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Basicity of a System of Exponents with a Piecewise Linear Phase in Variable Spaces

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Abstract. A system of exponents with a piecewise linear phase is considered in the paper. The criteria of basicity, completeness and minimality of this system in Lebesgue space of functions with variable summability exponent are established.

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1. Introduction

Consider the following system of exponents

$$\left\{e^{i(nt+\lambda_n(t))}\right\}_{n\in Z}, \tag{1.1}$$

where $\lambda_n(t) \equiv -sign\, n\,[\alpha t + \beta sign\, t]$, $t \in [-\pi, \pi]$, $\alpha, \beta \in C$ are complex parameters. We'll study basicity of this system in Lebesgue space of functions with variable summability exponent $p(t)$, denoted as L_{p_t}. Apparently, Paley - Wiener [17], N.Levinson [11] first paid attention to studying basis properties of the system of the form (1.1) in classic Lebesgue spaces (i.e. for $p(t) \equiv const$). In L_p, $1 \le p \le +\infty$, ($L_\infty \equiv C[-\pi, \pi]$), the basis properties of the system (1.1) were completely studied in [6;12;13] for $\beta = 0$ and in [1; 2] in general case.

The present paper studies basis properties of the system (1.1) in the spaces $L_{p_t} \equiv L_{p_t}(-\pi, \pi)$. In connection with consideration of some specific problems of mechanics and mathematical physics (see for instance [10;18-23]), there is a great interest to studying these or other problems of the spaces L_{p_t} and $W^k_{p_t}$. Study of bounded action of singular integral in the spaces allows to consider basis properties of systems in these spaces L_{p_t} related to Dirichlet or Hilbert type kernels.

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It should be noted that these problems have been studied well (see for instance [7;9;14]).

2. Necessary notion and facts

Let $p : [-\pi, \pi] \to [1, +\infty)$ be a Lebesgue measurable function. By $\mathcal{L}_0$ we denote a class of all functions measurable on $[-\pi, \pi]$ (with respect to Lebesgue measure). Accept the denotation

$$I_p(f) \equiv \int_{-\pi}^{\pi} |f(t)|^{p(t)}\, dt.$$

Assume $\mathcal{L} \equiv \{f \in \mathcal{L}_0 : I_p(f) < +\infty\}$. Let

$$p^+ = \sup \underset{[-\pi,\pi]}{vrai}\, p(t)\,;\, p^- = \inf \underset{[-\pi,\pi]}{vrai}\, p(t)\,.$$

For $p^+ < +\infty$, with respect to ordinary linear operations, $\mathcal{L}$ turns into a linear space. If we define the norm $\|\cdot\|_{p_t}$ as:

$$\|f\|_{p_t} \equiv \inf\left\{\lambda > 0 : I_p\left(\frac{f}{\lambda}\right) \le 1\right\},$$

then $\mathcal{L}$ is a Banach space (see for instance [15]) and we denote it by L_{p_t}. Everywhere $q(t)$ denotes the function $\frac{1}{p(t)} + \frac{1}{q(t)} \equiv 1$ conjugated to $p(t)$. We'll need the following class of functions:

$$H_\pi^{\ln} \equiv \Big\{p : p(-\pi) = p(\pi), \exists c > 0;\ \forall t_1, t_2 \in [-\pi, \pi]\,,$$
$$|t_1 - t_2| \le \frac{1}{2} \Rightarrow |p(t_1) - p(t_2)| \le \frac{c}{-\ln|t_1 - t_2|}\Big\}.$$

Basicity of the classic system of exponents $\left\{e^{int}\right\}_{n\in Z}$ (Z is a set of integers) in the spaces $L_{p_t}(-\pi, \pi)$ was studied in [16] and the necessity of the condition $p \in H_\pi^{\ln}$ for basicity was indicated.

It holds Hölder's generalized inequality:

$$\int_{-\pi}^{\pi} |f(t)\, g(t)|\, dt \le c\left(p^-, p^+\right) \|f\|_{p_t} \|g\|_{q_t}\,,$$

where $c(p^-, p^+) = 1 + \frac{1}{p^-} - \frac{1}{p^+}$. From definition it directly follows

Property A. *If* $|f(t)| \le |g(t)|$ *a.e. on* $(-\pi, \pi)$, *then* $\|f\|_{p_t} \le \|g\|_{p_t}$.

It is easily proved

Statement 1. *Let* $p \in H_\pi^{\ln}$, $p(t) > 0$, $\forall t \in [-\pi, \pi]$ *and* $\{\alpha_k\}_1^m \subset R$ *(R is a real axis). The function* $\omega(t) \equiv \prod_{k=1}^{m} |t - t_k|^{\alpha_k}$ *belongs to the space* L_{p_t} *iff* $\alpha_k > -\frac{1}{p(t_k)}$, $\forall k = \overline{1, m}$, *where* $\{t_k\}_1^m \subset [-\pi, \pi]$ *and* $t_i \neq j$.

In the sequel we'll use the following property.

Property B [18]. *If* $p(t) : 1 < p^- \leq p^+ < +\infty$, *then the class* $C_0^\infty(-\pi, \pi)$ *(finite and infinitely differentiable) is everywhere dense in* L_{p_t}.

Let Γ be a piecewise-Hölder curve on a complex plane C and f be a Lebesque summable function on Γ. Consider the singular integral S

$$Sf = \frac{1}{2\pi i} \int_\Gamma \frac{f(\tau)}{\tau - t} d\tau, \qquad t \in \Gamma.$$

Take some weight function $\rho : [-\pi, \pi] \to (0, +\infty)$ and define the weight class $L_{p_t,\rho_t} \equiv \{f : \rho f \in L_{p_t}\}$ with the norm $\|f\|_{p_t,\rho_t} \equiv \|\rho f\|_{p_t}$. The following theorem is valid.

Theorem [9]. *Let* $p \in H_\pi^{\ln}$, $p^- > 1$ *and* $\rho(t) \equiv \prod_{k=1}^{m} |t - \tau_k|^{\alpha_k}$, $\{\tau_k\}_1^m \subset [-\pi, \pi]$, $\tau_i \neq \tau_j$ *for* $i \neq j$. *The singular operator* S *boundedly acts from* L_{p_t,ρ_t} *to* L_{p_t,ρ_t} *iff* $-\frac{1}{p(\tau_k)} < \alpha_k < \frac{1}{q(\tau_k)}$, $k = \overline{1, m}$ *holds.*

In the sequel we'll need the Hardy classes of analytic functions with variable summability exponent.

3. Hardy classes with a variable summability exponent

Let $U \equiv \{z : |z| < 1\}$ be a unit ball on a complex plane and $\Gamma = \partial U$ be a unit circle. For a function $u(z)$ harmonic in U we accept $\|u\|_{h_{p_t}} \equiv \sup_{0<r<1} \left\|u\left(re^{it}\right)\right\|_{p_t}$, where $h_{p_t} \equiv \{u : \Delta u = 0 \text{ in } U \text{ and } \|u\|_{h_{p_t}} < +\infty\}$. The continuous imbeddings $h_{p^+} \subset h_{p_t} \subset h_{p^-}$ are true.

The Hardy class $H_{p_t}^+ \equiv \{f : f \text{ analytic in } U \text{ and } \|f\|_{H_{p_t}^+} < +\infty\}$, where $\|f\|_{H_{p_t}^+} \equiv \sup_{0<r<1} \left\|f\left(re^{it}\right)\right\|_{p_t}$ is introduced in the same way.

Define the Hardy class ${}_mH_{p_t}^-$ of functions analytic outside the unit circle and of order less or equal m at infinity. Let $f(z)$ be a function analytic on $C \backslash \overline{U}$ $\left(\overline{U} = U \bigcup \Gamma\right)$ of finite order $m_0 \leq m$, at infinity, i.e. $f(z) = f_1(z) + f_2(z)$, where $f_1(z)$ is a polynomial of power m_0, $f_2(z)$ is a tame part of the expansion of $f(z)$ in Lorentz series in the vicinity of the point at infinity. If the function $\varphi(z) \equiv \overline{f_2\left(\frac{1}{\bar{z}}\right)}$ ($(\bar{\cdot})$ is a complex conjugation) belongs to the class $H_{p_t}^+$, we'll say that the function $f(z)$ belongs to the class ${}_mH_{p_t}^-$.

4. Riemann problem in the classes $H_{p_t}^{\pm}$

Consider a Cartesian product $H_{p_t}^{+} \times H_{p_t}^{-}$. Let $G(t)$ and $g(t)$ be the functions given on $[-\pi, \pi]$. Under the solution of the Riemann problem in the class $H_{p_t}^{+} \times H_{p_t}^{-}$ we understand a pair of analytic functions $(F^+; F^-) \in H_{p_t}^{+} \times H_{p_t}^{-}$, whose non-tangential values $F^{\pm}(e^{it})$ on a unit circle Γ a.e. satisfy the relation:

$$F^+\left(e^{it}\right) + G(t)\, F^-\left(e^{it}\right) = g(t)\,,\ a.e.\ on\ [-\pi, \pi]\,, \tag{4.1}$$

where $g \in L_{p_t}$.

When the summability exponent is constant (i.e. $p(t) \equiv const$), the theory of these problems has been well studied (see for instance [5]). On the coefficient $G(t)$ of problem (4.1) we impose the following conditions:

1) $|G|^{\pm 1} \in L_\infty$;
2) The argument $\theta(t) \equiv \arg G(t)$ has a Jordan expansion $\theta(t) = \theta_0(t) + \theta_1(t)$, where $\theta_0 \in C[-\pi, \pi]$ and θ_1 is a function with bounded variation on $[-\pi, \pi]$. $\theta_1(t)$ has a finite number of discontinuity points of first kind $\{s_k\}_1^r : -\pi < s_1 < \ldots < s_r < \pi$;
3) $\left\{\dfrac{h_k}{2\pi} + \dfrac{1}{q(s_k)} : k = \overline{0, r}\right\} \bigcap Z = \emptyset$, where $h_k = \theta(s_k + 0) - \theta(s_k - 0)$, $h_0 = \theta(-\pi) - \theta(\pi)$, $\emptyset$ is an empty set.

Consider Riemann's homogeneous problem:

$$\left\{ \begin{array}{l} F^+(\tau) + G(arg\ \tau)\, F^-(\tau) = 0, \quad \tau \in \Gamma\,; \\[2ex] F^+ \in H_{p_t}^{+}, \quad F^- \in {}_mH_{p_t}^{-}. \end{array} \right. \tag{4.2}$$

Consider the functions $X_i^{\pm}(z)$ analytic inside (the sign "+") and outside (the sign "-") the unit circle:

$$X_1^{\pm}(z) \equiv \exp\left\{ \pm \frac{1}{4\pi} \int_{-\pi}^{\pi} \ln|G(t)|\, \frac{e^{it} + z}{e^{it} - z} dt \right\},$$

$$X_2^{\pm}(z) \equiv \exp\left\{ \pm \frac{i}{4\pi} \int_{-\pi}^{\pi} \theta(t)\, \frac{e^{it} + z}{e^{it} - z} dt \right\}.$$

Let

$$Z_i(z) \equiv \left\{ \begin{array}{ll} X_i^+(z)\,, & |z| < 1, \\[2ex] \left[X_i^-(z)\right]^{-1}, & |z| > 1, \quad i = 1, 2. \end{array} \right.$$

Assume $Z(z) \equiv Z_1(z) \cdot Z_2(z)$. Determine $\{n_i\}_{i=1}^r \subset Z$ from the inequalities:

$$\left\{ \begin{array}{l} -\dfrac{1}{q(s_k)} < \dfrac{h_k}{2\pi} + n_k - n_{k-1} < \dfrac{1}{p(s_k)}, \\[2ex] n_0 = 0, \qquad k = \overline{1, r}. \end{array} \right.$$

Denote $\omega_r = \frac{h_0}{2\pi} + n_r$.

The following theorem was established in [3].

Theorem [3]. *Let the conditions* 1)–2), $p \in H_\pi^{\ln}$, $p^- > 1$ *and* $\frac{1}{q(\pi)} < \omega_r < \frac{1}{p(\pi)}$ *be fulfilled. Then the general solution of the homogeneous problem* (4.2) *is of the form* $F(z) = Z(z) P_{m_0}(z)$, *where* $P_{m_0}(z)$ *is an arbitrary polynomial of power* $m_0 \le m$.

Corollary 1. *Let all the requirements of the previous theorem be fulfilled. Then under condition* $F^-(\infty) = 0$, *homogeneous problem* (5.1) *has only a trivial solution, i.e. zero solution.*

Now let's consider Riemann's homogeneous problem,

$$\begin{cases} F^+(\tau) + G(arg\ \tau) F^-(\tau) = g(\tau) e^{i[\alpha \arg \tau + \beta sign(\arg \tau)]}, \\ F^+ \in H_{p_t}^+, \quad F^- \in {}_m H_{p_t}^-, \end{cases} \tag{4.3}$$

where $g \in L_{p_t}(\Gamma)$ is an arbitrary function. It is obvious that the problem (4.3) has a unique solution (if it is solvable) iff the appropriate homogeneous problem (4.2) has only a trivial solution. In the general case, the solution $F(z)$ of the problem (4.3) is of the form $F(z) = F_0(z) + F_1(z)$, where $F_0(z)$ is one of particular solutions of the problem (4.3), $F_1(z)$ is a general solution of the homogeneous problem.

5. Basic results

Further we'll consider a more specific case, exactly, let $\gamma(t) \equiv \alpha t + \beta signt$. In the place of $G(t)$ we take $G(t) \equiv e^{i\gamma(t)}$. Assume that $\alpha, \beta \in R$. The complex case may be investigated similarly. Consider problem (4.3) and assume that the right hand side $g\left(e^{it}\right)$ is Hölderian on $[-\pi, \pi]$. In the sequel, for simplicity we'll denote $g\left(e^{it}\right)$ as $g(t)$. We'll solve this problem by the method worked out in [8]. For that we'll need the auxiliary functions.

Let $(z+1)_{-1}^{\gamma}$ and z_{-1}^{γ} $\left((z-1)_{+1}^{\gamma}\right.$ and $\left.z_{+1}^{\gamma}\right)$ be the branches of multi-valued analytic functions $(z+1)^{\gamma}$ and z^{γ} ($(z-1)^{\gamma}$ and z^{γ}) that are analytic on complex plane cut along negative (positive) part of a real axis, respectively. Accept

$$\left(\frac{z+1}{z}\right)_{-1}^{\gamma} = \frac{(z+1)_{-1}^{\gamma}}{z_{-1}^{\gamma}}, \left(\frac{z-1}{z}\right)_{+1}^{\gamma} = \frac{(z-1)_{+1}^{\gamma}}{z_{+1}^{\gamma}}.$$

Thus, a particular solution of problem (4.3) is of the form:

$$\left.\begin{aligned} F_0^+(z) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{i(\alpha t+\beta signt)}g\left(e^{it}\right)dt}{\left(e^{it}+1\right)_{-1}^{\gamma}\left(e^{it}-1\right)_{+1}^{\gamma_2}\left(1-ze^{it}\right)}(z+1)_{-1}^{\gamma_1}(z-1)_{+1}^{\gamma_2} \\ F_0^-(z) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{i(\alpha t+\beta signt)}g\left(e^{it}\right)dt}{\left(e^{it}+1\right)_{-1}^{\gamma}\left(e^{it}-1\right)_{+1}^{\gamma_2}\left(1-ze^{it}\right)}\left(\frac{z+1}{z}\right)_{-1}^{\gamma_1}\left(\frac{z-1}{z}\right)_{+1}^{\gamma_2} \end{aligned}\right\}, \tag{5.1}$$

where $\gamma_1 = 2\alpha + \frac{2\beta}{\pi}$; $\gamma_2 = -\frac{2\beta}{\pi}$. Consider the following systems:

$$h_n^+(t) = \frac{e^{i(\alpha t+\beta signt-2\beta)}}{2\pi}\left(e^{it}+1\right)_{-1}^{-\gamma_1}\left(e^{it}-1\right)_{+1}^{-\gamma_2} \times\sum_{k=0}^{n}(-1)^{n-k}C_{\gamma_2}^{n-k}\sum_{s=0}^{k}C_{\gamma_1}^{k-s}e^{-st}, \quad n\geq 0;$$

$$h_n^-(t) = \frac{e^{i(\alpha t+\beta signt-2\beta)}}{2\pi}\left(e^{it}+1\right)_{-1}^{-\gamma_1}\left(e^{it}-1\right)_{+1}^{-\gamma_2} \times\sum_{k=1}^{n}(-1)^{m-k}C_{\gamma_2}^{m-k}\sum_{s=1}^{k}C_{\gamma_1}^{k-s}e^{-st}, \quad m\geq 1;$$

where $C_\gamma^n = \frac{\gamma(\gamma-1)\cdot\ldots\cdot\gamma(\gamma-n+1)}{n!}$, $C_\gamma^0 = 1$ are binomial coefficients. The following lemma is proved in [4].

Lemma 1. *Let the inequalities*

$$0\leq\alpha+\frac{\beta}{\pi}<\frac{1}{2},\quad 0\leq\frac{\beta}{\pi}<\frac{1}{2},$$

be fulfilled. Then there hold the following relations:

$$< x_k^+, h_n^+ > = < x_k^-, h_n^- > = \delta_{nk},\quad \forall n,k\geq 0;$$
$$< x_k^+, h_{n+1}^- > = < x_{k+1}^-, h_n^+ > = 0,\quad \forall n,k\geq 0\,,$$

where

$$< x,y > = \int_{\pi}^{\pi}x(t)\,\overline{y(t)}dt,\quad x_n^{\pm} = e^{\pm i[(n-\alpha)t-\beta signt]},.$$

From the representations of the functions $F_0^{\pm}(z)$ it directly follows that $F_0^+ \in H_1^+$; $F_0^- \in {}_{-1}H_1^-$ if it holds $|\gamma_i| < 1$, $i = 1,2$. It follows from the known relations [5]

$$\int_{-\pi}^{\pi}\left|F_0^+\left(e^{it}\right)-F_0^+\left(re^{it}\right)\right|dt\to 0,\ \ r\to 1-0;$$

$$\int_{-\pi}^{\pi}\left|F_0^-\left(e^{it}\right)-F_0^-\left(re^{it}\right)\right|dt\to 0,\ \ r\to 1+0,$$

that

$$a_n^+ = \frac{1}{2\pi}\int_{-\pi}^{\pi} F_0^+\left(e^{it}\right)e^{int}dt, \quad \forall n \geq 0; a_k^- = \frac{1}{2\pi}\int_{-\pi}^{\pi} F_0^-\left(e^{it}\right)e^{ikt}dt, \quad \forall k \geq 1,$$

where

$$F_0^+(z) = \sum_{n=0}^{\infty} a_n^+ z^n \left(F_0^-(z) = \sum_{n=1}^{\infty} a_n^- z^{-n}\right),$$

is a Taylor expansion of the function $F_0^+(z)$ $\left(F_0^-(z)\right)$ in the vicinity of the zero (of a point at infinity).

Consider the case $0 \leq \gamma_k < 1$, $k = 1, 2$. Assume that the function $g(\tau)$ is Hölderian on $\Gamma : g(1) = g(-1) = 0$. Then, using the representation of the Cauchy type integral with power character peculiarity in the vicinity of the first order density discontinuity point (see [8], p.74), it is easy to show that the functions $F_0^{\pm}\left(e^{it}\right)$ satisfy some Hölderian conditions on Γ. Therefore, the Fourier series by the system of exponents $\left\{e^{int}\right\}_{n\in Z}$ converge to them uniformly on the segment $[-\pi, \pi]$. Thus,

$$F_0^+\left(e^{it}\right) = \sum_{n=0}^{\infty} a_n^+ e^{int}, \qquad F_0^-\left(e^{it}\right) = \sum_{n=1}^{\infty} a_n^- e^{-int},$$

uniformly on $[-\pi, \pi]$. Then, from the relation (4.3) we get that the function $g(t)$ expands in uniformly convergent series by the system (1.1).

$$g(t) = \sum_{n=0}^{\infty} a_n^+ e^{i[(n-\alpha)t-\beta signt]} + \sum_{n=1}^{\infty} a_n^- e^{-i[(n-\alpha)t-\beta signt]}.$$

If $p \in H_\pi^{\ln}$ and $p^- > 1$, then it directly follows from Property A that the system (1.1) belongs to the space L_{p_t}. As is known, the space conjugated to L_{p_t} is L_{q_t} (see for instance [18]). It follows from statement 1 and representations $\{h_n^{\pm}\}$ that, if the inequalities $\gamma_1 < \frac{1}{q(\pi)}$; $\gamma_2 < \frac{1}{q(0)}$ are fulfilled, the system $\left\{h_n^+;\ h_{n+1}^-\right\}_{n\geq 0}$ belongs to the space L_{q_t}. As a result, having paid attention to the Property B, we get that when the inequality

$$0 \leq \gamma_1 < \frac{1}{q(\pi)}, \quad 0 \leq \gamma_2 < \frac{1}{q(0)},$$

is fulfilled, the system (1) is complete and minimal in L_{p_t}. Accept the denotation

$$I(z) = \int_{-\pi}^{\pi} \frac{e^{i(\alpha t+\beta\cdot signt)}g(t)\,dt}{\left(e^{it}+1\right)_{-1}^{\gamma_1}\left(e^{it}-1\right)_{+1}^{\gamma_2}\left(1-ze^{-it}\right)}.$$

Consequently,

$$\left.\begin{aligned} F^+(z) &= \frac{1}{2\pi}I(z)(z+1)_{-1}^{\gamma_1}(z-1)_{+1}^{\gamma_2}, \ |z| < 1, \\ F^-(z) &= \frac{1}{2\pi}I(z)\left(1+z^{-1}\right)_{-1}^{\gamma_1}\left(1-z^{-1}\right)_{+1}^{\gamma_2}, \ |z| > 1. \end{aligned}\right\}$$

In the place of $g(\tau)$ we take a function that is finite in some vicinities of the points $z = \pm 1$. From the above cited reasoning we get that the Fourier series of functions $I\left(e^{\pm it}\right)$ uniformly converge to them on $[-\pi, \pi]$. As it follows from Statement 1, if $\gamma_1 > -\dfrac{1}{p(\pi)}$ and $\gamma_2 > -\dfrac{1}{p(0)}$ hold, the functions $\left(1+e^{\pm it}\right)_{-1}^{\gamma_1}$, $\left(1-e^{\pm it}\right)_{+1}^{\gamma_2}$ belong to L_{p_t}. By the result of the paper [16], the classic system of exponents $\left\{e^{int}\right\}_{n\in Z}$ forms a basis in L_{p_t}. Expanding these functions by this system and considering that the function $(z+1)_{-1}^{\gamma_1}(z-1)_{+1}^{\gamma_2}\ \left(\left(1+z^{-1}\right)_{-1}^{\gamma_1}\left(1-z^{-1}\right)_{+1}^{\gamma_2}\right)$ belongs to $H_{p_t}^{+}\left(H_{p_t}^{-}\right)$, from (4.3) we get that $g(t)$ expands in series in L_{p_t} by the system (1.1). Here we used the circumstance that the functions from $H_{p_t}^{+}\left(H_{p_t}^{-}\right)$ have Fourier zero coefficients for negative (positive) values of summation index. Again, having paid attention to the Property B, we get the completeness of the system (1.1) in L_{p_t} in this case as well. Thus, we proved the following

Statement 2. *Let $p \in H_{\pi}^{\ln}$, $p^- > 1$ and the inequalities*

$$-\frac{1}{p(\pi)} < \gamma_1 < \frac{1}{q(\pi)}, \quad -\frac{1}{p(0)} < \gamma_2 < \frac{1}{q(0)}, \tag{5.2}$$

be fulfilled. Then the system (1.1) *is complete and minimal in L_{p_t}.*

Consider the basicity of the system (1.1) in L_{p_t}. Let the inequalities (5.2) be fulfilled. So, the system (1.1) is complete and minimal in L_{p_t} and let $\left\{h_n^+;\ h_{n+1}^-\right\}_{n\geq 0} \subset L_{q_t}$ be an appropriate biorthogonal system. Take $\forall g \in L_{p_t}$ and consider

$$S_m[g] = \sum_{n=0}^{m} a_n^+ x_n^+ + \sum_{n=1}^{m} a_n^- x_n^-,$$

where $a_n^{\pm} = < g, h_n^{\pm} >$. Consider the problem (4.3) and require $F^-(\infty) = 0$. Then by Corollary 1 it has a unique solution $F^{\pm}(z)$ from the class $H_{p_t}^{+} \times {}_{-1}H_{p_t}^{-}$. It is clear that $F^{\pm}\left(e^{it}\right) \in L_{p_t}$. As we have established

$$a_n^+ = \frac{1}{2\pi}\int_{-\pi}^{\pi} F^+\left(e^{it}\right)e^{-int}dt, a_k^- = \frac{1}{2\pi}\int_{-\pi}^{\pi} F^-\left(e^{it}\right)e^{ikt}dt.$$

Since the system $\left\{e^{int}\right\}_{n\in Z}$ forms a basis in L_{p_t}, it is clear that $\exists M > 0$:

$$\left\|\sum_{n=0}^{m} a_n^+ e^{int}\right\|_{p_t} \leq M\left\|F^+\left(e^{it}\right)\right\|_{p_t},$$

$$\left\|\sum_{n=1}^{m} a_n^- e^{-int}\right\|_{p_t} \leq M\left\|F^-\left(e^{it}\right)\right\|_{p_t}, \quad \forall m \in N.$$

Considering these relations and having taken into attention the Property A, we get:

$$\begin{aligned}\left\|S_m\left[g\right]\right\|_{p_t} &\leq \left\|\sum_{n=0}^{m} a_n^{+} e^{int} e^{-i(\alpha t+\beta sign t)}\right\|_{p_t} + \left\|\sum_{n=1}^{m} a_n^{-} e^{-int} e^{-i(\alpha t+\beta sign t)}\right\|_{p_t} \\ &\leq \left\|\sum_{n=0}^{m} a_n^{+} e^{int}\right\|_{p_t} + \left\|\sum_{n=1}^{m} a_n^{-} e^{-int}\right\|_{p_t} \\ &\leq M\left(\left\|F^{+}\left(e^{it}\right)\right\|_{p_t} + \left\|F^{-}\left(e^{it}\right)\right\|_{p_t}\right).\end{aligned}$$

Applying the Sokhotskiy–Plemel formula to expressions (5.1), we get:

$$F^{\pm}\left(e^{it}\right) = ie^{\pm i(\alpha t+\beta sign t)} g\left(t\right) + S^{\pm}\left(g\right),$$

where $S^{\pm}\left(g\right)$ are singular type integrals

$$S^{\pm}\left(g\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i(\alpha t+\beta sign t)} g\left(t\right) dt}{\left(e^{it}+1\right)_{-1}^{\gamma_1}\left(e^{it}-1\right)_{+1}^{\gamma_2}\left(1-e^{i(s-t)}\right)}$$

$$\times\left(1+e^{\pm is}\right)_{-1}^{\gamma_1}\left(\pm e^{\pm is}\mp 1\right)_{+1}^{\gamma_2}.$$

Applying Theorem [9] to these expressions, we get $\exists M_2 > 0$:

$$\left\|S^{\pm}\left(g\right)\right\|_{p_t} \leq M_2\left\|g\right\|_{p_t}, \quad \forall g \in L_{p_t}.$$

Considering the above mentioned estimations, we have:

$$\begin{aligned}\left\|S_m\left[g\right]\right\|_{p_t} &\leq M\left(2\left\|g\right\|_{p_t} + \left\|S^{+}\left(g\right)\right\|_{p_t} + \left\|S^{-}\left(g\right)\right\|_{p_t}\right) \\ &\leq 2M\left(1+M_2\right)\left\|g\right\|_{p_t}, \quad \forall g \in L_{p_t}, \quad \forall m \in N.\end{aligned}$$

As a result, from this estimation and from basicity criterion we get that the system (1.1) forms a basis in L_{p_t}. So, we proved

Theorem 2. *Let $p \in H_{\pi}^{\ln}$, $p^{-} > 1$ and the inequalities* (5.2) *hold*

$$-\frac{1}{p(\pi)} < \gamma_1 < \frac{1}{q(\pi)}, \qquad -\frac{1}{p\left(0\right)} < \gamma_2 < \frac{1}{q\left(0\right)}.$$

Then, the system (1.1) *forms a basis in L_{p_t}.*

We separately consider the case $\gamma_1 = -\dfrac{1}{p(\pi)}$. It follows from the arguments cited above and from the expressions of the system $\left\{h_n^{+};\ h_{n+1}^{-}\right\}_{n\geq 0}$ that the system (1.1) is minimal in L_{p_t} in this case as well.

Represent the system (1.1) in the form

$$\left\{e^{i[(n+1)t-(\alpha+1)t+\beta sign\, t]};\ e^{-i[mt-\alpha t+\beta sign\, t]}\right\}_{n\geq 0;\, m\geq 1}. \tag{5.3}$$

It is obvious that multiplication of each term of the system (5.3) by the function $e^{i\frac{t}{2}}$ doesn't influence on its completeness in L_{p_t}. After multiplication we get the system $\left\{I_{n;m}^{\tilde{\alpha}}\left(t\right)\right\}_{n\geq 1;\, m\geq 1}$, where

$$I_{n;m}^{\tilde{\alpha}}\left(t\right) \equiv \left(e^{i[nt-\tilde{\alpha}t+\beta sign\, t]};\ e^{-i[mt-\tilde{\alpha}t+\beta sign\, t]}\right)$$

$\tilde{\alpha} = \alpha + \dfrac{1}{2}$. The appropriate parameter $\tilde{\gamma}_1$ of this system is $\tilde{\gamma}_1 = 2\left(\tilde{\alpha} + \dfrac{\beta}{\pi}\right) = \gamma_1 + 1$. Therefore,

$$\tilde{\gamma}_1 = 1 - \frac{1}{p(\pi)} = \frac{1}{q(\pi)} < 1,$$

is fulfilled for $\tilde{\gamma}_1$.

Then, from the previous arguments, we get that the system

$$\left\{I^{\tilde{\alpha}}_{n;m}(t)\right\}_{n\geq 0;\, m\geq 1},$$

is complete in L_{p_t}. By $\left\{\tilde{h}^+_n;\ \tilde{h}^-_{n+1}\right\}_{n\geq 0}$ we denote a system determined by the expressions $h^{\pm}_n$, wherein we take $\widetilde{\alpha}$ in the place of the parameter α. The systems $\left\{I^{\tilde{\alpha}}_{n;m}(t)\right\}_{n\geq 0;\, m\geq 1}$ and $\left\{\tilde{h}^+_n;\ \tilde{h}^-_{n+1}\right\}$ satisfy the relations of Lemma 1. It directly follows from the expressions for the system $\left\{\tilde{h}^+_n;\ \tilde{h}^-_{n+1}\right\}_{n\geq 0}$ that it doesn't belong to the space L_{q_t}, since $\tilde{\gamma}_1 \geq \frac{1}{q(\pi)}$ and the required one follows from Statement 1. From the uniqueness of the biorthogonal system to the complete system, hence we get that the system $\left\{I^{\tilde{\alpha}}_{n;m}(t)\right\}_{n\geq 0;\, m\geq 1}$ is not minimal in L_{p_t}. As a result, the system $\left\{I^{\tilde{\alpha}}_{n;m}(t)\right\}_{n;\, m\geq 1}$ is also complete and so the system (1.1) is complete in L_{p_t}. The fact that in this case it doesn't form a basis in L_{p_t}, is proved similar to the paper [12]. Consequently, it is valid

Theorem 3. *Let $p \in H^{\ln}_{\pi}$, $p^- > 1$ and the relations*

$$\alpha + \frac{\beta}{\pi} = -\frac{1}{2p(\pi)}, \qquad -\frac{1}{2q(0)} < \frac{\beta}{\pi} < \frac{1}{2p(0)},$$

be fulfilled. Then the system (1.1) *is complete and minimal in L_{p_t}, but it doesn't form a basis in it.*

The remaining cases for the values of the parameters α and β are considered similar to the paper [12]. As a result we get the following result.

Theorem 4. *Let $p \in H^{\ln}_{\pi}$, $p^- > 1$ and $-\dfrac{1}{2q(0)} < \dfrac{\beta}{\pi} < \dfrac{1}{2p(0)}$ hold. The system* (1.1) *forms a basis in L_{p_t} iff the inequality $-\dfrac{1}{2p(\pi)} < \alpha + \dfrac{\beta}{\pi} < \dfrac{1}{2q(\pi)}$ is fulfilled.*

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