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Von Neumann–Schatten Frames in Separable Banach Spaces

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Abstract. In this paper we introduce the notion of a von Neumann-Schatten p-frame in separable Banach spaces and obtain some of their characterizations. We show that p-frames and g-frames are a class of von Neumann-Schatten p-frames.

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1. Introduction and preliminaries

The concept of a frame in Hilbert space has been introduced by Duffin and Schaeffer [13] to study some deep problems in nonharmonic Fourier series. Since then various generalization of frames such as frame of subspaces [6], pseudo-frames [16], oblique frames [10], continuous frames [1, 4, 14], generalized frames [19] have been developed by several mathematicians. The concept of a frames in Banach space has been introduced by Christensen and Stoeva [9], Casazza, Han and Larson [7] and Gröchenig [15].

The p-frame and g-frame are two important generalizations of frames in Banach spaces and Hilbert spaces, which in this article we unify these two concepts. By utilizing von Neumann–Schatten frames many basic properties of frames can be derived in a more general setting. In fact, a von Neumann–Schatten frame is a sequence of bounded linear operators from a Banach space $\mathcal{X}$ into $\mathcal{C}_p \subseteq \mathcal{B}(\mathcal{H})$. The von Neumann–Schatten trace ideals $\mathcal{C}_p$ play an important role in the operator theory and mathematical physics [18]. The main goal of this paper is to define von Neumann-Schatten frames and to show that p-frames and g-frames can be considered as such frames. Throughout this paper, $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces and $\{\mathcal{K}_i : i \in \mathbb{N}\} \subset \mathcal{K}$ denotes a sequence of Hilbert spaces. Note that for any sequence $\{\mathcal{K}_i : i \in \mathbb{N}\}$ of

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Hilbert spaces, we can always find a larger space $\mathcal{K}$ containing all the Hilbert space $\mathcal{K}_i$ by setting $\mathcal{K} = \bigoplus_{i\in\mathbb{N}} \mathcal{K}_i$.

Definition 1.1. A sequence $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ is called a *frame* for $\mathcal{H}$ if there exist $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{i=1}^{\infty} |\langle f, f_i\rangle|^2 \leq B\|f\|^2 \qquad (f \in \mathcal{H}). \tag{1.1}$$

The numbers A and B are called *frame bounds*. If we can choose $A = B$, the frame is called *tight*. Given a frame $\{f_i\}_{i=1}^{\infty}$, the frame operator is defined by

$$Sf = \sum_{i=1}^{\infty} \langle f, f_i\rangle f_i\,.$$

The series defining Sf converges unconditionally for all $f \in \mathcal{H}$ and S is a bounded invertible self-adjoint operator. This leads to the following frame decomposition

$$f = S^{-1}Sf = \sum_{i=1}^{\infty} \langle f, S^{-1}f_i\rangle f_i \qquad (f \in \mathcal{H}).$$

The possibility of representing every $f \in \mathcal{H}$ in this way is the main feature of a frame. The coefficients $\{\langle f, S^{-1}f_i\rangle\}_{i=1}^{\infty}$ are called frame coefficients. A sequence satisfying the upper frame condition is called a Bessel sequence. A sequence $\{f_i\}_{i=1}^{\infty}$ is Bessel sequence if and only if the operator $T : l^2 \to \mathcal{H}$ given by $T\{c_i\} = \sum_{i=1}^{\infty} c_i f_i$ is a well-defined operator. In that case T, which is called the pre-frame operator, is automatically bounded. When $\{f_i\}_{i=1}^{\infty}$ is a frame, the pre-frame operator T is well-defined and $S = TT^*$. For general references on this theory, we refer the reader to [8, Section 5.1].

The notion of a frame extended to some various directions by several authors [1, 4, 7, 9, 16, 3]. We need the follow notion of a g-frame due to W. Sun [19].

Definition 1.2. A sequence $\{\Lambda_i \in \mathcal{B}(\mathcal{H}, \mathcal{K}_i) : i \in \mathbb{N}\}$ a *generalized frame*, or simply a *g-frame* for $\mathcal{H}$ with respect to $\{\mathcal{K}_i : i \in \mathbb{N}\}$ if there are two positive constant A and B such that

$$A\|f\|^2 \leq \sum_{i\in\mathcal{I}} \|\Lambda_i f\|^2 \leq B\|f\|^2 \qquad (f \in \mathcal{H}). \tag{1.2}$$

A sequence $\{g_i\}_{i=1}^{\infty} \subseteq \mathcal{X}^*$ is called a *p-frame* $(1 < p < \infty)$ if the norm $\|.\|$ of $\mathcal{X}$ is equivalent to the ℓ^p-norm of the sequence $\{g_i(.)\}$ or there exist constants A and B such that

$$A\|f\| \leq (\sum_{i=1}^{\infty} |g_i(f)|^p)^{\frac{1}{p}} \leq B\|f\| \qquad (f \in \mathcal{X}). \tag{1.3}$$

Christensen and Stoeva [9] without further assumption prove that a p-frame allows every element $f \in \mathcal{X}^*$ to be represented as an unconditionally series $g = \sum_{i=1}^{\infty} \alpha_i g_i$ for coefficients $\{\alpha_i\} \in \ell^q$.

We introduce the notion of a *von Neumann-Schatten p-frame* and show that every p-frame for a sparable Banach space $\mathcal{X}$ is a von Neumann-Schatten p-frame with respect to $\mathbb{C}$. Also, we give a characterization of *von Neumann-Schatten q-Riesz bases* for $\mathcal{X}^*$.

2. von Neumann-Schatten frames

Throughout this section we assume that $(\mathcal{X}, \|.\|)$ is a separable Banach space with dual $\mathcal{X}^*$ and $\mathcal{H}$ is a Hilbert space also $1 < p < \infty$ and q is the conjugate exponent to p, i.e., $\frac{1}{p} + \frac{1}{q} = 1$.

We first introduce some necessary definitions and notations and refer the reader to [17, 11] more information. Suppose $\{\mathcal{X}_i : i \in \mathcal{I}\}$ is a collection of normed spaces. Then $\Pi\{\mathcal{X}_i : i \in \mathcal{I}\}$ is a vector space if the linear operations are defined coordinatewise. For $1 \leq p < \infty$, define

$$\bigoplus_p \mathcal{X}_i \equiv \{x \in \Pi_i \mathcal{X}_i : \|x\| = (\sum_i \|x_i\|^p)^{\frac{1}{p}} < \infty\}.$$

It is known that $\bigoplus_p \mathcal{X}_i$ is a Banach space if and only if so is each $\mathcal{X}_i$.

Let $\mathcal{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators on a complex separable Hilbert space $\mathcal{H}$. For a compact operator $\mathcal{T} \in \mathcal{B}(\mathcal{H})$, let $s_1(\mathcal{T}) \geq s_2(\mathcal{T}) \geq ... \geq 0$ denote the singular values of $\mathcal{T}$, i.e., the eigenvalues of the positive operator $|\mathcal{T}| = (\mathcal{T}^*\mathcal{T})^{\frac{1}{2}}$, arranged in a decreasing order and repeated according to multiplicity. For $1 \leq p < \infty$, the von Neumann-Schatten p-class $\mathcal{C}_p$ is defined to be the set all compact operators $\mathcal{T}$ for which $\sum_{i=1}^{\infty} s_i^p(\mathcal{T}) < \infty$.

For $\mathcal{T} \in \mathcal{C}_p$, the von Neumann Schatten p-norm of $\mathcal{T}$ is defined by

$$\|\mathcal{T}\|_p = \left(\sum_{i=1}^{\infty} s_i^p(\mathcal{T})\right)^{\frac{1}{p}} = (\tau|\mathcal{T}|^p)^{\frac{1}{p}} \tag{2.1}$$

where τ is the usual trace functional which defines as $\tau(\mathcal{T}) = \sum_{e \in \mathcal{E}} \langle \mathcal{T}(e), e\rangle$, where $\mathcal{E}$ is any orthonormal basis of $\mathcal{H}$. It is convenient to let $\mathcal{C}_\infty$ denote the class of compact operators, and in this case $\|\mathcal{T}\|_\infty = s_1(\mathcal{T})$ is the usual operator norm.

Lemma 2.1. [17] *Suppose that* $1 \leq p \leq \infty$, *q is the index conjugate to p, and* $\mathcal{T} \in \mathcal{B}(\mathcal{H})$. *Then* $\mathcal{T} \in \mathcal{C}_p$ *if and only if*

$$\sup\{|\tau(\mathcal{S}\mathcal{T})| : \quad \mathcal{S} \in \mathcal{F}(\mathcal{H}), \|\mathcal{S}\|_q \leq 1\} < \infty \tag{2.2}$$

where $\mathcal{F}(\mathcal{H})$ *is the set of finite-rank operators on* $\mathcal{H}$. *When this is so, the value of the supremum is* $\|\mathcal{T}\|_p$.

The following theorem shows that the Banach space $\mathcal{C}_p$ is reflexive for $1 < p < \infty$. It is known that if $\mathcal{T} \in \mathcal{C}_p$ and $\mathcal{S} \in \mathcal{C}_q$, then $\mathcal{T}\mathcal{S}, \mathcal{S}\mathcal{T} \in \mathcal{C}_1$ and $\tau(\mathcal{T}\mathcal{S}) = \tau(\mathcal{S}\mathcal{T})$.

Theorem 2.2. [17] *Suppose that $1 < p < \infty$ and q is the index conjugate to p. Then for each $\mathcal{S} \in \mathcal{C}_q$ the function $\vartheta_{\mathcal{S}}(\mathcal{T}) = \tau(\mathcal{T}\mathcal{S})$, where $\mathcal{T} \in \mathcal{C}_p$, is a continuous linear functional on $\mathcal{C}_p$. Moreover the mapping $\mathcal{S} \to \vartheta_{\mathcal{S}}$ is an isometric isomorphism from $\mathcal{C}_q$ onto the dual space $(\mathcal{C}_p)^*$ of $\mathcal{C}_p$.*

If x, y are elements of a Hilbert spaces $\mathcal{H}$ we define the operator $x \otimes y$ on $\mathcal{H}$ by

$$(x \otimes y)(z) = \langle z, y\rangle x.$$

It is obvious that $\|x \otimes y\| = \|x\|\|y\|$ and the rank of $x \otimes y$ is one if x and y are non-zero. If $x_1, x_2, y_2, y_2 \in \mathcal{H}$, then the following equalities are easily verified:

$$\begin{aligned}(x_1 \otimes x_2)(y_1 \otimes y_2) &= \langle y_1, x_2\rangle (x_1 \otimes y_2)\\ (x_1 \otimes y_1)^* &= y_1 \otimes x_1.\end{aligned}$$

Note that if $x, y \in \mathcal{H}$, then $\|x \otimes y\|_p = \|x \otimes y\|_q = \|x\|\|y\|$ and $\tau(x \otimes y) = \langle x, y\rangle$ so $x \otimes y$ is in $\mathcal{C}_p$ for all $p \geq 1$. We recall that $\mathcal{C}_2$ is a Banach space with respect to the norm $\|.\|_2$. It is shown that the space $\mathcal{C}_2$ with the inner product

$$[\mathcal{T}, \mathcal{S}]_\tau = \tau(\mathcal{S}^*\mathcal{T})$$

is a Hilbert space. If $\{\eta_i : i \in I\}$ and $\{\zeta_i : i \in I\}$ are orthonormal bases in $\mathcal{H}$ and $\nu_{i,j} = \eta_i \otimes \zeta_j$, then $\{\nu_{i,j} : i, j \in I\}$ is an orthonormal basis of $\mathcal{C}_2$ [17].

Definition 2.3. A countable family $\{\mathcal{G}_i\}_{i=1}^\infty$ of bounded linear operators from $\mathcal{X}$ to $\mathcal{C}_p \subseteq \mathcal{B}(\mathcal{H})$ is said to be a *von Neumann-Schatten p-frame* for $\mathcal{X}$ with respect to $\mathcal{H}$ if there exist constants $A, B > 0$ such that

$$A\|f\| \leq \left(\sum_{i=1}^\infty \|\mathcal{G}_i(f)\|_p^p\right)^{\frac{1}{p}} \leq B\|f\| \qquad (f \in \mathcal{X}). \tag{2.3}$$

The sequence $\{\mathcal{G}_i\}$ is a *von Neumann-Schatten p-Bessel sequence* if at least upper von Neumann-Schatten p-frame condition is satisfied. In the other words, countable family $\{\mathcal{G}_i\}$ of bounded linear operators from $\mathcal{X}$ to $\mathcal{C}_p$ is a von Neumann-Schatten p-frame for $\mathcal{X}$ with respect to $\mathcal{H}$ if the norm $\|.\|$ of $\mathcal{X}$ is equivalent to ℓ^p-norm of the sequence $\{\|\mathcal{G}_i(.)\|_p\}_{i=1}^\infty$.

Now we define von Neumann-Schatten p-frame operators as follows. Let $\{\mathcal{G}_i\}$ be a von Neumann-Schatten p-frame for $\mathcal{X}$ with respect to $\mathcal{H}$. Define

$$\begin{aligned}U &: \mathcal{X} \to \bigoplus \mathcal{C}_P \qquad (2.4)\\ U(f) &= \{\mathcal{G}_i(f)\}\end{aligned}$$

and

$$\begin{aligned}T &: \bigoplus \mathcal{C}_q \to \mathcal{X}^* \qquad (2.5)\\ T(\{\mathcal{A}_i\}) &= \sum_{i=1}^\infty \mathcal{A}_i\mathcal{G}_i\,.\end{aligned}$$

The operator U is frequently called the analysis operator and T is the synthesis operator. It is clear that if $\{\mathcal{G}_i\}$ is a von Neumann-Schatten p-frame or just a von Neumann-Schatten p-Bessel sequence then U is a bounded operator.

If set $\mathcal{H} = \mathbb{C}$, then $\mathcal{B}(\mathcal{H}) = \mathcal{C}_p = \mathcal{C}_q = \mathbb{C}$, $\bigoplus \mathcal{C}_p = \ell^p$ and also $\bigoplus \mathcal{C}_q = \ell^q$. Hence every usual p-frame for $\mathcal{X}$ is a von Neumann-Schatten p-frame for $\mathcal{X}$ with respect to $\mathbb{C}$. Hence we have the following.

Lemma 2.4. *A p-frame for $\mathcal{X}$ can be considered as a von Neumann-Schatten p-frame for $\mathcal{X}$ with respect to $\mathbb{C}$.*

Lemma 2.5. *If $\{\mathcal{G}_i\}$ is a von Neumann-Schatten p-frame for $\mathcal{X}$ with respect to $\mathcal{H}$, then the operator U given by* (2.4) *has closed range, and $\mathcal{X}$ is reflexive.*

Proof. By the von Neumann-Schatten p-frame condition, the operator U is bounded below so U has closed range and $\mathcal{X}$ is isomorphic to $\mathrm{Ran}(U)$. But $\mathrm{Ran}(U)$ is reflexive because it is a closed subspace of the reflexive space $\bigoplus \mathcal{C}_p$. Therefore $\mathcal{X}$ is reflexive. □

Theorem 2.6. *A sequence $\{\mathcal{G}_i\} \subseteq \mathcal{B}(\mathcal{X}, \mathcal{C}_p)$ is a von Neumann-Schatten p-Bessel sequence with respect to $\mathcal{H}$ with a bound B if and only if the operator defined by* (2.5) *is a well defined bounded operator with $\|T\| \leq B$.*

Proof. First, let $\{\mathcal{G}_i\}$ be a von Neumann-Schatten p-Bessel sequence, $i \in \mathbb{N}$ and $\{A_i\} \in \mathcal{C}_q$. Then for $m > n$ we have

$$\begin{aligned}
\left|\sum_{i=1}^{m} \mathcal{A}_i\mathcal{G}_i - \sum_{i=1}^{n} \mathcal{A}_i\mathcal{G}_i\right| &= \sup_{f\in\mathcal{X}, \|f\|\leq 1} \left\{\left|\sum_{i=n+1}^{m} \tau\left(\mathcal{G}_i(f)\mathcal{A}_i\right)\right|\right\} \\
&\leq \sup_{f\in\mathcal{X}, \|f\|\leq 1} \left\{\sum_{i=n+1}^{m} \left|\tau\left(\mathcal{G}_i(f)\mathcal{A}_i\right)\right|\right\} \\
&\leq \sup_{f\in\mathcal{X}, \|f\|\leq 1} \left\{\sum_{i=n+1}^{m} \|\mathcal{G}_i(f)\mathcal{A}_i\|_1\right\} \\
&\leq \sup_{f\in\mathcal{X}, \|f\|\leq 1} \left\{\sum_{i=n+1}^{m} \|\mathcal{G}_i(f)\|_p\|\mathcal{A}_i\|_q\right\} \\
&\leq \sup_{f\in\mathcal{X}, \|f\|\leq 1} \left\{(\sum_{i=n+1}^{m} \|\mathcal{G}_i(f)\|_p^p)^{\frac{1}{p}}(\sum_{i=n+1}^{m} \|\mathcal{A}_i\|_q^q)^{\frac{1}{q}}\right\} \\
&\leq B\left(\sum_{i=n+1}^{m} \|\mathcal{A}_i\|_q^q\right)^{\frac{1}{q}}.
\end{aligned}$$

It follows from $\{\mathcal{A}_i\} \in \bigoplus \mathcal{C}_q$ that, $\sum_{i=1}^{\infty} \mathcal{A}_i\mathcal{G}_i$ is convergent and T is well-defined. It is obvious that with the same calculations as above now gives that

$$\|T(\{\mathcal{A}_i\})\| \leq B\|\{\mathcal{A}_i\}\|_q.$$

Therefore T is bounded and $\|T\| \leq B$. For the converse, assume that T is well defined. By the Banach Steinhaus theorem T is automatically bounded in this case. Given $f \in \mathcal{X}$, define

$$\varphi_f^j : \mathcal{C}_q \to \mathbb{C}, \quad \varphi_f^j(\mathcal{A}) = T(\{\mathcal{A}_i\})(f) = \mathcal{A}_j \mathcal{G}_j(f),$$

where $\{\mathcal{A}_i\}_{i=1}^{\infty}$ is a sequence with the property $\mathcal{A}_i = 0$ if $i \neq j$ and $\mathcal{A}_j = \mathcal{A}$. Then

$$\begin{aligned} \|\varphi_f^j\| &= \sup\{|\varphi_f^j(\mathcal{A})| : \quad \mathcal{A} \in \mathcal{C}_q, \|\mathcal{A}\| \leq 1\} \\ &= \sup\{|\tau(\mathcal{G}_j(f)\mathcal{A})| : \quad \mathcal{A} \in \mathcal{C}_q, \|\mathcal{A}\| \leq 1\} \\ &= \|\mathcal{G}_j(f)\|_p. \end{aligned}$$

It follows that φ_f^j is a continues linear functional on $\mathcal{C}_q$.

Now we define the linear functional $\varphi_f = \bigoplus \varphi_f^i$ on $\bigoplus \mathcal{C}_q$ as follows

$$\varphi(\{\mathcal{A}_i\}) \quad = \quad \bigoplus \varphi_f^i(\mathcal{A}_i) = T(\{\mathcal{A}_i\})(f) = \sum_{i=1}^{\infty} \mathcal{A}_i \mathcal{G}_i(f).$$

Therefore,

$$|\varphi_f\{\mathcal{A}_i\}| \quad = \quad |(T\{\mathcal{A}_i\})(f)| \leq \|f\|\|T\{\mathcal{A}_i\}\| \leq \|f\|\|T\|\|\{\mathcal{A}_i\}\|_q .$$

Whence φ_f is a continues linear functional. On the other hand,

$$\|\varphi_f\| = (\sum_{i=1}^{\infty} \|\varphi_f^i\|^p)^{\frac{1}{p}} = (\sum_{i=1}^{\infty} \|\mathcal{G}_i(f)\|^p)^{\frac{1}{p}},$$

since $(\bigoplus \mathcal{C}_q)^* = \bigoplus \mathcal{C}_p$. Now by the last inequality we get

$$\sum_{i=1}^{\infty} \|\mathcal{G}_i(f)\|^p)^{\frac{1}{p}} = \|\varphi_f\| \leq \|T\|\|f\|.$$

So $\{\mathcal{G}_i\}$ is a von Neumann–Schatten p-Bessel sequence with bound $\|T\|$. □

Now we can consider the following lemma which is similar to Lemma 2.3 in [9].

Lemma 2.7. *Suppose that $\{\mathcal{G}_i\}$ is a von Neumann–Schatten p-Bessel sequence. Then*

(i) $U^* = T$
(ii) $U \subseteq T^*$, *i.e., T^* is an extension of U. If $\mathcal{X}$ is reflexive, then $U = T^*$.*

In the following theorem we get an equivalent characterization of a von Neumann-Schatten p-frame for $\mathcal{X}$.

Theorem 2.8. *Let $\mathcal{X}$ be a reflexive Banach space. Then $\{\mathcal{G}_i\}_{i=1}^{\infty} \subseteq \mathcal{B}(\mathcal{X}, \mathcal{C}_p)$ is a von Neumann-Schatten p-frame for $\mathcal{X}$ if and only if the operator T defined as following*

$$T : \bigoplus \mathcal{C}_q \to \mathcal{X}^*$$

$$T(\{\mathcal{A}_i\}) = \sum_{i=1}^{\infty} \mathcal{A}_i \mathcal{G}_i$$

is well-defined and onto.

Definition 2.9. Let $1 < q < \infty$. A family $\{\mathcal{G}_i\}_{i=1}^{\infty} \subseteq \mathcal{B}(\mathcal{X}, \mathcal{C}_p)$ where $\frac{1}{p} + \frac{1}{q} = 1$, is called a von Neumann-Schatten q-Riesz basis for $\mathcal{X}^*$ with respect to $\mathcal{H}$ if

(i) $\{f \in \mathcal{X} : \quad \mathcal{G}_i(f) = 0 \quad \forall i \in \mathbb{N}\} = \{0\}$,

(ii) there are positive constant A and B such that for any finite subset $I_1 \subseteq \mathbb{N}$ and $\{\mathcal{A}_i\} \in \bigoplus \mathcal{C}_q$

$$A(\sum_{i \in I_1} \|\mathcal{A}_i\|_q^q)^{\frac{1}{q}} \leq \|\sum_{i \in I_1} \mathcal{A}_i \mathcal{G}_i\| \leq B(\sum_{i \in I_1} \|\mathcal{A}_i\|_q^q)^{\frac{1}{q}}.$$

The assumption of latter definition implies that $\sum_{i=1}^{\infty} \mathcal{A}_i \mathcal{G}_i$ converges unconditionally for all $\{\mathcal{A}_i\}_{i=1}^{\infty} \in \bigoplus \mathcal{C}_q$ and

$$A\|\{\mathcal{A}_i\}\|_q \leq \|\sum_{i=1}^{\infty} \mathcal{A}_i \mathcal{G}_i\| \leq B\|\{\mathcal{A}_i\}\|_q .$$

Thus $\{\mathcal{G}_i\}_{i=1}^{\infty} \subseteq \mathcal{B}(\mathcal{X}, \mathcal{C}_p)$ is a von Neumann-Schatten q-Riesz basis for $\mathcal{X}^*$ with respect to $\mathcal{H}$ if and only if the operator T defined in Theorem 2.8 is both bounded and bounded below. For $p = 2$ and $\mathcal{H} = \mathbb{C}$ this definition is consistent with the standard definition of a Riesz basis for the closed span of its elements.

Corollary 2.10. *Let $\{\mathcal{G}_i\}_{i=1}^{\infty} \subseteq \mathcal{B}(\mathcal{X}, \mathcal{C}_p)$ be a von Neumann-Schatten q-Riesz basis for $\mathcal{X}^*$ with respect to $\mathcal{H}$. Then $\{\mathcal{G}_i\}_{i=1}^{\infty}$ is a von Neumann-Schatten p-Bessel sequence for $\mathcal{X}$ with a bound $\|T\|$.*

Proof. The von Neumann-Schatten q-Riesz basis assumption implies that the operator T given as in Theorem 2.8 is well defined and bounded. By Proposition 2.6 we conclude that $\{\mathcal{G}_i\}_{i=1}^{\infty}$ is a von Neumann-Schatten p-Bessel sequence for $\mathcal{X}$ with a bound $\|T\|$. □

3. Hilbert-Schmidt frames

In this section, we introduce the *Hilbert-Schmidt frames* as a class of von Neumann-Schatten p-frames and present some examples. We also show that every g-frame can be considered as a Hilbert-Schmidt frame.

Definition 3.1. In the Definition 2.3 assume that $\mathcal{X} = \mathcal{K}$ is a Hilbert space and $p = 2$, then countable family $\{\mathcal{G}_i\}$ of bounded linear operators from $\mathcal{K}$ to $\mathcal{C}_2 \subseteq \mathcal{B}(\mathcal{H})$ is said to be a Hilbert Schmidt frame for $\mathcal{K}$ with respect to $\mathcal{H}$.

In other word a sequence $\{\mathcal{G}_i\}$ of bounded linear operators from $\mathcal{H}$ into $\mathcal{C}_2 \subseteq \mathcal{B}(\mathcal{K})$ is said to be a Hilbert-Schmidt frame, or simply a HS-frame for $\mathcal{H}$ with respect to $\mathcal{K}$, if there exist two positive number A and B such that

$$A\|f\|^2 \leq \sum_{i \in \mathcal{I}} \|\mathcal{G}_i(f)\|_2^2 \leq B\|f\|^2, \qquad (f \in \mathcal{H}).$$

Example 3.2. Let $\mathcal{H}$ be a separable Hilbert space and $\{f_i\}_{i=1}^{\infty}$ be a frame for $\mathcal{H}$. Let $\mathcal{G}_i$ be the functional induced by f_i, namely

$$\mathcal{G}_i(f) = \langle f, f_i \rangle \qquad (f \in \mathcal{H}).$$

It is obvious that $\{\mathcal{G}_i\}_{i=1}^{\infty}$ is a HS- frame for $\mathcal{H}$ with respect to $\mathbb{C}$ since

$$A\|f\|^2 \leq \sum_{i \in \mathcal{I}} \|\mathcal{G}_i(f)\|_2^2 = \sum_{i \in \mathcal{I}} |\langle f, f_i \rangle|^2 \leq B\|f\|^2,$$

Example 3.3. Let $\{e_i\}$ be an orthonormal basis in $\mathcal{H}$. Define $\mathcal{G}_i : \mathcal{C}_2 \to \mathcal{C}_2$ by

$$\mathcal{G}_i(\mathcal{A}) = \mathcal{A}(e_i \otimes e_i) = \mathcal{A}e_i \otimes e_i$$

Then $\sum \|\mathcal{G}_i\|_2^2 = \sum \|\mathcal{A}e_i\|^2 = \|\mathcal{A}\|_2^2$. Hence $\{\mathcal{G}_i\}$ is a HS-tight frame for $\mathcal{C}_2$ respect to $\mathcal{H}$.

Lemma 3.4. *Let $\{f_i\}_{i=1}^{\infty}$ be a frame with bounds A and B for a Hilbert space $\mathcal{H}$. Then the bounded operators $\{\mathcal{G}_i\}_{i=1}^{\infty}$ defined by*

$$\mathcal{G}_i : \mathcal{C}_2 \to \mathcal{C}_2, \quad \mathcal{G}_i(\mathcal{T}) = \mathcal{T}(f_i \otimes \frac{f_i}{\|f_i\|}) = \mathcal{T}f_i \otimes \frac{f_i}{\|f_i\|}$$

is a HS-frame for $\mathcal{C}_2$ respect to $\mathcal{H}$.

Proof. First for an orthonormal basis $\{e_i\}_{i=1}^{\infty}$ of $\mathcal{H}$ we would show that

$$A\sum_i \|\mathcal{T}^*e_i\|^2 \leq \sum_j \|\mathcal{T}f_j\|^2 \leq B\sum_i \|\mathcal{T}^*e_i\|^2, \tag{3.1}$$

where $\mathcal{T}^*$ is the adjoint of $\mathcal{T}$. Indeed,

$$\begin{aligned} \sum_j \|\mathcal{T}f_j\|^2 &= \sum_j \sum_i |\langle \mathcal{T}f_j, e_i \rangle|^2 \\ &= \sum_i \sum_j |\langle f_j, \mathcal{T}^*e_i \rangle|^2 \\ &\leq B\sum_i \|\mathcal{T}^*e_i\|^2. \end{aligned}$$

Also the first inequality in (3.1) is similar. It is well-known that $\mathcal{T}$ is an Hilbert Schmidt operator if and only if its adjoint is as well, and $\|\mathcal{T}\|_2 = \|\mathcal{T}^*\|_2$. Combining this with (3.1) we get

$$A\|\mathcal{T}\|_2^2 \leq \sum_j \|\mathcal{G}_i(\mathcal{T})\|_2^2 = \sum_j \|\mathcal{T}f_j\|^2 \leq B\|\mathcal{T}\|_2^2 .$$

Hence $\{\mathcal{G}_i\}$ is a HS-frame for $\mathcal{C}_2$ respect to $\mathcal{H}$ with the same bounds. □

Let y_0 be an unit vector in Hilbert space $\mathcal{K}$ then the operator $\mathcal{U}$ from $\mathcal{K}$ to $\mathcal{B}(\mathcal{K})$ defined by $\mathcal{U}x = x \otimes y_0$ is a linear isometry since $\|\mathcal{U}x\| = \|x \otimes y_0\| = \|x\|$, so we can consider $\mathcal{K}$ as subspace of $\mathcal{B}(\mathcal{K})$.

Lemma 3.5. *Let $\{\Lambda_i : i \in \mathbb{N}\}$ be a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_i : i \in \mathbb{N}\}$. Then $\{\Lambda_i : i \in \mathbb{N}\}$ is HS-frame for $\mathcal{H}$ with respect $\mathcal{K} = \bigoplus_{i \in \mathbb{N}} \mathcal{K}_i$.*

Proof. We can consider $\mathcal{K}_i \subseteq \mathcal{K} \subseteq \mathcal{C}_2 \subseteq \mathcal{B}(\mathcal{K})$. For every $f \in \mathcal{H}$ we have $\Lambda_i f \in \mathcal{K}_i \subseteq \mathcal{C}_2$ and $\|\Lambda_i f\|_2 = \|\Lambda_i f\|$, so

$$A\|f\|^2 \leq \sum_{i=1}^{\infty} \|\Lambda_i f\|_2^2 = \sum_{i=1}^{\infty} \|\Lambda_i f\|^2 \leq B\|f\|^2 \qquad (f \in \mathcal{H}). \qquad \square$$

Remark 3.6. W. Sun [19] shown that pseudo-frames (Li and Ogawa [16]), or similar, oblique frames (Christensen and Eldar [10]) or outer frames (Aldroubi, Cabrelli, and Molter [2]), frames of subspaces (Casazza and Kutyniok [6] and Asgari and Khosravi [5]), time-frequency localization operators Dörfler, Feichtinger and Gröchenig [12]) are a class of g-frames. Hence, Lemma 3.5 implies that all of them are a class of HS-frames.

Let $\{\mathcal{G}_i\}$ be a HS-frame for $\mathcal{H}$ with respect to $\mathcal{K}$. Define the HS- frame operator S as follows,

$$Sf = \sum_{i \in \mathcal{I}} \mathcal{G}_i^* \mathcal{G}_i(f) \qquad (f \in \mathcal{H}), \tag{3.2}$$

where $\mathcal{G}_i^*$ is the adjoint operator of $\mathcal{G}_i$.

Now we show that that the operator S is well define on $\mathcal{H}$. To see this, let $n < m$ be integers. We have

$$\begin{aligned}
\left\|\sum_{i=n}^{m} \mathcal{G}_i^* \mathcal{G}_i(f)\right\| &= \sup_{g \in \mathcal{H}, \|g\|=1} \left\{\left|\langle \sum_{i=n}^{m} \mathcal{G}_i^* \mathcal{G}_i(f), g\rangle\right|\right\} \\
&= \sup_{g \in \mathcal{H}, \|g\|=1} \left\{\left|\left[\sum_{i=n}^{m} \mathcal{G}_i f, \mathcal{G}_i g\right]_{\tau}\right|\right\} \\
&= \sup_{g \in \mathcal{H}, \|g\|=1} \left\{\left|\sum_{i=n}^{m} \tau(\mathcal{G}_i^*(g)\mathcal{G}_i(f))\right|\right\} \\
&\leq \sup_{g \in \mathcal{H}, \|g\|=1} \left\{\sum_{i=n}^{m} \|(\mathcal{G}_i^*(g)\mathcal{G}_i(f)\|_1\right\} \\
&\leq \sup_{g \in \mathcal{H}, \|g\|=1} \left\{\sum_{i=n}^{m} \|\mathcal{G}_i^*(g)\|_2 \|\mathcal{G}_i(f)\|_2\right\} \\
&\leq \sup_{g \in \mathcal{H}, \|g\|=1} \left\{\sum_{i=n}^{m} (\|\mathcal{G}_i(g)\|_2^2)^{\frac{1}{2}} \|(\sum_{i=n}^{m} \|\mathcal{G}_i(f)\|_2^2)^{\frac{1}{2}}\right\} \\
&\leq B^{\frac{1}{2}} \left(\sum_{i=n}^{m} \|\mathcal{G}_i(f)\|_2^2\right)^{\frac{1}{2}}.
\end{aligned}$$

Now we see that the series in (3.2) is convergent. Hence Sf is well defined for any $f \in \mathcal{H}$.

Lemma 3.7. *Let $\{\mathcal{G}_i\}$ be a HS-frame with frame operator S and frame bounds A and B. Then S is bounded invertible self-adjoint and positive.*

Proof. It is easy to check that for any $f, g \in \mathcal{H}$

$$\langle Sf, g\rangle = \langle \sum_{i=1}^{\infty} \mathcal{G}_i^* \mathcal{G}_i(f), g\rangle = \sum_{i=1}^{\infty} [\mathcal{G}_i(f), \mathcal{G}_i(g)]_{\mathcal{T}} = \sum_{i=1}^{\infty} \langle f, \mathcal{G}_i^* \mathcal{G}_i(g)\rangle = \langle f, Sg\rangle.$$

Hence S is self-adjoint. On the other hand,

$$\|S\| = \sup\{|\langle Sf, f\rangle| : \|f\| = 1\} = \sup\left\{\sum_{i\in\mathcal{I}} \|\mathcal{G}_i(f)\|_2^2 : \|f\| = 1\right\} \leq B.$$

Thus S is a bounded self-adjoint operator. Since $A\|f\|^2 \leq \langle Sf, f\rangle \leq \|Sf\|\|f\|$, we have $\|Sf\| \geq A\|f\|$, which implies that S is bounded below (injective) and $\mathrm{Ran}(S)$ is closed in $\mathcal{H}$. Let $g \in \mathcal{H}$ such that $\langle Sf, g\rangle = 0$ for every $f \in \mathcal{H}$. Then we have $\langle f, Sg\rangle = 0$ for every $f \in \mathcal{H}$. This implies that $Sg = 0$ and therefore $g = 0$. Hence $\mathrm{Ran}(S) = \mathcal{H}$. Consequently, S is invertible, and $\|f\| = \|S^{-1}Sf\| \leq A^{-1}\|Sf\|$ and $\|S^{-1}\| \leq A^{-1}$. □

Remark 3.8. By using the properties of the Hilbert-Schmidt operator in Lemma 3.7 we have

$$f = SS^{-1}f = \sum_{i\in\mathcal{I}} \mathcal{G}_i^* \mathcal{G}_i S^{-1} f \quad (f \in \mathcal{H})$$

and

$$f = S^{-1}Sf = \sum_{i\in\mathcal{I}} S^{-1} \mathcal{G}_i^* \mathcal{G}_i f \quad (f \in \mathcal{H}).$$

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