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A Filippov's Theorem, Some Existence Results and the Compactness of Solution Sets of Impulsive Fractional Order Differential Inclusions

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Abstract. In this paper, we first present an impulsive version of Filippov's Theorem for fractional differential inclusions of the form,

$$\begin{array}{rcl} D_*^\alpha y(t) & \in & F(t, y(t)), \quad \text{a.e. } t \in J\backslash\{t_1, \ldots, t_m\},\ \alpha \in (0,1], \\ y(t_k^+) - y(t_k^-) & = & I_k(y(t_k^-)), \quad k = 1, \ldots, m, \\ y(0) & = & a, \end{array}$$

where $J = [0, b]$, D_*^α denotes the Caputo fractional derivative and F is a set-valued map. The functions I_k characterize the jump of the solutions at impulse points t_k $(k = 1, \ldots, m)$. In addition, several existence results are established, under both convexity and nonconvexity conditions on the multivalued right-hand side. The proofs rely on a nonlinear alternative of Leray-Schauder type and on Covitz and Nadler's fixed point theorem for multivalued contractions. The compactness of solution sets is also investigated.

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1. Introduction

Differential equations with impulses were considered for the first time in the 1960's by Milman and Myshkis [43, 44]. A period of active research, primarily in Eastern Europe from 1960-1970, culminated with the monograph by Halanay and Wexler [27].

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The dynamics of many evolving processes are subject to abrupt changes, such as shocks, harvesting and natural disasters. These phenomena involve short-term perturbations from continuous and smooth dynamics, whose duration is negligible in comparison with the duration of an entire evolution. In models involving such perturbations, it is natural to assume these perturbations act instantaneously or in the form of "impulses". As a consequence, impulsive differential equations have been developed in modeling impulsive problems in physics, population dynamics, ecology, biotechnology, industrial robotics, pharmacokinetics, optimal control, and so forth. Again, associated with this development, a theory of impulsive differential equations has been given extensive attention. Works recognized as landmark contributions include [8, 39, 51, 55]. There are also many different studies in biology and medicine for which impulsive differential equations are good models (see, for example, [3, 36, 37] and the references therein).

In recent years, many examples of differential equations with impulses with fixed moments have flourished in several contexts. In the periodic treatment of some diseases, impulses correspond to administration of a drug treatment or a missing product. In environmental sciences, impulses correspond to seasonal changes of the water level of artificial reservoirs.

During the last ten years, impulsive ordinary differential inclusions and functional differential inclusions with different conditions have been intensely studied by many mathematicians. At present the foundations of the general theory are already laid, and many of them are investigated in detail in the books of Aubin [4], Benchohra *et al* [9] and Henderson and Ouahab [30] and the references therein.

Differential equations with fractional order have recently proved valuable tools in the modeling of many physical phenomena [19, 23, 24, 40, 41]. There has been a significant theoretical development in fractional differential equations in recent years; see the monographs of Kilbas *et al* [33], Miller and Ross [42], Podlubny [52], Samko *et al* [54], and the papers of Bai and Lu [7], Diethelm *et al* [18–20], El-Sayed and Ibrahim [21], Kilbas and Trujillo [34], Mainardi [40], Momani and Hadid, [45], Momani *et al* [46], Nakhushev [48], Podlubny *et al* [53], and Yu and Gao [57].

Very recently, some basic theory for initial value problems for fractional differential equations and inclusions involving the Riemann-Liouville differential operator was discussed by Benchohra *et al* [10], Lakshmikantham [38]. El-Sayed and Ibrahim [21] initiated the study of fractional multivalued differential inclusions.

Applied problems require definitions of fractional derivatives allowing a utilization that is physically interpretable for initial conditions containing $y(0)$, $y'(0)$, etc. The same requirements are true for boundary conditions. Caputo's fractional derivative satisfies these demands. For more details on the geometric and physical interpretation for fractional derivatives of both the Riemann-Liouville and Caputo types, see Podlubny [52].

Recently fractional functional differential equations and inclusions with standard Riemann-Liouville and Caputo derivatives with difference conditions were studied by Benchohra *et al* [10, 11], Henderson and Ouahab [28] and Ouahab [49].

When $\alpha \in (0,2]$, the impulsive differential equations and inclusions with Caputo fractional derivatives was studied by Agarwal *et al* [1, 2], Henderson and Ouahab [29] and Ouahab [50].

In this paper, we shall be concerned with Filippov's theorem and global existence of solutions for impulsive fractional differential inclusions with fractional order. More precisely, we will consider the following problem,

$$D^{\alpha}_{*}y(t) \in F(t, y(t)), \ a.e. \ t \in J = [0, b], \quad 0 < \alpha \leq 1, \tag{1.1}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \dots, m, \tag{1.2}$$

$$y(0) = a, \tag{1.3}$$

where D^{α}_{*} is the Caputo fractional derivative, $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map with compact values ($\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$), $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = b$, $I_k \in C(\mathbb{R}, \mathbb{R})\,(k = 1, \dots, m)$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, and $y(t_k^+) = \lim_{h\to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim_{h\to 0^+} y(t_k - h)$ stand for the right and the left limits of $y(t)$ at $t = t_k$, respectively.

The paper is organized as follows. We first collect some background material and basic results from multi-valued analysis and fractional calculus in Sections 2 and 3, respectively. Then, we shall be concerned with Filippov's theorem for impulsive differential inclusions with fractional order in Section 4. In Section 5, we present some existence results of the above problem, as well as compactness of solutions and upper semicontinuity of the operator solution for problem (1.1)–(1.3).

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts that will be used in the remainder of this paper. Let $AC^i([0, b], \mathbb{R}^n)$ be the space of functions $y : [0, b] \to \mathbb{R}^n$ i-differentiable and whose i^{th} derivative, $y^{(i)}$, is absolutely continuous.

We take $C(J, \mathbb{R})$ to be the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|y\|_{\infty} = \sup\{|y(t)| : 0 \leq t \leq b\}.$$

$L^1(J, \mathbb{R})$ refers to the Banach space of measurable functions $y : J \longrightarrow \mathbb{R}$ which are Lebesgue integrable; it is normed by

$$|y|_1 = \int_0^b |y(s)| ds.$$

Let $(X, \|\cdot\|)$ be a separable Banach space, and denote:

$$\begin{array}{rcl} \mathcal{P}(X) & = & \{Y \subset X : Y \neq \emptyset\}, \\ \mathcal{P}_{cv}(X) & = & \{Y \in \mathcal{P}(X) : Y \text{convex}\}, \\ \mathcal{P}_{cl}(X) & = & \{Y \in \mathcal{P}(X) : Y \text{closed}\}, \\ \mathcal{P}_{b}(X) & = & \{Y \in \mathcal{P}(X) : Y \text{bounded}\}, \\ \mathcal{P}_{cp}(X) & = & \{Y \in \mathcal{P}(X) : Y \text{compact}\}, \\ \mathcal{P}_{cv,cp}(X) & = & \mathcal{P}_{cv}(X) \cap \mathcal{P}_{cp}(X). \end{array}$$

A multi-valued map $G : X \longrightarrow \mathcal{P}(X)$ has *convex (closed) values* if $G(x)$ is convex (closed) for all $x \in X$. We say that G is *bounded on bounded sets* if $G(B)$ is bounded in X for each bounded set B of X (i.e., $\sup_{x\in B}\{\sup\{\|y\| : y \in G(x)\}\} < \infty$). The map G is *upper semi-continuous (u.s.c.)* on X if for each $x_0 \in X$ the set $G(x_0)$ is a nonempty, closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood M of x_0 such that $G(M) \subseteq N$. Finally, we say that G is *completely continuous* if $G(B)$ is relatively compact for every bounded subset $B \subseteq X$.

If the multi-valued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e., $x_n \longrightarrow x_*,\ y_n \longrightarrow y_*,\ y_n \in G(x_n)$ imply $y_* \in G(x_*)$). We say that G has a *fixed point* if there exists $x \in X$ such that $x \in G(x)$.

The following two results are easily deduced from the limit properties.

Lemma 2.1. (see e.g. [6], Theorem 1.4.13) *If $G : X \longrightarrow \mathcal{P}_{cp}(X)$ is u.s.c., then for any $x_0 \in X$,*

$$\limsup_{x\to x_0} G(x) = G(x_0).$$

Lemma 2.2. (see e.g. [6], Lemma 1.1.9) *Let $(K_n)_{n\in\mathbb{N}} \subset K \subset X$ be a sequence of subsets where K is compact in the separable Banach space X. Then*

$$\overline{co}\,(\limsup_{n\to\infty} K_n) = \bigcap_{N>0} \overline{co}\,(\bigcup_{n\geq N} K_n),$$

where $\overline{co}\,A$ refers to the closure of the convex hull of A.

A multi-valued map $G : J \longrightarrow \mathcal{P}_{cp}(X)$ is said to be *measurable* if for each $x \in \mathbb{R}$ the function $Y : J \longrightarrow \mathbb{R}$ defined by

$$Y(t) = d(x, G(t)) = \inf\{\|x - z\| : z \in G(t)\}$$

is measurable.

Lemma 2.3. (see [25], Thm 19.7) *Let E be a separable metric space and G a multi-valued map with nonempty closed values. Then G has a measurable selection.*

Lemma 2.4. (see [58], Lemma 3.2) *Let $G : [0, b] \to \mathcal{P}(E)$ be a measurable multifunction and $u : [0, b] \to E$ a measurable function. Then for any measurable $v : [0, b] \to \mathbb{R}^+$ there exists a measurable selection g of G such that* *for a.e.* $t \in [0, b]$,

$$|u(t) - g(t)| \leq d(u(t), G(t)) + v(t).$$

Lemma 2.5. [50] *Let* $G : [0, b] \to \mathcal{P}_{cl}(\mathbb{R})$ *be a measurable multifunction and* $u : [0, b] \to \mathbb{R}$ *a measurable function. Assume that there exist* $p \in L^1(J, \mathbb{R})$ *such that* $G(t) \subseteq p(t)B(0, 1)$, *where* $B(0, 1)$ *denotes the closed ball in* $\mathbb{R}$. *Then there exists a measurable selection* g *of* G *such that for a.e.* $t \in [0, b]$,

$$|u(t) - g(t)| \leq d(u(t), G(t)).$$

Lemma 2.6. (Mazur's Lemma, [47], Theorem 21.4) *Let* E *be a normed space and* $\{x_k\}_{k\in\mathbb{N}} \subset E$ *be a sequence weakly converging to a limit* $x \in E$. *Then there exists a sequence of convex combinations* $y_m = \sum_{k=1}^{m} \alpha_{mk} x_k$ *with* $\alpha_{mk} > 0$ *for* $k = 1, 2, \ldots, m$ *and* $\sum_{k=1}^{m} \alpha_{mk} = 1$, *which converges strongly to* x.

Definition 2.7. The multivalued map $F : J \times X \longrightarrow \mathcal{P}(X)$ is L^1-Carathéodory if

(i) $t \longmapsto F(t, y)$ is measurable for each $y \in X$;
(ii) $y \longmapsto F(t, y)$ is upper semi-continuous for almost all $t \in J$;
(iii) For each $q > 0$, there exists $\phi_q \in L^1(J, \mathbb{R}_+)$ such that

$$\|F(t, y)\|_{\mathcal{P}} = \sup\{\|v\| : v \in F(t, y)\} \leq \phi_q(t)$$
$$\text{for all } \|y\| \leq q \text{ and for almost all } t \in J.$$

Let (X, d) be a metric space induced from the normed space $(X, |\cdot|)$. Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \longrightarrow \mathbb{R}_+ \cup \{\infty\}$ given by

$$H_d(A, B) = \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\right\},$$

where $d(A, b) = \inf_{a\in A} d(a, b)$, $d(a, B) = \inf_{b\in B} d(a, b)$. Then $(\mathcal{P}_{b,cl}(X), H_d)$ is a metric space and $(\mathcal{P}_{cl}(X), H_d)$ is a generalized metric space; see [35].

Definition 2.8. A multivalued operator $N : X \to \mathcal{P}_{cl}(X)$ is called

a) γ-Lipschitz if and only if there exists $\gamma > 0$ such that

$$H_d(N(x), N(y)) \leq \gamma d(x, y), \quad \text{for each } x,\ y \in X;$$

b) a contraction if and only if it is γ-Lipschitz with $\gamma < 1$.

Lemma 2.9. (Covitz-Nadler, [16]) *Let* (X, d) *be a complete metric space. If* $N : X \to \mathcal{P}_{cl}(X)$ *is a contraction, then* $FixN \neq \emptyset$.

For more details on multi-valued maps we refer to the books by Aubin *et al* [5,6], Deimling [17], Gorniewicz [25], Hu and Papageorgiou [32], Kisielewicz [35] and Tolstonogov [56].

3. Fractional Calculus

According to the Riemann-Liouville approach to fractional calculus the notation of fractional integral of order α ($\alpha > 0$) is a natural consequence of the well known formula (usually attributed to Cauchy), that reduces the calculation of the $n-$fold primitive of a function $f(t)$ to a single integral of convolution type. In our notation, the Cauchy formula reads

$$J^n f(t) := \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1} f(s)ds,\ t>0, \qquad n\in\mathbb{N}.$$

Definition 3.1. The fractional integral of order $\alpha > 0$ of a function $f \in L^1([a,b],\mathbb{R})$ is defined by

$$J^\alpha_{a^+} f(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds,$$

where Γ is the gamma function. When $a = 0$, we write $J^\alpha f(t) = f(t) * \phi_\alpha(t)$, where $\phi_\alpha(t) = \dfrac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t > 0$ and $\phi_\alpha(t) = 0$ for $t \leq 0$, and $\phi_\alpha \to \delta(t)$ as $\alpha \to 0$, where δ is the delta function and Γ is the Euler gamma function defined by

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt,\ \alpha > 0.$$

Also $J^0 = I$ (Identity operator), i.e. $J^0 f(t) = f(t)$. Furthermore, by $J^\alpha f(0^+)$ we mean the limit (if it exists) of $J^\alpha f(t)$ for $t \to 0^+$; this limit may be infinite.

After the notion of fractional integral, that of fractional derivative of order α ($\alpha > 0$) becomes a natural requirement and one is attempted to substitute α with $-\alpha$ in the above formulas. However, this generalization needs some care in order to guarantee the convergence of the integral and preserve the well known properties of the ordinary derivative of integer order. Denoting by D^n with $n \in \mathbb{N}$, the operator of the derivative of order n, we first note that

$$D^n J^n = I, \qquad J^n D^n \neq I, \qquad n\in\mathbb{N},$$

i.e. D^n is the left-inverse (and not the right-inverse) to the corresponding integral operator J^n. We can easily prove that

$$J^n D^n f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(a^+)\frac{(t-a)^k}{k!},\ t>0.$$

As consequence, we expect that D^α is defined as the left-inverse to J^α. For this purpose, introducing the positive integer n such that $n-1 < \alpha \leq n$, one defines the fractional derivative of order $\alpha > 0$:

Definition 3.2. For a function f given on interval $[a,b]$, the αth Riemann-Liouville fractional-order derivative of f is defined by

$$D^\alpha f(t) \quad = \quad \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{-\alpha+n-1} f(s)ds,$$

where $n = [\alpha] + 1$ and $[\alpha]$ is the integer part of α.

Also, we define $D^0 = J^0 = I$. Then we easily recognize that

$$D^\alpha J^\alpha = I, \qquad \alpha \geq 0, \tag{3.1}$$

and

$$D^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)} t^{\gamma-\alpha},\ \alpha > 0,\ \gamma - 1,\ t > 0. \tag{3.2}$$

Of course, the properties (3.1) and (3.2) are natural generalizations of those known when the order is a positive integer.

Note the remarkable fact that the fractional derivative $D^\alpha f$ is not zero for the constant function $f(t) = 1$ if $\alpha \notin \mathbb{N}$. In fact, (3.2) with $\gamma = 0$ teaches us that

$$D^\alpha 1 = \frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)},\ \alpha > 0, \quad t > 0. \tag{3.3}$$

It is clear that $D^\alpha 1 = 0$ for $\alpha \in \mathbb{N}$, due to the poles of the gamma function at the points $0, -1, -2, \ldots,$.

We now observe an alternative definition of fractional derivative, originally introduced by Caputo [12,13] in the late sixties and adopted by Caputo and Mainardi [14] in the framework of the theory of Linear Viscoelasticity (see a review in [40]).

Definition 3.3. Let $f \in AC^n([a, b])$. The Caputo fractional-order derivative of f is defined by

$$(D_*^\alpha f)(t) := \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds.$$

This definition is of course more restrictive than the Riemann-Liouville definition, in that it requires the absolute integrability of the derivative of order n. Whenever we use the operator D_*^α we (tacitly) assume that this condition is met. We easily recognize that, in general,

$$D^\alpha f(t) := D^m J^{m-\alpha} f(t) \neq J^{m-\alpha} D^m f(t) := D_*^\alpha f(t), \tag{3.4}$$

unless the function $f(t)$ along with its first $m-1$ derivatives vanishes at $t = a^+$. In fact, assuming that the passage of the $m-$derivative under the integral is legitimate, one recognizes that, for $m - 1 < \alpha < m$ and $t > 0$,

$$D^\alpha f(t) = D_*^\alpha f(t) + \sum_{k=0}^{m-1} \frac{(t-a)^{k-\alpha}}{\Gamma(k-\alpha+1)} f^{(k)}(a^+), \tag{3.5}$$

and therefore, recalling the fractional derivative of the power function (3.2),

$$D^\alpha \left(f(t) - \sum_{k=0}^{m-1} \frac{(t-a)^{k-\alpha}}{\Gamma(k-\alpha+1)} f^{(k)}(a^+) \right) = D_*^\alpha f(t). \tag{3.6}$$

The alternative definition, that is, Definition 3.3, for the fractional derivative thus incorporates the initial values of the function and of order lower than α. The subtraction of the Taylor polynomial of degree $m-1$ at $t = a^+$ from $f(t)$

means a sort of regularization of the fractional derivative. In particular, according to this definition, a relevant property is that the fractional derivative of a constant is sill zero, i.e.

$$D_*^\alpha 1 = 0, \quad \alpha > 0. \tag{3.7}$$

We now explore the most relevant differences between Definitions 3.2 and 3.3 for the two fractional derivatives. From the Riemann-Liouville fractional derivative, we have

$$D^\alpha (t-a)^{\alpha-j} = 0, \quad \text{for } j = 1, 2, \ldots, [\alpha]+1. \tag{3.8}$$

From (3.7) and (3.8) we thus recognize the following statements about functions, which for $t > 0$ admit the same fractional derivative of order α, with $n-1 < \alpha \le n$, $n \in \mathbb{N}$,

$$D^\alpha f(t) = D^\alpha g(t) \Leftrightarrow f(t) = g(t) + \sum_{j=1}^{m} c_j (t-a)^{\alpha-j}, \tag{3.9}$$

and

$$D_*^\alpha f(t) = D_*^\alpha g(t) \Leftrightarrow f(t) = g(t) + \sum_{j=1}^{m} c_j (t-a)^{n-j}. \tag{3.10}$$

In these formulas the coefficients c_j are arbitrary constants. For proving all our main results, we present the following auxiliary lemmas.

Lemma 3.4. [33] *Let $\alpha > 0$ and let $y \in L^\infty(a,b)$ or $C([a,b])$. Then*

$$(D_*^\alpha J^\alpha y)(t) = y(t).$$

Lemma 3.5. [33] *Let $\alpha > 0$ and $n = [\alpha]+1$. If $y \in AC^n[a,b]$ or $y \in C^n[a,b]$, then*

$$(J^\alpha D_*^\alpha y)(t) = y(t) - \sum_{k=0}^{n-1} \frac{y^{(k)}(a)}{k!}(t-a)^k.$$

Now we state the following generalization of Gronwall's lemma for singular kernels (whose proof can be found in Lemma 7.1.1 in [31]). This will be essential for the main result of Section 5.1.

Lemma 3.6. *Let $v : [0,b] \to [0,\infty)$ be a real function and $w(\cdot)$ is a nonnegative, locally integrable function on $[0,b]$, and suppose there are constants $a > 0$ and $0 < \alpha < 1$ such that*

$$v(t) \le w(t) + a\int_0^t \frac{v(s)}{(t-s)^\alpha} ds.$$

Then, there exists a constant $K = K(\alpha)$ such that

$$v(t) \le w(t) + Ka\int_0^t \frac{w(s)}{(t-s)^\alpha} ds,$$

for every $t \in [0,b]$.

For further reading and details on fractional calculus, we refer to the books and papers by Kilbas [33], Podlubny [52], Samko [54] and Caputo [12–14].

4. Filippov's Theorem

Let $J_k = (t_k, t_{k+1}]$, $k = 0, \ldots, m$, and let y_k be the restriction of a function y to J_k. In order to define mild solutions for problem (1.1)–(1.3), consider the space

$$\begin{aligned} PC &= \{y\colon\ J \to \mathbb{R} \mid y_k \in C(J_k, \mathbb{R}),\ k = 0, \ldots, m, \text{ and } y(t_k^-) \\ &\quad \text{and } y(t_k^+) \text{ exist and satisfy } y(t_k^-) = y(t_k) \text{ for } k = 1, \ldots, m\}. \end{aligned}$$

Endowed with the norm

$$\|y\|_{PC} = \max\{\|y_k\|_\infty :\ k = 0, \ldots, m\},$$

this is a Banach space.

Definition 4.1. A function $y \in PC$ is said to be a solution of (1.1)–(1.3) if there exists $v \in L^1(J, \mathbb{R})$ with $v(t) \in F(t, y(t))$ for a.e. $t \in J$ such that y satisfies the fractional differential equation $D_*^\alpha y(t) = v(t)$ a.e. on J, and the conditions (1.2)–(1.3).

Let $\overline{a} \in \mathbb{R}$, $g \in L^1(J, \mathbb{R})$ and let $x \in PC$ be a solution of the impulsive differential problem with fractional order,

$$\left\{ \begin{array}{rcll} D_*^\alpha x(t) &=& g(t), & \text{a.e. } t \in J\backslash\{t_1, \ldots, t_m\},\ \alpha \in (0, 1], \\ \Delta x_{t=t_k} &=& I_k(x(t_k^-)), & k = 1, \ldots, m, \\ x(0) &=& \overline{a}, & \end{array} \right. \tag{4.1}$$

where $\sup\{\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |g(s)| ds :\ t \in [0, b]\} < \infty$. We will need the following two assumptions:

($\mathcal{H}_1$). The function $F\colon\ J \times \mathbb{R} \to \mathcal{P}_{cl}(\mathbb{R})$ is such that
- (a) for all $y \in \mathbb{R}$, the map $t \mapsto F(t, y)$ is measurable,
- (b) the map $\gamma : t \mapsto d(g(t), F(t, x(t))$ is integrable.

($\mathcal{H}_2$). There exists a function $p \in L^1(J, \mathbb{R}^+)$ such that

$$H_d(F(t, z_1), F(t, z_2)) \leq p(t)|z_1 - z_2|, \quad \text{for all } z_1,\ z_2 \in \mathbb{R}.$$

Remark 4.2. From Assumptions ($\mathcal{H}_1(a)$) and ($\mathcal{H}_2$), it follows that the multifunction $t \mapsto F(t, x(t))$ is measurable and by Lemmas 1.4 and 1.5 from [22], we deduce that $\gamma(t) = d(g(t), F(t, x(t))$ is measurable (see also the Remark p. 400 in [6]).

Theorem 4.3. *Suppose that hypotheses* ($\mathcal{H}_1$)–($\mathcal{H}_2$) *are satisfied. If*

$$\|I^\alpha p\|_* = \frac{1}{\Gamma(\alpha)} \sup\left\{ \int_0^t (t-s)^{\alpha-1} p(s) ds :\ t \in [0, b] \right\} < 1,$$

then Problem (1.1)–(1.3) *has at least one solution y satisfying, for a.e. $t \in [0, b]$, the estimates*

$$|y(t) - x(t)| \quad \leq \quad \sum_{0 \leq t_k < t} \eta_k(t),$$

and

$$|D_*^\alpha y(t) - g(t)| \leq p(t) \sum_{0<t_k<t} H_k(t) + \sum_{0<t_k<t} \gamma_k(t), \text{ for } k = 1, \dots, m,$$

where

$$\eta_k(t) = \frac{\delta_k + \|\gamma_k\|_\infty \|I^\alpha p\|_*}{1 - \|I^\alpha p\|_*}, \quad t \in (t_k, t_{k+1}],$$

$$H_k(t) = \frac{\delta_k + \|\gamma_k\|_\infty}{1 - \|I^\alpha p\|_*},$$

$$\delta_k := |x(t_k) - y(t_k)| + |I_1(y(t_k)) - I_k(x(t_k))|, \quad \gamma_k = \gamma|_{J_k}, \ k = 1, \dots, m.$$

and

$$\|\gamma\|_\infty = \frac{1}{\Gamma(\alpha)} \sup \left\{ \int_{t_k}^{t_{k-1}} (t-s)^{\alpha-1} \gamma(s) ds \right\} < \infty, \ k = 0, 1, \dots, m.$$

Proof. We are going to study Problem (1.1)–(1.3) respectively in the intervals $[0, t_1]$, $(t_1, t_2]$, $\dots, (t_m, b]$. The proof will be given in three steps and then continued by induction.

Step 1. In this first step, we construct a sequence of functions $(y_n)_{n\in\mathbb{N}}$ which will be shown to converge to some solution of Problem (1.1)–(1.3) on the interval $[0, t_1]$, namely to

$$\left\{ \begin{array}{rcl} D_*^\alpha y(t) & \in & F(t, y(t)), \quad t \in J_0 = [0, t_1], \ \alpha \in (0, 1], \\ y(0) & = & a. \end{array} \right. \tag{4.2}$$

Let $f_0 = g$ on $[0, t_1]$ and $y_0(t) = x(t)$, $t \in [0, t_1]$, i.e.

$$y_0(t) = \overline{a} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_0(s) ds, \qquad t \in [0, t_1].$$

Then define the multi-valued map $U_1 \colon [0, t_1] \to \mathcal{P}(\mathbb{R})$ by $U_1(t) = F(t, y_0(t)) \cap (g(t) + \gamma(t)B(0,1))$. Since g and γ are measurable, Theorem III.4.1 in [15] tells us that the ball $(g(t) + \gamma(t)B(0,1))$ is measurable. Moreover $F(t, y_0(t))$ is measurable (see Remark 4.2). We claim that U_1 is nonempty. It is clear that

$$\begin{array}{rcl} d(0, F(t,0)) & \leq & d(0, g(t)) + d(g(t), F(t, y_0(t))) + H_d(F(t, y_0(t)), F(t, 0)) \\ & \leq & |g(t)| + \gamma(t) + p(t)|y_0(t)|, \ \text{a.e. } t \in [0, t_1]. \end{array}$$

Hence for all $w \in F(t, y_0(t))$ we have

$$\begin{array}{rcl} |w| & \leq & d(0, F(t,0)) + H_d(F(t,0), F(t, y_0(t))) \\ & \leq & |g(t)| + \gamma(t) + 2p(t)|y_0(t)| := M(t), \ \text{a.e.} t \in [0, t_1]. \end{array}$$

This implies that

$$F(t, y_0(t)) \subseteq M(t)B(0,1), \ t \in [0, t_1].$$

From Lemma 2.5, there exists a function u which is a measurable selection of $F(t, y_0(t))$ such that

$$|u(t) - g(t)| \leq d(g(t), F(t, y_0(t))) = \gamma(t).$$

Then $u \in U_1(t)$, proving our claim. We deduce that the intersection multivalued operator $U_1(t)$ is measurable (see [6,15,25]). By Lemma 2.3 (Kuratowski-Ryll-Nardzewski selection theorem), there exists a function $t \to f_1(t)$ which is a measurable selection for U_1. Consider

$$y_1(t) = a + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_1(s)ds, \qquad t \in [0, t_1].$$

For each $t \in [0, t_1]$, we have

$$|y_1(t) - y_0(t)| \le |a - \overline{a}| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f_0(s) - f_1(s)|\, ds. \tag{4.3}$$

Hence

$$|y_1(t) - y_0(t)| \le \delta + \|\gamma\|_\infty, \quad t \in [0, t_1] \text{ with } |a - \overline{a}| = \delta.$$

Now Lemma 1.4 in [22] tells us that $F(t, y_1(t))$ is measurable. The ball $(f_1(t) + p(t)|y_1(t) - y_0(t)|B(0,1))$ is also measurable by Theorem III.4.1 in [15]. The set $U_2(t) = F(t, y_1(t)) \cap (f_1(t) + p(t)|y_1(t) - y_0(t)|)B(0,1)$ is nonempty. Indeed, since f_1 is a measurable function, Lemma 2.5 yields a measurable selection u of $F(t, y_1(t))$ such that

$$|u(t) - f_1(t)| \le d(f_1(t), F(t, y_1(t))).$$

Then using $(\mathcal{H}_2)$, we get

$$\begin{aligned} |u(t) - f_1(t)| &\le d(f_1(t), F(t, y_1(t))) \\ &\le H_d(F(t, y_0(t)), F(t, y_1(t))) \\ &\le p(t)|y_0(t) - y_1(t)|, \end{aligned}$$

i.e. $u \in U_2(t)$, proving our claim. Now, since the intersection multi-valued operator U_2 defined above is measurable (see [6, 15, 25]), there exists a measurable selection $f_2(t) \in U_2(t)$. Hence

$$|f_1(t) - f_2(t)| \le p(t)|y_1(t) - y_0(t)|. \tag{4.4}$$

Define

$$y_2(t) = a + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_2(s)ds, \qquad t \in (0, t_1].$$

Using (4.3) and (4.4), a simple integration of the following estimates, valid for every $t \in [0, t_1]$,

$$\begin{aligned} |y_2(t) - y_1(t)| &\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f_2(s) - f_1(s)| ds \\ &\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p(s) \left(\delta + \|\gamma\|_\infty\right) ds \\ &\le \|I^\alpha p\|_* (\delta + \|\gamma\|_\infty), \ t \in [0, t_1]. \end{aligned}$$

Let $U_3(t) = F(t, y_2(t)) \cap (f_2(t) + p(t)|y_2(t) - y_1(t)|)B(0,1)$. Arguing as for U_2, we can prove that U_3 is a measurable multi-valued map with nonempty values; so there exists a measurable selection $f_3(t) \in U_3(t)$. This allows us to define

$$y_3(t) = a + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_3(s)ds, \qquad t \in [0, t_1].$$

For $t \in [0, t_1]$, we have

$$|y_3(t) - y_2(t)| \leq \int_0^t (t-s)^{\alpha-1} p(s)|y_2(s) - y_1(s)|\, ds.$$

Then

$$|y_3(s) - y_2(s)| \leq \|I^\alpha p\|_*^2 (\delta + \|\gamma\|_\infty), \ t \in [0, t_1].$$

Repeating the process for $n = 0, 1, 2, 3, \ldots$, we arrive at the following bound

$$|y_n(t) - y_{n-1}(t)| \leq \|I^\alpha p\|_*^{n-1} (\delta + \|\gamma\|_\infty), \quad t \in [0, t_1]. \tag{4.5}$$

By induction, suppose that (4.5) holds for some n and check (4.5) for $n+1$. Let $U_{n+1}(t) = F(t, y_n(t)) \cap (f_n + p(t)|y_n(t) - y_{n-1}(t)|B(0,1))$. Since U_{n+1} is a nonempty measurable set, there exists a measurable selection $f_{n+1}(t) \in U_{n+1}(t)$, which allows us to define for $n \in \mathbb{N}$

$$y_{n+1}(t) = a + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_{n+1}(s)ds, \ \ t \in [0, t_1]. \tag{4.6}$$

Therefore, for a.e. $t \in [0, t_1]$, we have

$$\begin{aligned} |y_{n+1}(t) - y_n(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f_{n+1}(s) - f_n(s)|\, ds \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p(s)|y_n(s) - y_{n-1}(s)|\, ds. \end{aligned}$$

Then

$$|y_{n+1}(t) - y_n(t)| \leq \|I^\alpha p\|_*^n (\delta + \|\gamma\|_\infty), \ t \in [0, t_1].$$

Hence

$$\|y_{n+1} - y_n\|_\infty \leq \|I^\alpha p\|_*^n (\delta + \|\gamma\|_\infty).$$

Consequently, (4.5) holds true for all $n \in \mathbb{N}$. We infer that $\{y_n\}$ is a Cauchy sequence in PC_1, converging uniformly to a limit function $y \in PC_1$, where

$$PC_1 = C([0, t_1], \mathbb{R}).$$

Moreover, from the definition of $\{U_n\}$, we have

$$|f_{n+1}(t) - f_n(t)| \leq p(t)|y_n(t) - y_{n-1}(t)|, \quad \text{a.e } t \in [0, t_1].$$

Hence, for almost every $t \in [0, t_1]$, $\{f_n(t)\}$ is also a Cauchy sequence in $\mathbb{R}$ and then converges almost everywhere to some measurable function $f(\cdot)$ in

$\mathbb{R}$. In addition, since $f_0 = g$, we have for a.e. $t \in [0, t_1]$

$$\begin{aligned} |f_n(t)| &\leq \sum_{i=1}^{n} p(t)|f_i(t) - f_{i-1}(t)| + |f_0(t)| \\ &\leq \sum_{i=2}^{n} p(t)|y_{i-1}(t) - y_{i-2}(t)| + |g(t)| \\ &\leq p(t)\sum_{i=1}^{\infty} |y_i(t) - y_{i-1}(t)| + \gamma(t) + |g(t)|. \end{aligned}$$

Hence

$$|f_n(t)| \leq H_0(t)p(t) + \gamma(t) + |g(t)|,$$

where

$$H_0(t) := \frac{(\delta + \|\gamma\|_\infty)}{1 - \|I^\alpha p\|_*}. \tag{4.7}$$

Then

$$\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f_n(s)|ds \leq \|I^\alpha g\|_* + H_0\|I^\alpha p\|_* < \infty.$$

Hence for every t fixed in $(0, t_1]$, we have that

$$|t - .|^{\alpha-1} f_n(.) \in L^1([0,t], \mathbb{R}),$$

and

$$|t - .|^{\alpha-1} f_n(.) \to |t - .|^{\alpha-1} f(.), \text{ a.e. on } [0,t].$$

Put

$$h_*(t) = a + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)ds \quad t \in [0, t_1].$$

Let $t = 0$, then

$$\lim_{t\to 0} y_n(t) = a = h_*(0).$$

If $t \in (0, t_1]$ we have

$$\begin{aligned} |y_n(t) - h_*(t)| &= \left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(f_n(s) - f(s))ds\right| \\ &\leq \frac{1}{\Gamma(\alpha)}\int_0^t |t-s|^{\alpha-1}|f_n(s) - f(s)|ds. \end{aligned}$$

By the Lebesgue dominated convergence theorem

$$|y_n(t) - h_*(t)| \to 0, \text{ as } n \to \infty.$$

Consequently,

$$y(t) = a + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)ds, \ t \in [0, t_1], \quad y(0) = a,$$

is a solution of the problem (1.1)–(1.2) with condition $y(0) = a$, $y \in S_{[0,t_1]}(a)$. Moreover, for a.e. $t \in (0, t_1]$, we have

$$\begin{aligned} |x(t) - y(t)| &= \Big| a + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)ds \\ &\quad -\bar{a} - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds \Big| \\ &\le \delta + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s) - f_0(s)|ds \\ &\le \delta + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s) - f_n(s)|ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}) |f_n(s) - f_0(s)|ds \\ &\le \delta + \frac{1}{\Gamma(\alpha)} \int_0^t |(t-s)^{\alpha-1}| |f(s) - f_n(s)|ds \\ &\quad + \frac{(\delta + \|\gamma\|_\infty)\|I^\alpha p\|_*}{1 - \|I^\alpha p\|_*}. \end{aligned}$$

Passing to the limit as $n \to \infty$, we get

$$|x(t) - y(t)| \le \eta_0(t), \quad \text{a.e. } t \in [0, t_1], \tag{4.8}$$

with

$$\eta_0(t) := \frac{(\delta + \|\gamma\|_\infty)\|I^\alpha p\|_*}{1 - \|I^\alpha p\|_*}.$$

Next, we give an estimate for $|D_*^\alpha y(t) - g(t)|$, for $t \in [0, t_1]$. We have

$$\begin{aligned} |D_*^\alpha y(t) - g(t)| &= |f(t) - f_0(t)| \\ &\le |f_n(t) - f_0(t)| + |f_n(t) - f(t)| \\ &\le p(t) \sum_{i=0}^{\infty} |y_{i+1}(t) - y_i(t)| + \gamma(t) + |f_n(t) - f(t)|. \end{aligned}$$

Arguing as in (4.7) and passing to the limit as $n \to +\infty$, we deduce that

$$|D_*^\alpha y(t) - g(t)| \le H_0(t)p(t) + \gamma(t), \quad t \in [0, t_1].$$

The obtained solution is denoted by $y_1 := y_{|[0,t_1]}$.

Step 2: Consider now Problem (1.1)–(1.3) on the second interval $(t_1, t_2]$, i.e.

$$\begin{cases} D_*^\alpha y(t) \in F(t, y(t)), & \text{a.e. } t \in (t_1, t_2], \\ y(t_1^+) = y_1(t_1) + I_1(y_1(t_1)). \end{cases} \tag{4.9}$$

Let $f_0 = g$ and set

$$y^0(t) = x(t_1) + I_1(x(t_1)) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} f_0(s)ds, \quad t \in (t_1, t_2].$$

Notice that (4.8) allows us to use Assumption $(\mathcal{H}_2)$, apply again Lemma 1.4 in [22] and argue as in Step 1 to prove that the multi-valued map $U_1 : [t_1, t_2] \to \mathcal{P}(\mathbb{R})$, defined by $U_1(t) = F(t, y^0(t)) \cap (g(t) + \gamma(t)B(0, 1))$, is measurable. Hence, there exists a function $t \mapsto f_1(t)$ which is a measurable selection for U_1. Define

$$y^1(t) = y_1(t_1) + I_1(y_1(t_1)) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} f_1(s)ds, \quad t \in (t_1, t_2].$$

Next define the measurable multi-valued map $U_2(t) = F(t, y^1(t)) \cap (f_1(t) + p(t)|y^1(t) - y^0(t)|B(0, 1))$. It has a measurable selection $f_2(t) \in U_2(t)$ by the Kuratowski-Ryll-Nardzewski selection theorem. Repeating the process of selection as in Step 1, we can define by induction a sequence of multi-valued maps $U_n(t) = F(t, y^{n-1}(t)) \cap (f_{n-1}(t) + p(t)|y^{n-1}(t) - y^{n-2}(t)|B(0, 1))$ where $\{f_n\} \in U_n$ and $(y^n)_{n\in\mathbb{N}}$ is as defined by

$$y^n(t) = y_1(t_1) + I_1(y_1(t_1)) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} f_n(s)ds, t \in (t_1, t_2],$$

and we can easily prove that

$$|y^{n+1}(t) - y^n(t)| \leq \|I^\alpha p\|_*^n (\delta + \|\gamma\|_\infty), \quad t \in (t_1, t_2].$$

Let

$$PC_2 = \{y\colon\ y \in C((t_1, t_2], \mathbb{R}) \text{ and } y(t_1^+) \text{ exists}\}.$$

As in Step 1, we can prove that the sequence $\{y^n\}$ converges to some $y \in PC_2$, a solution to Problem (4.9), such that, for a.e. $t \in (t_1, t_2]$, we have

$$\begin{aligned} |x(t) - y(t)| &\leq |x_1(t_1) - y_1(t_1)| + |I_1(x(t_1)) - I_1(y_1(t_1))| \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} |f(s) - g(s)|ds. \end{aligned}$$

Hence

$$|x(t) - y(t)| \leq \frac{(\delta + \|\gamma_1\|_\infty)\|I^\alpha p\|_*}{1 - \|I^\alpha p\|_*},$$

and

$$|D_*^\alpha y(t) - g(t)| := |f(t) - f_0(t)| \leq H_1(t)p(t) + \gamma(t),\ t \in (t_1, t_2].$$

Denote the restriction $y_{|(t_1,t_2]}$ by y_2.

Step 3: We continue this process until we arrive at the function $y_{m+1} := y\Big|_{(t_m,b]}$ as a solution of the problem

$$\begin{cases} D_*^\alpha y(t) \in F(t, y(t)), & \text{a.e. } t \in (t_m, b], \\ y(t_m^+) = y_{m-1}(t_m) + I_m(y_{m-1}(t_m)). \end{cases}$$

Then, for a.e. $t \in (t_m, b]$, the following estimates are easily derived:

$$\begin{aligned} |x(t) - y(t)| &\leq |y_m(t_m) - x(t_m)| + | + |I_m(x(t_m)) - I_m(y(t_m))| \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_m}^t (t-s)^{\alpha-1} |f(s) - g(s)|ds. \end{aligned}$$

Then

$$|x(t)-y(t)| \leq \frac{(\delta+\|\gamma\|_{\infty})\|I^{\alpha}p\|_{*}}{1-\|I^{\alpha}p\|_{*}}$$

and

$$|D_{*}^{\alpha}y(t)-g(t)| \leq H_m(t)p(t)+\gamma(t)).$$

Step 4: Summarizing, a solution y of Problem (1.1)–(1.3) can be defined as follows

$$y(t)=\begin{cases} y_1(t), & \text{if } t\in[0,t_1],\\ y_2(t), & \text{if } t\in(t_1,t_2],\\ \dots & \dots\\ y_{m+1}(t), & \text{if } t\in(t_m,b]. \end{cases}$$

From Steps 1 to 3, we have that, for a.e. $t\in[0,t_1]$,

$$|x(t)-y(t)|\leq \eta_0(t), \quad \text{and } |D_{*}^{\alpha}y(t)-g(t)|\leq H_0(t)p(t)+\gamma(t),$$

as well as the following estimates, valid for $t\in(t_1,b]$

$$|x(t)-y(t)| \quad \leq \quad \sum_{k=0}^{m}\eta_k(t).$$

Similarly

$$|D_{*}^{\alpha}y(t)-g(t)|\leq p(t)\sum_{0<t_k<t}H_k(t)+\sum_{0<t_k<t}\gamma_k(t),$$

where $\gamma_k:=\gamma_{|J_k}$. The proof of Theorem 4.3 is complete. □

4.1. Filippov's Theorem on the Half-Line

We may consider Filippov's Problem on the half-line as given by,

$$\begin{cases} D_{*}^{\alpha}y(t) & \in & F(t,y(t)), \quad \text{a.e. } t\in\widetilde{J}\backslash\{t_1,\dots\},\\ \Delta y_{t=t_k} & = & I_k(y(t_k^{-})), \quad k=1,\dots,\\ y(t) & = & a, \end{cases} \tag{4.10}$$

where $\widetilde{J}=[0,\infty)$, $0=t_0<t_1<\ldots<t_m<\ldots,\ \lim_{m\to\infty}t_m=+\infty$, $F\colon\ \widetilde{J}\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ is a multifunction, and $a\in\mathbb{R}$. Let x be the solution of Problem (4.1) but on the half-line. We will need the following assumptions:

($\widetilde{\mathcal{H}}_1$). The function $F\colon\ \widetilde{J}\times\mathbb{R}\to\mathcal{P}_{cl}(\mathbb{R})$ is such that
  (a) for all $y\in\mathbb{R}$, the map $t\mapsto F(t,y)$ is measurable,
  (b) the map $t\mapsto\ \gamma(t)=d(g(t),F(t,x(t))\in L^1([0,\infty),\mathbb{R}_+)$

($\widetilde{\mathcal{H}}_2$). There exists a function $p\in L^1([0,\infty),\mathbb{R}^+)$ such that

$$H_d(F(t,z_1),F(t,z_2))\leq p(t)|z_1-z_2|, \quad \text{for all } z_1,\, z_2\in\mathbb{R}.$$

($\widetilde{\mathcal{H}}_3$). For every $x\in\mathbb{R}$, we have

$$\sum_{k=1}^{\infty}|I_k(x)|<\infty.$$

Then we can extend Filippov's Theorem to the half-line.

Theorem 4.4. *Let* $\gamma_k := \gamma_{|_{J_k}}$ *and assume* $(\widetilde{\mathcal{H}}_1)$–$(\widetilde{\mathcal{H}}_3)$ *hold. If*

$$\sup\left\{\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}p(s)ds : t\in[0,\infty)\right\} < 1,$$

then, Problem (4.10) *has at least one solution* y *satisfying, for* $t\in[0,\infty)$, *the estimates*

$$|y(t)-x(t)| \quad\leq\quad \sum_{0<t_k<t}\eta_k(t),$$

and

$$|D_*^\alpha y(t)-g(t)| \leq p(t)\sum_{0<t_k<t}H_k(t)+\sum_{0<t_k<t}\gamma_k(t).$$

Proof. The solution will be sought in the space

$$\widetilde{PC} = \{y\colon\ [0,\infty)\to\mathbb{R},\ y_k\in C(J_k,\mathbb{R}),\ k=0,\ldots,\ \text{ such that } y(t_k^-) \text{ and } y(t_k^+) \text{ exist and satisfy } y(t_k^-)=y(t_k) \text{ for } k=1,\ldots\},$$

where y_k is the restriction of y to $J_k = (t_k,t_{k+1}],\ k\geq 0$. Theorem 4.3 yields estimates of y_k on each one of the bounded intervals $J_0=[0,t_1]$, and $J_k=(t_{k-1},t_k],\ k=2,\ldots$. Let y_0 be solution of Problem (1.1)–(1.3) on J_0.

Then, consider the problem,

$$\begin{cases} D_*^\alpha y(t) &\in& F(t,y(t)), & \text{a. e. } t\in(t_1,t_2],\\ y(t_1^+) &=& y_0(t_1)+I_1(y_0(t_1)). \end{cases}$$

From Theorem 4.3, this problem has a solution y_1. We continue this process taking into account that $y_m := y\Big|_{(t_m,t_{m+1}]}$ is a solution to the problem,

$$\begin{cases} D_*^\alpha y(t) &\in& F(t,y(t)), & \text{a. e. } t\in(t_m,t_{m+1}],\\ y(t_m^+) &=& y_{m-1}(t_m)+I_m(y_{m-1}(t_m^-)). \end{cases}$$

Then a solution y of Problem (4.10) may be rewritten as

$$y(t)=\begin{cases} y_1(t), & \text{if } t\in[0,t_1],\\ y_2(t), & \text{if } t\in(t_1,t_2],\\ \ldots & \ldots\\ y_m(t), & \text{if } t\in(t_m,t_{m+1}],\\ \ldots & \ldots \end{cases}$$

□

5. Existence results

5.1. Convex case

In a main consideration of the problem (1.1)–(1.3), a nonlinear alternative of Leray Schauder type is used to investigate the existence of solutions for first order impulsive fractional differential inclusions.

Theorem 5.1 (Convex case). *Assume the hypotheses:*

($\mathcal{B}_1$) *The function $F : J \times \mathbb{R} \to \mathcal{P}_{cp,cv}(\mathbb{R})$ is an Carathéodory.*

($\mathcal{B}_2$) *There exist $M_1, M_2 > 0$ such that*

$$\|F(t,z)\|_{\mathcal{P}} \leq M_1 + M_2|z| \quad \textit{for a.e. } t \in J \textit{ and each } z \in \mathbb{R}.$$

Then the set of solutions for Problem (1.1)–(1.3) *is nonempty and compact. Moreover the operator solution $S(\cdot) : \mathbb{R} \to \mathcal{P}(PC)$, defined by*

$$S(a) = \{y \in PC : y \textit{ solution of the problem with } y(0) = a\},$$

is u.s.c.

Proof. Transform the problem into a fixed point problem. Consider first the problem (1.1)–(1.3) on the interval $[0, t_1]$, that is, the problem

$$D_*^\alpha y(t) \in F(t, y(t)), \quad a.e. \ t \in [0, t_1], \ \alpha \in (0, 1] \tag{5.1}$$

$$y(0) = a. \tag{5.2}$$

It is clear that the solutions of the problem (5.1)–(5.2) are fixed points of the multivalued operator, $N_0 : C([0, t_1], \mathbb{R}) \to \mathcal{P}(C([0, t_1], \mathbb{R}))$ defined by

$$N_0(y) = \{h \in C([0, t_1], \mathbb{R}) : h(t) = a + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) ds, t \in [0, t_1] \ \}$$

where

$$g \in S_{F,y} = \Big\{g \in L^1([0, t_1], \mathbb{R}) : g(t) \in F(t, y(t)) \ \text{ for a.e. } \ t \in [0, t_1]\Big\}.$$

As in [11,28,49], we can show that N_0 is completely continuous, with compact and convex values.

Now we show that N_0 is *upper semi-continuous.* Since N_0 is completely continuous, it suffices to prove that N_0 has a closed graph. Let $y_n \longrightarrow y_*$, $h_n \in N_0(y_n)$ and $h_n \longrightarrow h_*$, $y_n \longrightarrow y_*$ as $n \to \infty$. We will prove that $h_* \in N_0(y_*)$. Now $h_n \in N_0(y_n)$ implies that there exists $g_n \in S_{F,y_n}$ such that for each $t \in J$,

$$h_n(t) = a + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g_n(s) ds.$$

We must prove that there exists $g_* \in S_{F,y_*}$ such that for each $t \in J$,

$$h_*(t) = a + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g_*(s) ds.$$

Since $g_n(\cdot) \in F(\cdot, y_n(\cdot))$, then

$$|g_n(t)| \leq M_1 + M_2|y_n(t)|, \quad t \in [0, t_1] \Rightarrow g_n(t) \in \overline{M}B(0, 1),$$

where

$$\|y_n\|_\infty < \overline{M}, \ \text{for all } n \in \mathbb{N}.$$

It is clear that $\overline{M}B(0, 1)$ is a compact set in $\mathbb{R}$, and then there exists a subsequence of $\{g_n(\cdot)\}$ converging to $g_*(\cdot)$.

It remains to prove that $g_*(t) \in F(t, y(t))$, for a.e. $t \in J$. Lemma 2.6 yields the existence of $\alpha_i^n \geq 0,\ i = n, \ldots, k(n)$ such that $\sum_{i=1}^{k(n)} \alpha_i^n = 1$ and the sequence of convex combinations $\bar{g}_n(\cdot) = \sum_{i=1}^{k(n)} \alpha_i^n g_i(\cdot)$ converges strongly to v in L^1. Since F takes convex values, using Lemma 2.2, we obtain that

$$\begin{aligned} g_*(t) &\in \bigcap_{n\geq 1} \overline{\{\bar{g}_n(t)\}}, \quad \text{a.e. } t \in J \\ &\subset \bigcap_{n\geq 1} \overline{co}\{g_k(t),\ k \geq n\} \\ &\subset \bigcap_{n\geq 1} \overline{co}\{\bigcup_{k\geq n} F(t, y_k(t))\} \\ &= \overline{co}(\limsup_{k\to\infty} F(t, y_k(t))). \end{aligned} \tag{5.3}$$

Since F is u.s.c. with compact values, then by Lemma 2.1, we have

$$\limsup_{n\to\infty} F(t, y_n(t)) = F(t, y(t), \quad \text{for a.e. } t \in J.$$

This with (5.10) imply that $g_*(t) \in \overline{co}\, F(t, y(t))$. Since $F(.,.)$ has closed, convex values, we deduce that $g_*(t) \in F(t, y(t))$, for a.e. $t \in J$. prove that Let $t \in (0, t_1]$, then

$$\begin{aligned} \frac{1}{\Gamma(\alpha}\int_0^t (t-s)^{\alpha-1}|g_n(s)|ds &\leq \frac{1}{\Gamma(\alpha}\int_0^t (t-s)^{\alpha-1}[M_1 + \overline{M}M_2]ds \\ &\leq \frac{t_1^\alpha(M_1 + \overline{M}M_2)}{\Gamma(\alpha+1)} \end{aligned}$$

and

$$\frac{1}{\Gamma(\alpha}\int_0^t (t-s)^{\alpha-1}|g_*(s)|ds \leq \frac{t_1^\alpha(M_1 + \overline{M}M_2)}{\Gamma(\alpha+1)}.$$

Set

$$h_*(t) = a + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g_*(s)ds,$$

where $g_* \in S_{F,y_*}$. As in Theorem 4.3 we can prove that

$$y_*(t) = a + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g_*(s)ds,\ t \in [0, t_1].$$

A priori bounds. *We now show there exists an open set $U \subseteq C([0, t_1], \mathbb{R})$ with $y \in \lambda N_0(y)$, for $\lambda \in (0, 1)$ and $y \in \partial U_0$.*

Let $y \in C([0, t_1], \mathbb{R})$ and $y \in \lambda N_0(y)$ for some $0 < \lambda < 1$. Thus there exists $g \in S_{F,y}$ such that, for each $t \in [0, b]$, we have

$$y(t) = \lambda\Big[a + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s)ds\Big], \tag{5.4}$$

and

$$|y(t)| \leq |a| + \frac{t_1^\alpha M_1}{\Gamma(\alpha+1)} + \frac{M_2}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}|y(s)|ds,\ t \in [0,t_1].$$

Lemma 3.6 implies

$$|y(t)| \leq R + K_0(\alpha)R \int_0^t (t-s)^{\alpha-1}ds$$

where

$$R = |a| + \frac{t_1^\alpha M_1}{\Gamma(\alpha+1)}.$$

Hence

$$\|y\|_\infty \leq R + \frac{RK_0(\alpha)t_1^\alpha}{\Gamma(\alpha+1)} := \widetilde{M}_0.$$

Set

$$U_0 = \{y \in C([0,t_1],\mathbb{R}) : \|y\|_\infty < \widetilde{M}_0 + 1\}.$$

$N : \overline{U} \to \mathcal{P}(C([0,t_1],\mathbb{R}))$ is continuous and completely continuous. From the choice of U_0, there is no $y \in \partial U_0$ such that $y \in \lambda N_0(y)$, for some $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of Leray-Schauder type [26], we deduce that N_0 has a fixed point y_0 in U_0, which is a solution to (5.1)–(5.2).

Now, let $y_2 := y\Big|_{(t_1,t_2]}$ be a solution to the problem

$$D_*^\alpha y(t) \in F(t,y(t)), \quad a.e.\ t \in (t_1,t_2], \tag{5.5}$$

$$y(t_1^+) = y_0(t_1) + I_1(y_0(t_1^-)). \tag{5.6}$$

Then y_1 is a fixed point of the multivalued operator $N_1 : PC_1 \to \mathcal{P}(PC_1)$ defined by, for $t \in [t_1,t_2]$,

$$N_1(y) := \{h \in PC_1 : h(t) = y_1(t_1) + I_1(y_1(t_1)) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1}g(s)ds\},$$

where

$$g \in S_{F,y} = \Big\{g \in L^1([t_1,t_2],\mathbb{R}) : g(t) \in F(t,y(t)) \ \text{ for a.e. } \ t \in [t_1,t_2]\Big\}.$$

Clearly, N_1 is completely continuous, u.s.c. with compact and convex valued.

We now show there exists an open set $U_1 \subseteq PC_1$ with $z \in \lambda N_1(y)$ for $\lambda \in (0,1)$ and $y \in \partial U_1$.

Let $y \in PC_1$ and $y \in \lambda N_1(y)$ for some $0 < \lambda < 1$.

$$y(t) = \lambda \left[y_0(t_1) + I_1(y_0(t_1)) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1}g(s)ds\right],$$

for some $\lambda \in (0,1)$. Then

$$\begin{aligned} |y(t)| &\leq |y_0(t_1)| + |I_1(y_0(t_1))| \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1}[M_1 + M_2|y(s)|]ds,\ t \in (t_1,t_2]. \end{aligned} \tag{5.7}$$

Then, for $t \in (t_1, t_2]$,

$$|y(t)| \leq \widetilde{M}_0 + \sup_{q \in B(0,\widetilde{M}_0)} |I_1(q)| + \frac{(t_2 - t_1)^\alpha M_1}{\Gamma(\alpha+1)} + \frac{M_2}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} |y(s)| ds.$$

Lemma 3.6 implies

$$|y(t)| \leq R_1 + K_1(\alpha) R \int_0^t (t-s)^{\alpha-1} ds$$

where

$$R_1 = \widetilde{M}_0 + \sup_{q \in B(0,\widetilde{M}_0)} |I_1(q)| + \frac{(t_2 - t_1)^\alpha M_1}{\Gamma(\alpha+1)}.$$

Hence

$$\|y\|_\infty \leq R_1 + \frac{R K_1(\alpha)(t_2 - t_1)^\alpha}{\Gamma(\alpha+1)} := \widetilde{M}_1.$$

Set

$$U_1 = \{y \in PC_1 : \|y\|_{PC_1} < \widetilde{M}_1 + 1\}.$$

$N_1 : \overline{U} \to \mathcal{P}(PC_1)$ is continuous and completely continuous. From the choice of U_1, there is no $y \in \partial U_1$ such that $y \in \lambda N_1(y)$, for some $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of Leray-Schauder type [26], we deduce that N_1 has a fixed point y_1 in U_1, which is solution to (5.5)–(5.6). We continue this process and taking into account that $y_m := y\Big|_{(t_m,b]}$ is a solution to the problem

$$D_*^\alpha y(t) \in F(t, y(t)), \quad a.e.\ t \in (t_m, b],\ \alpha \in (0,1], \tag{5.8}$$

$$y(t_m^+) = y_{m-1}(t_1) + I_m(y_{m-1}(t_m^-)). \tag{5.9}$$

A solution y of the problem (1.1)–(1.3) is then defined by

$$y(t) = \begin{cases} y_0(t), & \text{if } t \in [0, t_1], \\ y_1(t), & \text{if } t \in (t_1, t_2], \\ \vdots & \\ y_m(t), & \text{if } t \in (t_m, b]. \end{cases}$$

Now, we will show that $S(\cdot)$ is u.s.c. by proving that the graph of $S(\cdot)$,

$$\Gamma(a) := \{(a, y) \mid y \in S(a)\},$$

is closed. Let $(a_n, y_n) \in \Gamma$, i.e., $y_n \in S(a_n)$, and let $(a_n, y_n) \to (a, y)$ as $n \to \infty$. Since $y_n \in S(a_n)$, there exists $v_n \in L^1(J, \mathbb{R})$ such that

$$y_n(t)=\begin{cases} a_n+\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1}v_n(s)ds, & \text{if } t\in[0,t_1],\\ y_n(t_1)+I_1(y_n(t_1)) \\ +\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{t_1}^t (t-s)^{\alpha-1}v_n(s)ds, & \text{if } t\in(t_1,t_2],\\ \vdots \\ y_n(t_m)+I_m(y_n(t_m)) \\ +\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{t_1}^t (t-s)^{\alpha-1}v_n(s)ds, & \text{if } t\in(t_m,b]. \end{cases}$$

Since (a_n, y_n) converges to (a, y), there exists $M > 0$ such that

$$|a_n| \leq M \quad \text{for} \quad \text{all } n \in \mathbb{N}.$$

By using $(\mathcal{B}_2)$, we can easily prove that there exist $\overline{M}_* > 0$ such that

$$\|y_n\|_{PC} \leq \overline{M}_* \quad \text{for all} \quad n \in \mathbb{N}.$$

From the definition of y_n, we have $D^\alpha_* y_n(t) = v_n(t)$ a.e. $t \in J$, and so

$$|v_n(t)| \leq M_1 + M_2\overline{M}_*, \ t \in J.$$

Thus, $v_n(t) \in (M_1 + M_2\overline{M}_*)\overline{B}(0,1) := \chi(t)$ a.e. $t \in J$. It is clear that $\chi : J \to \mathcal{P}_{cp,cv}(\mathbb{R})$ is a multivalued map that is integrably bounded. Since $\{v_n(\cdot) : n \geq 1\} \in \chi(\cdot)$, we may pass to a subsequence if necessary to obtain that v_n converges to v in L^1.

It remains to prove that $v(t) \in F(t, y(t))$, for a.e. $t \in J$. Lemma 2.6 yields the existence of $\alpha_i^n \geq 0$, $i = n, \ldots, k(n)$ such that $\sum_{i=1}^{k(n)} \alpha_i^n = 1$ and the sequence of convex combinaisons $g_n(\cdot) = \sum_{i=1}^{k(n)} \alpha_i^n v_i(\cdot)$ converges strongly to v in L^1. Since F takes convex values, using Lemma 2.2, we obtain that

$$\begin{aligned} v(t) &\in \bigcap_{n\geq 1} \overline{\{g_n(t)\}}, \quad \text{a.e. } t \in J \\ &\subset \bigcap_{n\geq 1} \overline{co}\{v_k(t),\ k \geq n\} \\ &\subset \bigcap_{n\geq 1} \overline{co}\{\bigcup_{k\geq n} F(t, y_k(t))\} \\ &= \overline{co}(\limsup_{k\to\infty} F(t, y_k(t))). \end{aligned} \tag{5.10}$$

Since F is u.s.c. with compact values, then by Lemma 2.1, we have

$$\limsup_{n\to\infty} F(t, y_n(t)) = F(t, y(t), \quad \text{for a.e. } t \in J.$$

This with (5.10) imply that $v(t) \in \overline{co}\, F(t, y(t))$. Since $F(.,.)$ has closed, convex values, we deduce that $v(t) \in F(t, y(t))$, for a.e. $t \in J$,

$$z(t) = \begin{cases} a + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s)ds, & \text{if } t \in [0, t_1], \\ y(t_1) + I_1(y(t_1)) \\ + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} v(s)ds, & \text{if } t \in (t_1, t_2], \\ \vdots \\ y(t_m) + I_m(y(t_m)) \\ + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} v(s)ds, & \text{if } t \in (t_m, b]. \end{cases}$$

Since the functions $I_k,\ k = 1, \dots, m$, are continuous, we obtain the estimates

$$|y_n(t) - z(t)| \le |a_n - a| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |v_n(s) - v(s)|ds,\ t \in [0, t_1].$$

Let $t \in (t_1, t_2]$. Then we have

$$\begin{aligned} |y_n(t) - z(t)| &\le |y_n(t_1) - y(t_1)| + |I_1(y_n(t_1)) - I_1(y(t_1))| \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} |v_n(s) - v(s)|ds \\ &\le \|y_n - y\|_{PC} + |I_1(y_n(t_1)) - I_1(y(t_1))| \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} |v_n(s) - v(s)|ds. \end{aligned}$$

We continue this process taking into account that

$$y_n(t) = y_n(t_m) + I_m(y_n(t_m)) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} v(s)ds,\ \ t \in (t_m, b].$$

Thus

$$\begin{aligned} |y_n(t) - z(t)| &\le |y_n(t_m) - y(t_m)| + |I_m(y_n(t_{m-1})) - I_m(y(t_{m-1}))| \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_m}^t (t-s)^{\alpha-1} |v_n(s) - v(s)|ds,\ t \in (t_m, b]. \end{aligned}$$

Hence

$$\begin{aligned} |y_n(t) - z(t)| &\le \|y_n - y\|_{PC} + |I_m(y_n(t_{m-1})) - I_m(y(t_{m-1}))| \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_m}^t (t-s)^{\alpha-1} |v_n(s) - v(s)|ds. \end{aligned}$$

The right-hand side of the above expression tends to 0, as $n \to +\infty$. Hence,

$$y(t) = \begin{cases} a + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} v(s)ds, & \text{if } t \in [0, t_1], \\ y(t_1) + I_1(y(t_1)) \\ + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{t_1}^t (t-s)^{\alpha-1} v(s)ds, & \text{if } t \in (t_1, t_2], \\ \vdots \\ y(t_m) + I_m(y(t_m)) \\ + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{t_1}^t (t-s)^{\alpha-1} v(s)ds, & \text{if } t \in (t_m, b]. \end{cases}$$

Thus, $y \in S(a)$. Now, we show that $S(\cdot)$ maps bounded sets into relatively compact sets of PC. Let B be a bounded set in $\mathbb{R}$ and let $\{y_n\} \subset S(B)$. Then there exist $\{a_n\}_{n\in\mathbb{N}} \subset B$ such that

$$y_n(t) = \begin{cases} a_n + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} v_n(s)ds, & \text{if } t \in [0, t_1], \\ y_n(t_1) + I_1(y_n(t_1)) \\ + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{t_1}^t (t-s)^{\alpha-1} v_n(s)ds, & \text{if } t \in (t_1, t_2], \\ \vdots \\ y_n(t_m) + I_m(y_n(t_m)) \\ + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{t_1}^t (t-s)^{\alpha-1} v_n(s)ds, & \text{if } t \in (t_m, b], \end{cases}$$

where $v_n \in S_{F,y_n},\ n \in \mathbb{N}$. Since $\{a_n\}$ is a bounded sequence, there exists a subsequence of $\{a_n\}$ converging to a, and so from $(\mathcal{B}_2)$, there exist $M_* > 0$ such that

$$\|y_n\|_{PC} \leq M_*,\ n \in \mathbb{N}.$$

As in [2, 11, 49], we can show that $\{y_n : n \in \mathbb{N}\}$ is equicontinuous in PC. As a consequence of the Arzelá-Ascoli Theorem, we conclude that there exists a subsequence of $\{y_n\}$ converging to y in PC. By a similar argument to the one above, we can prove that

$$y(t) = \begin{cases} a + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} v(s)ds, & \text{if } t \in [0, t_1], \\ y(t_1) + I_1(y(t_1)) \\ + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{t_1}^t (t-s)^{\alpha-1} v(s)ds, & \text{if } t \in (t_1, t_2], \\ \vdots \\ y(t_m) + I_m(y(t_m)) \\ + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{t_1}^t (t-s)^{\alpha-1} v(s)ds, & \text{if } t \in (t_m, b]. \end{cases}$$

where $v \in S_{F,y}$. Thus, $y \in S(a)$, and this implies that $S(\cdot)$ is u.s.c. Also for every $a \in \mathbb{R}$ we have $S(a) \in \mathcal{P}_{cp}(\mathbb{R})$. □

5.2. Nonconvex case

In this section, we present a second result for the problem (1.1)–(1.3) with a non-convex valued right-hand side. We will make use of some new conditions.

(A1) $F : [0, b] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathcal{P}_{cp}(\mathbb{R})$; $t \longmapsto F(t, x)$ is measurable for each $x \in \mathbb{R}$.

(A2) There exists a function $p \in L^1([0, b], \mathbb{R}^+)$ such that, for *a.e.* $t \in [0, b]$ and all $x, y \in \mathbb{R}$,

$$H_d(F(t, x), F(t, y) \leq p(t)|x - y|,$$

and

$$H_d(0, F(t, 0)) \leq p(t) \;\text{ for a.e. }\; t \in [0, b].$$

Theorem 5.2. *Suppose that hypotheses* (A1) *and* (A2) *are satisfied. If*

$$\frac{1}{\Gamma(\alpha)} \sup \left\{ \int_0^t (t-s)^{\alpha-1} p(s)ds : \; t \in [0, b] \right\} < 1,$$

then the IVP (1.1)–(1.3) *has at least one solution.*

Proof. We are going to study Problem (1.1)–(1.3) in the respective intervals $[0, t_1]$, $(t_1,\ t_2]$, $\ldots, (t_m, b]$. The proof will be given in three steps and then continued by induction.

Step 1. Consider the problem on the interval $[0, t_1]$, that is,

$$\begin{cases} D_*^\alpha y(t) & \in & F(t, y(t)), & \text{a.e. } t \in [0, t_1], \\ y(0) & = & a. \end{cases} \tag{5.11}$$

It is clear that all solutions of Problem (5.11) are fixed points of the multi-valued operator $N_0 : PC_1 \to \mathcal{P}(PC_1)$ defined by

$$N_0(y) := \left\{ h \in PC_1 : h(t) = a + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)ds, t \in [0, t_1] \right\}$$

where

$$g \in S_{F,y} = \Big\{ g \in L^1([0,t_1],\mathbb{R}) : g(t) \in F(t,y(t)) \text{ for a.e. } t \in (0,t_1] \Big\}$$

and $PC_1 = C([0,t_1],\mathbb{R})$. To show that N_0 satisfies the assumptions of Lemma 2.9, the proof will be given in two claims.

Claim 1. $N_0(y) \in \mathcal{P}_{cl}(PC_1)$, let $y \in PC_1$: Indeed, let $\{y_n\} \in N_0(y)$ be such that $y_n \to \tilde{y}$ in PC_1, as $n \to +\infty$. Then $\tilde{y} \in PC_1$ and there exists a sequence $g_n \in S_{F,y}$ such that

$$y_n(t) = a + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g_n(s)ds, \quad t \in [0,t_1].$$

Then $\{g_n\}$ is integrably bounded. Since $F(\cdot,\cdot)$ has compact values, let $w(\cdot) \in F(\cdot,0)$ be such that $|w(t)| = d(0,F(t,0))$. From $(A1)$ and $(A2)$, we infer that for a.e. $t \in [0,t_1]$,

$$\begin{array}{rcl} |g_n(t)| & \leq & d(0,F(t,0)) + H_d(F(t,0),F(t,y(t)) \\ & \leq & p(t)\|y\|_{PC_1} + d(0,F(t,0)) := M_*(t), \ \forall\, n \in \mathbb{N}. \end{array}$$

that is,

$$g_n(t) \in M_*(t)B(0,1), \text{ a.e.} t \in [0,t_1].$$

Since $B(0,1)$ is compact in $\mathbb{R}$, there exists a subsequence still denoted $\{g_n\}$ which converges to g. Then the Lebesgue dominated convergence theorem implies that, as $n \to \infty$,

$$\|g_n - g\|_{L^1} \to 0 \text{ as and } n \to \infty.$$

As in Theorem 4.3, we can prove that

$$\tilde{y}(t) = a + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)ds, \ t \in [0,t_1],$$

proving that $\tilde{y} \in N(y)$.

Claim 2. There exists $\gamma < 1$ such that $H_d(N(y),N(\overline{y})) \leq \gamma \|y-\overline{y}\|_{PC_1}$ for each $y,\overline{y} \in PC_1$ where the norm $\|y-\overline{y}\|_{PC_1}$ will be chosen conveniently: Indeed, let $y,\overline{y} \in PC([0,t_1])$ and $h_1 \in N_0(y)$. Then there exists $g_1(t) \in F(t,y(t))$ such that for each $t \in [0,t_1]$

$$h_1(t) = a + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g_1(s)ds.$$

Since, for each $t \in [0,t_1]$,

$$H_d(F(t,y_t),F(t,\overline{y}(t))) \leq p(t)|y(t)-\overline{y}(t)|,$$

then there exists some $w(t) \in F(t,\overline{y}(t))$ such that

$$|g_1(t) - w(t)| \leq p(t)|y(t)-\overline{y}(t)|, \quad t \in [0,t_1].$$

Consider the multi-map $U_1 : [0,t_1] \to \mathcal{P}(\mathbb{R})$ defined by

$$U_1(t) = \{w \in \mathbb{R} : |g_1(t) - w| \leq p(t)|y(t)-\overline{y}(t)|\}.$$

As in the proof of Theorem 4.3, we can show that the multi-valued operator $V_1(t) = U_1(t) \cap F(t,\overline{y}(t))$ is measurable and takes nonempty values. Then there exists a function $g_2(t)$, which is a measurable selection for V_1. Thus, $g_2(t) \in \overline{co}\, F(t, \overline{y}(t))$ and

$$|g_1(t) - g_2(t)| \leq p(t)|y - \overline{y}|, \quad \text{for a.e. } \ t \in [0, t_1].$$

For each $t \in [0, t_1]$, let

$$h_2(t) = a + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g_2(s)ds.$$

Therefore, for each $t \in [0, t_1]$, we have

$$\begin{aligned} |h_1(t) - h_2(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |g_1(s) - g_2(s)|\, ds \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p(s)|y(s) - \bar{y}(s)|ds \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha} p(s)ds \|y - \bar{y}\|_{PC_1}. \end{aligned}$$

Hence,

$$\|h_1 - h_2\|_{PC_1} \leq p_0 \|y - \bar{y}\|_{PC_1},$$

where

$$p_0 = \sup\left\{ \int_0^t (t-s)^{\alpha-1} p(s)ds : t \in [0, t_1] \right\} < 1.$$

By an analogous relation, obtained by interchanging the roles of y and $\bar{y}$, we find that

$$H_d(N_0(y), N_0(\bar{y})) \leq p_0 \|y - \bar{y}\|_{PC_1}.$$

Then N_0 is a contraction and hence, by Lemma 2.9, N_0 has a fixed point y_0, which is solution to Problem (5.11).

Step 2. Let $y_2 := y\Big|_{(t_1, t_2]}$ be a possible solution to the problem

$$\begin{cases} D_*^\alpha y(t) \in F(t, y(t)), & t \in (t_1, t_2], \\ y(t_1^+) = y_0(t_1) + I_1(y_0(t_1^-)), \end{cases} \tag{5.12}$$

Then y_2 is a fixed point of the multivalued operator $N_1 : PC_2 \to \mathcal{P}(PC_2)$ defined by, for $t \in (t_1, t_2]$,

$$N_1(y) := \left\{ h \in PC_2 : h(t) = \left\{ \begin{array}{l} y_0(t_1) + I_1(y_0(t_1)) \\ + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{t_1}^t (t-s)^{\alpha-1} g(s)ds, \end{array} \right. \right.$$

where

$$g \in S_{F,y} = \{ g \in L^1([t_1, t_2], \mathbb{R}) : \ g(t) \in \ F(t, y(t)) \ \text{ for a.e. } \ t \in [t_1, t_2] \}.$$

Again, we show that N_1 satisfies the assumptions of Lemma 2.9. Clearly, $N_1(y) \in \mathcal{P}_{cl}(PC_2)$ for each $y \in PC_2$. It remains to show that there exists $0 < \gamma < 1$ such that

$$H_d(N_1(y), N_1(\overline{y})) \leq \gamma \|y - \overline{y}\|_{PC_2},$$

for each $y, \overline{y} \in PC_2$. For this purpose, let $y, \overline{y} \in PC_2$ and $h_1 \in N_1(y)$. Then there exists $g_1(t) \in F(t, y(t))$ such that, for each $t \in [0, t_2]$,

$$h_1(t) = y_0(t_1) + I_1(y_0(t_1)) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} (t-s)^{\alpha-1} g_1(s) ds.$$

Since from $(A2)$

$$H_d(F(t, y(t)), F(t, \overline{y}_t)) \leq p(t)|y(t) - \overline{y}(t)|, \ t \in [t_1, t_2],$$

we deduce that there is a $w(\cdot) \in F(\cdot, \overline{y}(\cdot))$ such that

$$|g_1(t) - w(t)| \leq p(t)|y(t) - \overline{y}(t)|, \quad t \in [t_1, t_2].$$

Consider the multi-valued map $U_2 : [t_1, t_2] \to \mathcal{P}(\mathbb{R})$ defined by

$$U_2(t) = \{w \in \mathbb{R} : \ |g_1(t) - w| \leq p(t)|y(t) - \overline{y}(t)|\}.$$

Since the multivalued operator $V_2(t) = U_2(t) \cap F(t, \overline{y}(t))$ is measurable with nonempty values, there exists $g_2(t)$ which is a measurable selection for V_2. Then $g_2(t) \in F(t, \overline{y}(t))$ and

$$|g_1(t) - g_2(t)| \leq p(t)|y(t) - \overline{y}(t)|, \quad \text{for a.e. } \ t \in [t_1, t_2].$$

For a.e. $t \in [t_1, t_2]$, define

$$h_2(t) = y_0(t_1) + I_1(y_0(t_1)) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g_2(s) ds.$$

Then for $t \in (t_1, t_2]$ we have the estimates

$$\begin{aligned} |h_1(t) - h_2(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} (t-s)^{\alpha-1} |g_1(s) - g_2(s)| \, ds \\ &\leq \frac{t_2^{\alpha}}{\Gamma(\alpha)} \int_{t_1}^{t} p(s)|y(s) - \overline{y}(s)| ds \\ &\leq \frac{1}{\Gamma(\alpha)} \sup \left\{ \int_{t_1}^{t} (t-s)^{\alpha-1} p(s) ds : \ t \in [t_1, t_2] \right\} \|y - \overline{y}\|_{PC_2}. \end{aligned}$$

By an analogous relation, obtained by interchanging the roles of y and $\overline{y}$, we obtain

$$H_d(N_1(y), N_1(\overline{y})) \leq p_1 \|y - \overline{y}\|_{PC_2},$$

where

$$p_1 := \frac{1}{\Gamma(\alpha)} \sup \left\{ \int_{t_1}^{t} (t-s)^{\alpha-1} p(s) ds : \ t \in [t_1, t_2] \right\}.$$

Therefore N_1 is a contraction and thus, by Lemma 2.9, N_1 has a fixed point y_1 solution of Problem (5.12).

Step 3. We continue this process taking into account that $y_m := y\Big|_{[t_m, b]}$ is a solution of the problem,

$$\begin{cases} D_*^{\alpha} y(t) \in F(t, y(t)), & t \in (t_m, b], \\ y(t_m^+) = y_{m-1}(t_m) + I_m(y_{m-1}(t_m^-)). \end{cases} \tag{5.13}$$

Then a solution y of Problem (1.1)–(1.3) may be defined by

$$y(t) = \begin{cases} y_0(t), & \text{if } t \in [0, t_1], \\ y_2(t), & \text{if } t \in (t_1, t_2], \\ \dots & \dots \\ y_m(t), & \text{if } t \in (t_m, b]. \end{cases}$$

□

Lemma 5.3. *Assume that the conditions of Theorem* 5.2 *are satisfied and* $F : [0, b] \times \mathbb{R} \to \mathcal{P}_{cv,cp}(\mathbb{R})$. *Then the solution set of the problem* (1.1)–(1.3) *is compact.*

Proof. Using the fact $F(\cdot, \cdot) \in \mathcal{P}_{cv,cp}(\mathbb{R})$ and Mazur's lemma, and by Ascoli's theorem, we can prove that the solutions set of the problem (1.1)–(1.3) is compact. □

6. Concluding remarks

In this paper, we investigated Problem (1.1)–(1.3) under various assumptions on the multi-valued right hand-side nonlinearity, and we obtained a number of new results regarding existence of solutions. We first proved a Filippov's result for impulsive differential inclusions with fractional order. The main assumptions on the nonlinearity are the Carathéodory and the Lipschitz conditions with respect to the Hausdorf distance in generalized metric spaces. In the case where $\alpha \in (0, 1)$ and the convex case of the problem (1.1), (1.3) or the problem (1.1), (1.2), (1.3) some authors assume the growth conditions along with Lipchitz conditions, and there are some errors in the proofs of the u.s.c of the operator solution of the problem; for example, in the paper [[11], Theorem 3.4 step 4], it is assumed that $|f_n(t) - f(t)| \leq d(f_n(t), F(t, y_t))$ with $f \in F(t, y)$, but that result is not correct. But the correct proof of Step 4 follows from the integral operator of the problem having compact values, and then by the Lipchitz conditions, the operator integral is H_d- u.s.c, or the same technique of Theorem 5.1 of this paper can be used.

This paper is a generalization of the papers [11, 28] and as well as of other works in the literature.

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