [28], we define a semi-inner-product reproducing kernel Banach space (hereafter s.i.p. RKBS) on Ω as a uniformly convex and uniformly Fréchet differentiable RKBS on Ω.

An RKHS is an s.i.p. RKBS. Also, the dual of an s.i.p. RKBS remains an s.i.p. RKBS. An s.i.p. RKBS $\mathcal{B}$ is by definition uniformly Fréchet differentiable. Therefore, it has a unique semi-inner-product which represents all the interaction between $\mathcal{B}$ and $\mathcal{B}^*$. This leads to a more specific representation of the reproducing kernel [28, Theorem 9]:

Theorem 1.3. *Let $\mathcal{B}$ be an s.i.p. RKBS on Ω and k its reproducing kernel. Then there exists a unique function $G : \Omega \times \Omega \longrightarrow \mathbb{C}$ such that $\{G(t,\cdot) : t \in \Omega\} \subset \mathcal{B}$ and*

$$f(t) = [f, G(t,\cdot)]_{\mathcal{B}} \quad \text{for all } f \in \mathcal{B}, \quad t \in \Omega .$$

Moreover, there holds the relationship

$$k(\cdot,t) = (G(t,\cdot))^* , \quad t \in \Omega$$

and

$$f^*(t) = [k(t,\cdot), f]_{\mathcal{B}} \quad \text{for all } f \in \mathcal{B}, \quad t \in \Omega .$$

We call the unique function G in the Theorem above the s.i.p. kernel of the s.i.p. RKBS $\mathcal{B}$. It coincides with the reproducing kernel k when $\mathcal{B}$ is an RKHS. In general, when $G = k$ in Theorem 1.3, we call G an s.i.p. reproducing kernel. An s.i.p. reproducing kernel G satisfies that $G(t,s) = [G(t,\cdot), G(s,\cdot)]_{\mathcal{B}}$ for every $t, s \in \Omega$.

2. Sampling in a s.i.p. RKBS Induced by a Banach Space Valued Kernel

Consider a separable complex uniform (i.e., both uniformly Fréchet differentiable and uniformly convex) Banach space $\mathcal{B}$ and denote by $[\cdot,\cdot]_{\mathcal{B}}$ the unique compatible semi-inner product on $\mathcal{B}$. Note that its dual $\mathcal{B}^*$ is also a uniform Banach space [6].

Let X_d be a BK-space on $\mathbb{N}$, i.e., a Banach space of sequences $c = \{c_n\}_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ such that the linear functionals $c \mapsto c_n$ are continuous on X_d for $n \in \mathbb{N}$. It is known [20] that its dual space X_d^* is also a BK-space such that the series $\sum_{n=1}^{\infty} c_n d_n$ converges for every $c \in X_d$ and $d \in X_d^*$. We suppose that if the series above converges for every $c \in X_d$, then $d \in X_d^*$ and if it converges for every $d \in X_d^*$, then $c \in X_d$. We also assume that X_d is reflexive, and that the sequence of the canonical unit vectors $\{\delta_n\}_{n=1}^{\infty}$ is a Schauder basis for both X_d and X_d^*. An example of such BK-spaces is $X_d = \ell^p(\mathbb{N})$ for $1 < p < \infty$; in this case, $X_d^* = \ell^q(\mathbb{N})$ with $1/p + 1/q = 1$.

Let $\{x_n^*\}_{n=1}^\infty \subset \mathcal{B}^*$ be an X_d^*-Riesz basis for $\mathcal{B}^*$. Remember that this means that:

1. $\overline{\text{span}}\{x_n^* : n \in \mathbb{N}\} = \mathcal{B}^*$;
2. $\sum_{n=1}^{\infty} c_n x_n^*$ converges in $\mathcal{B}^*$ for all $c \in X_d^*$;
3. there exist $0 < A \le B < \infty$ such that

$$A\|c\|_{X_d^*} \le \Big\|\sum_{n=1}^{\infty} c_n x_n^*\Big\|_{\mathcal{B}^*} \le B\|c\|_{X_d^*} \quad \text{for all } c \in X_d^* . \tag{2.1}$$

By [29, Theorem 2.15], there exists a unique (dual) X_d-Riesz basis $\{y_n\}_{n=1}^\infty$ for $\mathcal{B}$ such that $[y_m, x_n]_{\mathcal{B}} = \delta_{m,n}$ for $m, n \in \mathbb{N}$, and satisfying the expansions:

$$x = \sum_{n=1}^{\infty} [x, x_n]_{\mathcal{B}}\, y_n \qquad \text{and} \qquad x^* = \sum_{n=1}^{\infty} [y_n, x]_{\mathcal{B}}\, x_n^* , \tag{2.2}$$

for all $x \in \mathcal{B}$ and $x^* \in \mathcal{B}^*$, respectively. If the spaces X_d and X_d^* possess the additional property that for all $c \in X_d$ and $d \in X_d^*$ the series $\sum_{n=1}^{\infty} c_n d_n$ converges absolutely, then the expansions in (2.2) are unconditionally convergent, i.e., independent of the summation order (see [29, p. 7]). In particular, it is true for ℓ^p-Riesz bases due to Hölder inequality.

Following the steps in [28, Theorem 10], in the next section we give a s.i.p. RKBS with explicit s.i.p. reproducing kernel where a sampling theory holds. It is the Banach counterpart of the sampling theory given in [11, 18] for the RKHS introduced by Saitoh in [24].

2.1. A Sampling Theory in $\mathcal{B}_K$

Consider a $\mathcal{B}$-valued function $K : \Omega \subset \mathbb{C} \to \mathcal{B}$ and define, for each $x \in \mathcal{B}$, the function

$$f_x \;:\; z \in \Omega \longmapsto [x, K(z)]_{\mathcal{B}} \in \mathbb{C}.$$

Then, we have a linear transform $\mathcal{T}_K$ on $\mathcal{B}$ with values in $\mathbb{C}^\Omega$ such that $\mathcal{T}_K x = f_x$. Indeed, for $x, y \in \mathcal{B}$ and $\lambda, \mu \in \mathbb{C}$, we have

$$f_{\lambda x+\mu y}(z) = [\lambda x+\mu y, K(z)]_{\mathcal{B}} = \lambda[x, K(z)]_{\mathcal{B}}+\mu[y, K(z)]_{\mathcal{B}} = \lambda f_x(z)+\mu f_y(z) ,$$

for all $z \in \Omega$.

Having in mind (2.2), for each $z \in \Omega$, we can write

$$\big[K(z)\big]^* = \sum_{n=1}^{\infty} [y_n, K(z)]_{\mathcal{B}}\, x_n^* .$$

We denote $S_n(z) := [y_n, K(z)]_{\mathcal{B}} = f_{y_n}(z)$. Suppose that there exists a sequence $\{z_n\}_{n=1}^\infty$ in Ω and $\{a_n\}_{n=1}^\infty$ in $\mathbb{C} \setminus \{0\}$ such that the interpolatory condition

$$S_n(z_m) = a_n \delta_{n,m} , \tag{2.3}$$

holds. Then, we have that $\big[K(z_m)\big]^* = a_m x_m^*$ and that $\mathcal{T}_K$ is one-to-one. Indeed, if $f_x(z) = 0$ for all $z \in \Omega$,

$$0 = f_x(z) = [x, K(z)]_{\mathcal{B}} = \Big[\sum_{n=1}^{\infty}[x, x_n]_{\mathcal{B}}\, y_n, K(z)\Big]$$
$$= \sum_{n=1}^{\infty}[x, x_n]_{\mathcal{B}}\, [y_n, K(z)]_{\mathcal{B}} = \sum_{n=1}^{\infty}[x, x_n]_{\mathcal{B}}\, S_n(z)\,,$$

where we have used that $x \mapsto [x, y]_{\mathcal{B}}$ is a continuous functional for any fixed $y \in \mathcal{B}$. Evaluating in z_m for $m \in \mathbb{N}$, we have that $[x, x_m]_{\mathcal{B}} = 0$ for $m \in \mathbb{N}$. Having in mind (2.2), this implies that $x = 0$ and $\mathcal{T}_K$ is one-to-one.

Denote $\mathcal{B}_K := \operatorname{rg}(\mathcal{T}_K)$, the range of the operator $\mathcal{T}_K$. If we define $\|f_x\|_{\mathcal{B}_K} := \|x\|_{\mathbb{B}}$ we obtain that $\mathcal{B}_K$ is a Banach space of functions defined on Ω and valued on $\mathbb{C}$. Moreover, $[f_x, f_y]_{\mathcal{B}_K} := [x, y]_{\mathcal{B}}$ defines a compatible semi-inner product on $\mathcal{B}_K$. The space $\mathcal{B}_K$ becomes a s.i.p. RKBS whose s.i.p. reproducing kernel is given by

$$k(z, w) = [K(z), K(w)]_{\mathcal{B}}\,, \quad z, w \in \Omega\,.$$

Indeed, for each $z \in \Omega$, the evaluation functional $\mathcal{E}_z : \mathcal{B}_K \longrightarrow \mathbb{C}$ is continuous:

$$|\mathcal{E}_z(f_x) = |f_x(z)| = |[x, K(z)]_{\mathcal{B}}| \le \|x\|_{\mathcal{B}}\|K(z)\|_{\mathcal{B}} = \|K(z)\|_{\mathcal{B}}\|f_x\|_{\mathcal{B}_K}\,.$$

Observe that, by definition, $k_z := k(z, \cdot) = f_{K(z)} \in \mathcal{B}_K$ for all $z \in \Omega$. Hence we deduce that

$$f_x(z) = [x, K(z)]_{\mathcal{B}} = [f_x, k_z]_{\mathcal{B}_K} = [f_x, k(z, \cdot)]_{\mathcal{B}_K}\,,$$

being k the s.i.p. reproducing kernel for $\mathcal{B}_K$. See [28, Theorem 10] for more details.

Note also that convergence in the norm of $\mathcal{B}_K$ implies pointwise convergence which is uniform on subsets of Ω where the function $z \mapsto \|K(z)\|_{\mathcal{B}}$ is bounded.

Proposition 2.1. *For every $z \in \Omega$, the sequence $\{S_n(z)\}_{n=1}^{\infty}$ is an element of X_d^*.*

Proof. Consider $c \in X_d$ and $z \in \Omega$. We must prove that $\sum_{n=1}^{\infty} c_n S_n(z)$ is convergent. Indeed, by using that, for each $z \in \mathcal{B}$, the mapping $x \mapsto [x, z]_{\mathcal{B}}$ is a continuous linear functional on $\mathcal{B}$, and $\{y_n\}_{n=1}^{\infty}$ is an X_d-Riesz basis for $\mathcal{B}$,

$$\Big|\sum_{n=1}^{\infty} c_n S_n(z)\Big| = \Big|\sum_{n=1}^{\infty} c_n [y_n, K(z)]_{\mathcal{B}}\Big| = \Big|\Big[\sum_{n=1}^{\infty} c_n y_n, K(z)\Big]_{\mathcal{B}}\Big|$$
$$\le \|K(z)\|_{\mathcal{B}}\Big\|\sum_{n=1}^{\infty} c_n y_n\Big\|_{\mathcal{B}} \le B\|K(z)\|_{\mathcal{B}}\|c\|_{X_d}$$

which completes the proof. □

Theorem 2.2 (A Kramer-Type Sampling Theorem for $\mathcal{B}_K$). *Suppose that, for each $z \in \Omega$, we have the expansion $\big[K(z)\big]^* = \sum_{n=1}^{\infty} S_n(z) x_n^*$, where $\{x_n^*\}_{n=1}^{\infty} \subset \mathcal{B}^*$ is an X_d^*-Riesz basis for $\mathcal{B}^*$ and $S_n(z) = [y_n, K(z)]_{\mathcal{B}}$, being $\{y_n\}_{n=1}^{\infty}$ the dual X_d-Riesz basis for $\mathcal{B}$ of $\{x_n^*\}_{n=1}^{\infty}$. Assume also the existence of sequences $\{z_m\}_{m=1}^{\infty} \subset \mathbb{C}$ and $\{a_m\}_{m=1}^{\infty} \subset \mathbb{C} \setminus \{0\}$ such that the interpolatory condition (2.3) holds. Then, the sequence $\{S_n\}_{n=1}^{\infty}$ is an X_d-Riesz basis for $\mathcal{B}_K$ and, for each $f \in \mathcal{B}_K$, we have the sampling expansion*

$$f(z) = \sum_{n=1}^{\infty} f(z_n) \frac{S_n(z)}{a_n}, \quad z \in \Omega .$$

The convergence of the series above is in the norm of $\mathcal{B}_K$, and uniform on subsets of Ω where the function $z \mapsto \big\|K(z)\big\|_{\mathcal{B}}$ is bounded.

Proof. First, we prove that the sequence $\{S_n\}_{n=1}^{\infty}$ is an X_d-Riesz basis for $\mathcal{B}_K$.

1. Consider $z \in \Omega$ and $x \in \mathcal{B}$. Then

$$f_x(z) = [x, K(z)]_{\mathcal{B}} = \Big[\sum_{n=1}^{\infty} [x, x_n]_{\mathcal{B}}\, y_n, K(z)\Big]_{\mathcal{B}} = \sum_{n=1}^{\infty} [x, x_n]_{\mathcal{B}}\, S_n(z) ,$$

hence, $\overline{\text{span}}\{S_n\}_{n=1}^{\infty} = \mathcal{B}_K$.

2. Let c be in X_d. As $\mathcal{T}_K$ is an isometry and $\{y_n\}_{n=1}^{\infty}$ is an X_d-Riesz basis for $\mathcal{B}$,

$$\Big\|\sum_{n=1}^{\infty} c_n S_n\Big\|_{\mathcal{B}_K} = \Big\|\sum_{n=1}^{\infty} c_n y_n\Big\|_{\mathcal{B}} ,$$

and thus it is convergent for all $c \in X_d$.

3. As $\mathcal{T}_K$ is an isometry, for every $c \in X_d$,

$$A\big\|c\big\|_{X_d} \le \Big\|\sum_{n=1}^{\infty} c_n y_n\Big\|_{\mathcal{B}} = \Big\|\sum_{n=1}^{\infty} c_n S_n\Big\|_{\mathcal{B}_K} \le B\big\|c\big\|_{X_d} .$$

Now consider $m \in \mathbb{N}$. We have that

$$f_x(z_m) = [x, K(z_m)]_{\mathcal{B}} = \sum_{n=1}^{\infty} [x, x_n]_{\mathcal{B}}\, S_n(z_m) = a_m [x, x_m]_{\mathcal{B}} .$$

Thus,

$$f_x(z) = \sum_{n=1}^{\infty} [x, x_n]_{\mathcal{B}}\, S_n(z) = \sum_{n=1}^{\infty} \frac{f_x(z_n)}{a_n} S_n(z) ,$$

in the norm of $\mathcal{B}_K$. The pointwise and uniform convergence comes from the fact that $\mathcal{B}_K$ is a s.i.p. RKBS. □

The above result generalizes the classical Kramer sampling result (see, for instance, [17, 27]).

2.2. Average Sampling in the Space $\mathcal{B}_K$

Consider an X_d^*-Riesz basis for $\mathcal{B}^*$ expressed as $\{x^*_{1,n}\}_{n\in\mathbb{N}} \cup \cdots \cup \{x^*_{M,n}\}_{n\in\mathbb{N}}$; denote as $\{y_{1,n}\}_{n\in\mathbb{N}} \cup \cdots \cup \{y_{M,n}\}_{n\in\mathbb{N}}$ its dual X_d-Riesz basis for $\mathcal{B}$. For $0 \le \ell \le L$, consider functions $K_\ell : \Omega \to \mathcal{B}$ and define, for each $x \in \mathcal{B}$, the functions

$$f_{\ell,x}(z) = [x, K_\ell(z)]_{\mathcal{B}}\,, \quad 0 \le \ell \le L\,.$$

Thus we have $L+1$ linear transforms $\mathcal{T}_\ell : \mathcal{B} \to \mathbb{C}^\Omega$ such that $\mathcal{T}_\ell\, x = f_{\ell,x}$ for $0 \le \ell \le L$. Assume that $M \le L$ and that, for each $z \in \Omega$, we have the expansions

$$K_\ell(z)^* = \sum_{n=1}^{\infty}\sum_{m=1}^{M} S^\ell_{m,n}(z) x^*_{m,n}\,, \quad 0 \le \ell \le L\,,$$

where $S^\ell_{m,n}(z) := [y_{m,n}, K_\ell(z)]_{\mathcal{B}} = f_{\ell, y_{m,n}}(z)$.

Suppose that there exist L sequences $\{z^\ell_n\}_{n=1}^\infty$ in Ω, $\ell \in \{1,2,\ldots,L\}$, such that

$$S^\ell_{m,n}(z^\ell_k) = a^n_{\ell,m}\delta_{n,k}\,, \quad n,k \in \mathbb{N}\,, \tag{2.4}$$

where $1 \le m \le M$, $1 \le \ell \le L$ and the coefficents $a^n_{\ell,m}$ are complex numbers such that the matrices

$$A_n := \begin{pmatrix} a^n_{1,1} & a^n_{1,2} & \cdots & a^n_{1,M} \\ a^n_{2,1} & a^n_{2,2} & \cdots & a^n_{2,M} \\ \vdots & \vdots & \ddots & \vdots \\ a^n_{L,1} & a^n_{L,2} & \cdots & a^n_{L,M} \end{pmatrix} \in \mathbb{C}^{L\times M}\,, \qquad (n \in \mathbb{N}) \tag{2.5}$$

have full rank for $n \in \mathbb{N}$, i.e., $\operatorname{rank}(A_n) = M$ for every $n \in \mathbb{N}$.

Suppose the compatibility condition: $\ker \mathcal{T}_0 \subseteq \bigcap_{\ell=1}^L \ker \mathcal{T}_\ell$ which implies that the mapping $\mathcal{T}_0$ is one-to-one. Indeed, if $\mathcal{T}_0\, x = 0$, then $\mathcal{T}_\ell\, x = 0$ for any $\ell \in \{1,2,\ldots,L\}$. Hence, for every $z \in \Omega$ we have

$$\begin{aligned} 0 = \mathcal{T}_\ell\, x(z) = f_{\ell,x}(z) = [x, K_\ell(z)]_{\mathcal{B}} &= \Big[\sum_{n=1}^{\infty}\sum_{m=1}^{M} [x, x_{m,n}]_{\mathcal{B}}\, y_{m,n}, K_\ell(z)\Big]_{\mathcal{B}} \\ = \sum_{n=1}^{\infty}\sum_{m=1}^{M} [x, x_{m,n}]_{\mathcal{B}}\, [y_{m,n}, K_\ell(z)]_{\mathcal{B}} &= \sum_{n=1}^{\infty}\sum_{m=1}^{M} [x, x_{m,n}]_{\mathcal{B}}\; S^\ell_{m,n}(z)\,, \end{aligned}$$

where we have used that $x \mapsto [x,y]_{\mathcal{B}}$ is a continuous functional for each $y \in \mathcal{B}$. By using (2.4), we have, for each $n \in \mathbb{N}$, the linear system

$$\sum_{m=1}^{M} a^n_{\ell,m}[x, x_{m,n}]_{\mathcal{B}} = 0\,, \qquad (1 \le \ell \le L).$$

As the matrices A_n have full rank, we conclude that $x = 0$ (see (2.2)). If $x \in \mathcal{B}$, then we have proved that

$$\sum_{m=1}^{M} a^n_{\ell,m}[x, x_{m,n}]_{\mathcal{B}} = f_\ell(z^\ell_n)\,, \qquad (1 \le \ell \le L)\,,$$

where $\{f_\ell(z_n^\ell)\}_{n=1}^\infty$ denotes the sequence of samples of the function $f_\ell(z) = \big[x, K_\ell(z)\big]_{\mathcal{B}}$, $z \in \Omega$. Notice that this system is consistent. As the matrices A_n has full rank for every $n \in \mathbb{N}$, we always can choose a regular $M \times M$ submatrix $A_n^{[M]}$ of A_n for obtaining the coefficients $[x, x_{m,n}]_{\mathcal{B}}$ in terms of the samples. Thus we have proved the following result:

Theorem 2.3. *Suppose that* $\ker \mathcal{T}_0 \subseteq \bigcap_{\ell=1}^L \ker \mathcal{T}_\ell$ *and that the matrix* A_n *given in* (2.5) *has full rank* M *for all* $n \in \mathbb{N}$. *Then, each function* $f \in \mathcal{B}_{K_0}$ *can be recovered from the* L *sequences of samples* $\{f_\ell(z_n^\ell)\}_{n=1}^\infty$, $1 \le \ell \le L$, *by means of the following sampling formula*

$$f(z) = \sum_{n=1}^\infty \mathbb{S}_n(z)^\top \Big(A_n^{[M]}\Big)^{-1} \mathbf{F}_n\,, \quad z \in \Omega\,, \tag{2.6}$$

where $\mathbf{F}_n := \big[f_1(z_n^1), \dots, f_L(z_n^L)\big]^\top$ *and* $\mathbb{S}_n(z) := \big[S_{0,1}^n(z), \dots, S_{0,M}^n(z)\big]^\top$. *The convergence of the series in* (2.6) *is uniform in subsets of* Ω *where the function* $z \mapsto \big\|K(z)\big\|_{\mathcal{B}}$ *is bounded.*

The above abstract sampling result includes, as particular cases, sampling with samples of the derivative of the function to recover or its Hilbert transform, and Hermite-type interpolation series among others (see [11, 12]).

2.3. An Illustrative Example

Consider $p \in (1, 2]$ and its conjugate index $q \in \mathbb{R}$, i.e., $1/p + 1/q = 1$. We consider the compatible semi-inner-product for $\mathcal{B} := L^p[-1/2, 1/2]$ given by

$$[f, g]_p := \big\|g\big\|_p^{2-p} \int_{-1/2}^{1/2} f(x)\overline{g(x)}|g(x)|^{p-2}\,dx\,.$$

Remember that $\mathcal{B}^* = L^q[-1/2, 1/2]$. We take $X_d := \ell^q(\mathbb{Z})$, then, $X_d^* = \ell^p(\mathbb{Z})$.

Define $e_n(\xi) := \mathrm{e}^{2\pi i n \xi}$ for $n \in \mathbb{Z}$. Easy computations show that $\|e_n\|_p = 1$ for any $n \in \mathbb{Z}$. On the other hand, by [28], we have that $e_n^*(\xi) = \mathrm{e}^{-2\pi i n \xi}$ and that $\|e_n^*\|_q = \|e_n\|_p = 1$. We know that (see [25, p. 20]):

$$\overline{\operatorname{span}}\big\{\mathrm{e}^{-2\pi i n \xi} :\ n \in \mathbb{Z}\big\} = L^q[-1/2, 1/2]\,.$$

We define the linear operator

$$U\ :\ F \in L^p[-1/2, 1/2] \longmapsto \big\{[F, e_n]_p\big\}_{n \in \mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}.$$

The Hausdorff-Young theorem (see [30, p. 101]) ensures that U is a bounded operator with values on $\ell^q(\mathbb{Z})$ (and thus a closed operator). We have that $U(L^p[-1/2, 1/2])$ is a closed subspace of $\ell^q(\mathbb{Z})$ and thus, a Banach space with the metric induced by $\ell^q(\mathbb{Z})$. Remember that $\{\delta_n\}_{n \in \mathbb{Z}}$ where $\delta_n(m) = 0$ if $n \ne m$ and $\delta_n(n) = 1$, is a Schauder basis for $\ell^q(\mathbb{Z})$. Thus, as $\delta_n = U(e_n)$, we obtain that U is a surjective mapping. By using [29, Proposition 2.12], the sequence $\big\{e_n^*\big\}_{n \in \mathbb{Z}}$ is an $\ell^p(\mathbb{Z})$-Riesz basis for $\mathcal{B}^* = L^q[-1/2, 1/2]$.

Now, we define $K(z) := \mathrm{e}^{2\pi i z \xi} \in L^p[-1/2, 1/2]$ for every $z \in \mathbb{C}$. Thus, we obtain the following s.i.p. RKBS

$$\mathcal{B}_K := \Big\{f(z) = \big[F, \mathrm{e}^{2\pi i z \xi}\big]_p,\ z \in \mathbb{C}\,,\ \text{where}\ \ F \in L^p[-1/2, 1/2]\Big\}\,,$$

endowed with the norm $\|f\|_{\mathcal{B}_K} := \|F\|_{L^p[-1/2,1/2]}$.

Next, we compute $S_n(z) = [e_n, K(z)]_p$. First, observe that, if we write $z = x + iy$,

$$\big\|K(z)\big\|_p = \left(\int_{-1/2}^{1/2} \left|e^{2\pi i z\xi}\right|^p d\xi\right)^{1/p} = \left(\int_{-1/2}^{1/2} e^{-2\pi y\xi p} d\xi\right)^{1/p} = \operatorname{sinc}^{1/p}(iyp)\,.$$

Thus, we have that

$$\begin{aligned} S_n(z) = [e_n, K(z)]_p &= \operatorname{sinc}^{(2-p)/p}(iyp) \int_{-1/2}^{1/2} e^{2\pi i n\xi} e^{-2\pi i z\xi} e^{-2\pi y\xi(p-2)} d\xi \\ &= \operatorname{sinc}^{(2-p)/p}(iyp) \int_{-1/2}^{1/2} e^{-2\pi i[(z-n)-iy(p-2)]\xi} d\xi \\ &= \operatorname{sinc}^{(2-p)/p}(iyp)\operatorname{sinc}\big[(z-n) - iy(p-2)\big]\,, \quad z \in \mathbb{C}\,, \end{aligned}$$

for $n \in \mathbb{Z}$. Moreover, $S_n(m) = \delta_{m,n}$ for every $m, n \in \mathbb{Z}$. Finally, Theorem 2.2 gives the following sampling formula for any $f \in \mathcal{B}_K$:

$$f(z) = \operatorname{sinc}^{(2-p)/p}(iyp) \sum_{n\in\mathbb{Z}} f(n)\operatorname{sinc}\big[(z-n) - iy(p-2)\big]\,, \tag{2.7}$$

where $z = x + iy \in \mathbb{C}$. The convergence of the series in (2.7) is uniform on horizontal strips of $\mathbb{C}$. Observe that, if $p = 2$ or $z \in \mathbb{R}$, formula (2.7) coincides with the cardinal series (1.1). Compare the sampling result for $\mathcal{B}_K$ given by (2.7) with the one obtained in [7, Theorem 1] for slightly different functional spaces.

3. Sampling in a RKBS: the Case of L^p Shift-Invariant Spaces

In this section we obtain an average sampling result valid in the L^p shift-invariant space $V_\varphi^p := \overline{\operatorname{span}}_{L^p(\mathbb{R})}\big\{\varphi(t-n)\big\}_{n\in\mathbb{Z}}$ where $1 \le p < \infty$. Under appropriate hypotheses on the generator φ we see that these spaces are RKBS sharing the same reproducing kernel k. Next we introduce some technical background:

3.1. Preliminaries on the L^p Shift-Invariant Space V_φ^p

First, we introduce the following Banach spaces: A measurable function $f : \mathbb{R} \to \mathbb{C}$ belongs to $\mathcal{L}^p(\mathbb{R})$, where $1 \le p \le \infty$, whenever the function $\widetilde{f}(t) := \sum_{n\in\mathbb{Z}} |f(t-n)|$ belongs to the Lebesgue space $L^p[0,1]$. In this case, we define $|f|_p := \|\widetilde{f}\|_{L^p[0,1]}$. Endowed with this norm, the space $(\mathcal{L}^p(\mathbb{R}), |\cdot|_p)$ becomes a Banach space (see [19]).

Given a function φ in the Banach space $\mathcal{L}^\infty(\mathbb{R})$, for $1 \le p < \infty$ we consider the L^p shift-invariant space

$$V_\varphi^p := \overline{\operatorname{span}}_{L^p(\mathbb{R})}\big\{\varphi(t-n)\big\}_{n\in\mathbb{Z}} \subset L^p(\mathbb{R})\,.$$

If in addition we assume that the sequence $\{\varphi(t-n)\}_{n\in\mathbb{Z}}$ is an ℓ^p-Riesz basis for V_φ^p, i.e., there exist constants $0 < A \le B$ such that for any $a := \{a_n\}_{n\in\mathbb{Z}} \in \ell^p(\mathbb{Z})$ we have

$$A\|a\|_{\ell^p} \le \Big\| \sum_{n\in\mathbb{Z}} a_n\, \varphi(t-n) \Big\|_{L^p(\mathbb{R})} \le B\|a\|_{\ell^p}\,, \tag{3.1}$$

then the space V_φ^p can be expressed as

$$V_\varphi^p = \Big\{ \sum_{n\in\mathbb{Z}} a_n\, \varphi(t-n) : \{a_n\} \in \ell^p(\mathbb{Z}) \Big\} \subset L^p(\mathbb{R})\,.$$

Since V_φ^p is a closed subspace of $L^p(\mathbb{R})$, it is a uniformly Fréchet differentiable and uniformly convex Banach space [29].

Following [19], a necessary and sufficient condition for the sequence $\{\varphi(t-n)\}_{n\in\mathbb{Z}}$ to be an ℓ^p-Riesz basis (regardless p) is that there exists a sequence $b \in \ell^1(\mathbb{Z})$ such that the function $\varphi^*(t) := \sum_{n\in\mathbb{Z}} b_n\, \varphi(t-n)$ is dual to the function φ in the sense that

$$\int_{-\infty}^{\infty} \varphi(t-n)\, \varphi^*(t-m) dt = \delta_{n,m}\,, \quad n, m \in \mathbb{Z}\,,$$

where $\delta_{n,m}$ denotes the Kronecker symbol.

We assume throughout this section that the functions in the shift-invariant space V_φ^p are continuous on $\mathbb{R}$. Equivalently, that the generator φ is continuous on $\mathbb{R}$ and the function $\sum_{n\in\mathbb{Z}} |\varphi(t-n)|^q$, where $1/p + 1/q = 1$, is uniformly bounded on $\mathbb{R}$; this is a consequence of Banach-Steinhaus theorem (see [10, Theorem 1]).

Thus, the shift-invariant space V_φ^p becomes a RKBS, and the convergence in the L^p sense implies pointwise convergence which is uniform on $\mathbb{R}$ since Hölder's inequality shows that

$$|f(t)| \le \|a\|_{\ell^p} \|\{\varphi(t-n)\}_{n\in\mathbb{Z}}\|_{\ell^q} \le A^{-1} K \|f\|_{L^p(\mathbb{R})}\,, \quad f \in V_\varphi^p \text{ and } t \in \mathbb{R}\,.$$

In the sense of Theorem 1.2, we have that the function

$$k\ :\ (t,s) \in \mathbb{R}\times\mathbb{R} \longmapsto k(t,s) := \sum_{n\in\mathbb{Z}} \varphi(s-n)\, \varphi^*(t-n) \in \mathbb{C}$$

is the reproducing kernel for V_φ^p. Notice that all the spaces V_φ^p, $1 < p < \infty$, have the same reproducing kernel k although they are not isomorphic.

3.2. An Average Sampling Formula in V_φ^p

For any function $f \in V_\varphi^p$, throughout this section we consider the sequence of samples $\{(\mathcal{C}f)(n)\}_{n\in\mathbb{Z}}$, where the convolution system $\mathcal{C}$ satisfies:

(a) $(\mathcal{C}f)(t) := [f * \mathsf{h}](t) = \int_{\mathbb{C}} f(x)\mathsf{h}(t-x)dx\,,\ t \in \mathbb{R}\,,$ with $\mathsf{h} \in \mathcal{L}^q(\mathbb{R})$ and q satisfying $1/p + 1/q = 1$; or

(b) $(\mathcal{C}f)(t) := f(t+a)$ for some fixed $a \in \mathbb{R}$.

Note that the sequence of samples $\{(\mathcal{C}f)(n)\}_{n\in\mathbb{Z}} \in \ell^p(\mathbb{Z})$ since the inequality $\big\|\{\mathsf{h} * f(n)\}_{n\in\mathbb{Z}}\big\|_p \le |\mathsf{h}|_q \,\|f\|_p$ (see [19, p. 220]) in the first case, and the inequality $\|a*b\|_{\ell^p} \le \|a\|_{\ell^p}\|b\|_{\ell^1}$ in the second one.

Let $\mathcal{A}$ be the Wiener algebra of the functions of the form

$$f(x) = \sum_{n\in\mathbb{Z}} a_n e^{-2\pi i n x} \quad \text{with } a := \{a_n\}_{n\in\mathbb{Z}} \in \ell^1(\mathbb{Z}).$$

The space $\mathcal{A}$, normed by $\|f\|_{\mathcal{A}} := \|a\|_1$ and with pointwise multiplication becomes a commutative Banach algebra. If $f \in \mathcal{A}$ and $f(x) \neq 0$ for every $x \in \mathbb{R}$, the function $1/f$ is also in $\mathcal{A}$ by Wiener's lemma (see, for instance, [15]).

Theorem 3.1. *Assume that the function $G(x) := \sum_{n\in\mathbb{Z}} (\mathcal{C}\varphi)(n) e^{-2\pi i n x}$ does not vanish for any $x \in [0,1]$. Then, there exists a function $S \in \mathcal{L}^\infty(\mathbb{R}) \cap V_\varphi^p$ such that, for any $f \in V_\varphi^p$, the following sampling formula holds:*

$$f(t) = \sum_{n\in\mathbb{Z}} (\mathcal{C}f)(n) S(t-n)\,, \qquad t \in \mathbb{R}\,. \tag{3.2}$$

The convergence of the series is in the L^p-sense and uniform on $\mathbb{R}$.

Proof. First of all, notice that the sequence $\{(\mathcal{C}\varphi)(n)\}_{n\in\mathbb{Z}}$ belongs to $\ell^1(\mathbb{Z})$; for systems of type (a) we have used that $\big\|\{\mathsf{h} * \varphi(n)\}_{n\in\mathbb{Z}}\big\|_{\ell^1} \le |\mathsf{h}|_q |\varphi|_\infty$ [19, p. 220]. Therefore, the function $G \in \mathcal{A}$ and, as a consequence of Wiener's lemma, the function $1/G$ belongs to $\mathcal{A}$, i.e.,

$$\frac{1}{G(x)} = \sum_{n\in\mathbb{Z}} d_n e^{-2\pi i n x} \quad \text{with } \{d_n\} \in \ell^1(\mathbb{Z})\,. \tag{3.3}$$

Consider the operator $\mathcal{T}_\varphi$ defined by

$$\mathcal{T}_\varphi : \sum_{n\in\mathbb{Z}} a_n e^{-2\pi i n x} \in \mathcal{A} \longmapsto \sum_{n\in\mathbb{Z}} a_n\ \varphi(t-n) \in V_\varphi^p. \tag{3.4}$$

Note that the operator $\mathcal{T}_\varphi$ is bounded since $\Big\|\sum_{n\in\mathbb{Z}} a_n\ \varphi(t-n)\Big\|_p \le |\varphi|_\infty \|a\|_{\ell^1}$ (see [19, p. 212]). Next we prove that formula (3.2) holds for any function $f \in \operatorname{span}\{\varphi(\cdot - n)\}_{n\in\mathbb{Z}}$. Indeed, consider a function $f(t) = \sum_{\text{finite}} a_n \varphi(t-n)$ in $\operatorname{span}\{\varphi(\cdot - n)\}_{n\in\mathbb{Z}}$ and its corresponding $F(x) = \sum_{\text{finite}} a_n e^{-2\pi i n x} \in \mathcal{A}$. Since

$$(\mathcal{C}f)(m) = \sum_{\text{finite}} a_n (\mathcal{C}\varphi)(m-n) = \int_0^1 F(x) G(x) e^{2\pi i m x} dx\,,$$

and having in mind that the sequence $\{(\mathcal{C}f)(m)\}_{m\in\mathbb{Z}} \in \ell^1(\mathbb{Z})$, we obtain that

$$F(x)G(x) = \sum_{m\in\mathbb{Z}} (\mathcal{C}f)(m) e^{-2\pi i m x} \quad \text{ in } \mathcal{A}\,.$$

In other words, $F(x) = \sum_{m\in\mathbb{Z}} (\mathcal{C}f)(m)\big(1/G(x)\big)e^{-2\pi imx}$ in $\mathcal{A}$. By applying the operator $\mathcal{T}_\varphi$ we get

$$f(t) = \sum_{m\in\mathbb{Z}} (\mathcal{C}f)(m)\mathcal{T}_\varphi\Big(\frac{e^{-2\pi imx}}{G(x)}\Big)(t) = \sum_{m\in\mathbb{Z}} (\mathcal{C}f)(m)S(t-m)\,, \quad t\in\mathbb{C}\,,$$

where $S = \mathcal{T}_\varphi\Big(1/G(x)\Big)$. Notice that $S(t) = \sum_{n\in\mathbb{Z}} d_n\varphi(t-n)$ where $\{d_n\}$ is the sequence in $\ell^1(\mathbb{Z})$ given in (3.3). Moreover, the function S belongs to $\mathcal{L}^\infty(\mathbb{C})$ since $\big|\sum_{n\in\mathbb{Z}} d_n\varphi(t-n)\big|_\infty \le |\varphi|_\infty \|\{d_n\}\|_{\ell^1}$ (see [19, p. 212]).

Finally, we prove that the sampling formula (3.2) holds for every $f\in V_\varphi^p$. To this end, for $f\in V_\varphi^p$ consider a sequence $\{f_N\}\subset \operatorname{span}\{\varphi(\cdot-n)\}_{n\in\mathbb{Z}}$ such that $\|f_N - f\|_p \underset{N\to\infty}{\longrightarrow} 0$. The sampling operator

$$\Gamma \;:\; f\in V_\varphi^p \longmapsto \Gamma f := \sum_{n\in\mathbb{Z}} (\mathcal{C}f)(n)S(\cdot-n) \in V_\varphi^p,$$

is a well-defined bounded operator. Indeed, for a system of type (a) we have

$$\Big\|\sum_{n\in\mathbb{Z}} (\mathcal{C}f)(n)S(\cdot-n)\Big\|_p \le |S|_\infty \big\|\{(\mathcal{C}f)(n)\}_{n\in\mathbb{Z}}\big\|_{\ell^p} \le |S|_\infty |\mathsf{h}|_q \|f\|_p\,. \quad (3.5)$$

where we have used the inequalities $\big\|\sum_{n\in\mathbb{Z}} a_n\,\varphi(t-n)\big\|_p \le |\varphi|_\infty \|a\|_{\ell^p}$ (see [19, p. 212]) and $\big\|\{\mathsf{h} * f(n)\}_{n\in\mathbb{Z}}\big\|_{\ell^p} \le |\mathsf{h}|_q\,\|f\|_p$ (see [19, p. 220]). For a system of type (b) we use the inequality $\|a*b\|_{\ell^p}\le \|a\|_{\ell^p}\|b\|_{\ell^1}$ for sequences, and the left inequality in (3.1). Moreover, $\Gamma f = f$ for each $f\in \operatorname{span}\{\varphi(\cdot-n)\}_{n\in\mathbb{Z}}$. Thus,

$$0 \le \|f-\Gamma f\|_p \le \|f - f_N + \Gamma f_N - \Gamma f\|_p \le (1+\|\Gamma\|)\|f_N - f\|_p \underset{N\to\infty}{\longrightarrow} 0\,.$$

As a consequence, $\Gamma f = f$ for each $f\in V_\varphi^p$. □

Corollary 3.2. *The sequence of reconstruction functions* $\{S(\cdot-n)\}_{n\in\mathbb{Z}}$ *is a* ℓ^p*-Riesz basis for the Banach space* $(V_\varphi^p, \|\cdot\|_p)$.

Proof. Having in mind the arguments in (3.5), we conclude that there exist two positive constants $0 < A_p \le B_p$ such that

$$A_p\|f\|_p \le \|\{(\mathcal{C}f)(n)\}_{n\in\mathbb{Z}}\|_{\ell^p} \le B_p\|f\|_p\,, \quad f\in V_\varphi^p\,.$$

In other words, we have

$$\frac{1}{B_p}\|\{(\mathcal{C}f)(n)\}_{n\in\mathbb{Z}}\|_{\ell^p} \le \Big\|\sum_{n\in\mathbb{Z}^d} (\mathcal{C}f)(n)S(\cdot-n)\Big\|_p \le \frac{1}{A_p}\|\{(\mathcal{C}f)(n)\}_{n\in\mathbb{Z}}\|_{\ell^p}\,.$$

As a consequence, it is sufficient to prove that the mapping $f\mapsto \{(\mathcal{C}f)(n)\}_{n\in\mathbb{Z}}$ is surjective from $V_\varphi^p \to \ell^p(\mathbb{Z})$. Given $\{b_n\}_{n\in\mathbb{Z}}\in \ell^p(\mathbb{Z})$, the function $g =$

$\sum_{n\in\mathbb{Z}} b_n S(t-n)$ belongs to V_φ^p. Besides, the interpolatory condition

$$(\mathcal{C}S)(m) = \int_0^1 \frac{1}{G(x)} G(x) e^{2\pi i m x} dx = \delta_{m,0}$$

gives that $(\mathcal{C}g)(m) = b_m$ for each $m \in \mathbb{Z}$. □

As a consequence of the above corollary, the convergence of the series in (3.2) is also absolute due to the unconditional character of an ℓ^p-Riesz basis expansion.

References

[1] E. Acosta-Reyes, A. Aldroubi and I. Krishtal, *On stability of sampling-reconstruction models*. Adv. Comput. Math. **31** (2009), 5–34.

[2] A. Aldroubi and K. Gröchenig, *Non-uniform sampling and reconstruction in shift-invariant spaces*. SIAM Rev. **43** (2001), 585–620.

[3] A. Aldroubi, Q. Sun and W-S. Tang, *Convolution, average sampling, and a Calderon resolution of the identity for shift-invariant spaces*. J. Fourier Anal. Appl. **11** (2005), 215–244.

[4] N. Aronszajn, *Theory of reproducing kernels*. Trans. Amer. Math. Soc. **68** (1950), 337–404.

[5] O. Christensen, *An Introduction to Frames and Riesz Bases*. Birkhäuser, Boston, 2003.

[6] D. F. Cudia, *On the localization and directionalization of uniformly convexity*. Bull. Amer. Math. Soc. **69** (1963), 265–267.

[7] G. Fang, *Whittaker-Kotelnikov-Shannon sampling theorem and aliasing error*. J. Approx. Theory **85** (1996), 115–131.

[8] G. D. Faulkner, *Representations of linear functionals in a Banach space*. Rocky Mountain J. Math. **7** (1977), 789–792.

[9] A. G. García, *Orthogonal sampling formulas: a unified approach*. SIAM Rev. **42** (2000), 499–512.

[10] A. G. García and G. Pérez-Villalón, *Approximation from shift-invariant spaces by generalized sampling formulas*. Appl. Comput. Harmon. Anal. **24** (2008), 58–69.

[11] A. G. García and A. Portal, *Sampling in the functional Hilbert space induced by a Hilbert space valued kernel*. Appl. Anal. **82** (2003), 1145–1158.

[12] A. G. García and A. Portal, *A general sampling theory in the functional Hilbert space induced by a Hilbert space valued kernel*. J. Appl. Funct. Anal. **3** (2008), 299–313.

[13] A. G. García, M. J. Muñoz-Bouzo and G. Pérez-Villalón, *Regular multivariate sampling and approximation in L^p shift-invariant spaces*. J. Math. Anal. Appl. **380** (2011), 607–627.

[14] J. R. Giles, *Classes of semi-inner-product spaces*. Trans. Amer. Math. Soc. **129** (1967), 436–446.

[15] K. H. Gröchenig, *Wiener's Lemma: Theme and Variations. An Introduction to Spectral Invariance and Its Applications.* In: "Four Short Courses on Harmonic Analysis" (B. Forster & P. Massopust Eds), Ch. 5, Birkhäuser, Boston, 2010.

[16] D. Han, M. Z. Nashed, and Q. Sun, *Sampling expansions in reproducing kernel Hilbert and Banach spaces.* Numer. Funct. Anal. Optim. **30** (2009), 971–987.

[17] J. R. Higgins, *Sampling Theory in Fourier and Signal Analysis: Foundations.* Oxford University Press, Oxford, 1996.

[18] J. R. Higgins, *A sampling principle associated with Saitoh's fundamental theory of linear transformations.* In: "Analytic Extension Formulas and their Applications" (S. Saitoh, N. Hayashi & M. Yamamoto Eds), 73–86, Int. Soc. Anal. Appl. Comput. **9**, Kluwer Acad. Publ., Dordrecht, 2001.

[19] R. Q. Jia and C. A. Micchelli, *Using the refinement equations for the construction of pre-waveles II: Powers of two.* In: "Curves and Surfaces" (P. J. Laurent, A. Le Méhauté & L. L. Schumaker Eds), 209–246, Academic Press, Boston, 1991.

[20] L. V. Kantorovich and G. P. Akilov, *Functional Analysis in Normed Spaces.* Macmillan, New York, 1964.

[21] G. Lumer, *Semi-inner-product spaces.* Trans. Amer. Math. Soc. **100** (1961), 29–43.

[22] M. Z. Nashed and G. G. Walter, *General sampling theorems in reproducing kernel Hilbert spaces.* Math. Control Signals Systems **4** (1991), 373–412.

[23] W. Rudin, *Functional Analysis.* 2nd Edition. McGraw–Hill Inc., New York, 1991.

[24] S. Saitoh, *Integral Transforms, Reproducing Kernels and Their Applications.* Pitman Research Notes in Mathematics Series **369**, Longman, Harlow, 1997.

[25] R. M. Young, *An Introduction to Nonharmonic Fourier Series.* Academic Press, New York, 1980.

[26] S. Zaremba, *L' équation biharmonique et une classe remarquable de fonctions fondamentales harmoniques.* Bulletin International de l'Académie des Sciences de Cracovie **3** (1907), 147–196.

[27] A. I. Zayed, *Advances in Shannon's Sampling Theory.* CRC Press, Boca Raton FL, 1993.

[28] H. Zhang, Y. Xu, and J. Zhang, *Reproducing kernel Banach spaces for machine learning.* J. Mach. Learn. Res. **10** (2009), 2741–2775.

[29] H. Zhang and J. Zhang, *Frames, Riesz bases, and sampling expansions in Banach spaces via semi-inner products.* Appl. Comput. Harmon. Anal. **31** (2011), 1–25.

[30] A. Zygmund, *Trigonometric Series.* 2nd Ed., Vol. II. Cambridge University Press, New York, 1959.

Antonio G. García
Departamento de Matemáticas
Universidad Carlos III de Madrid
Avda. de la Universidad 30
28911 Leganés – Madrid
Spain
e-mail: `agarcia@math.uc3m.es`

Alberto Portal
Departamento de Matemática Aplicada
E.T.S.I.T.
Universidad Politécnica de Madrid
Avda. Complutense 30
28040 Madrid
Spain
e-mail: alberto.portal@upm.es

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