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Property (R) for Bounded Linear Operators

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Abstract. We introduce the spectral property (R), for bounded linear operators defined on a Banach space, which is related to Weyl type theorems. This property is also studied in the framework of polaroid ,or left polaroid, operators.

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1. Introduction and basic results

In this paper we shall consider a property which is related to Weyl type theorems for bounded linear operators $T \in L(X)$, defined on a complex Banach space X. This property, that we call *property* (R), means that the isolated points of the spectrum $\sigma(T)$ of T which are eigenvalues of finite multiplicity are exactly those points λ of the approximate point spectrum for which $\lambda I - T$ is upper semi-Browder (see later for definitions). Property (R) is strictly related to a strong variant of classical Weyl's theorem, the so-called *property* (w) introduced by Rakočević in [26], and more extensively studied in recent papers ([11], [3], [6], [8], [10]). We shall characterize property (R) in several ways and we shall also describe the relationships of it with the other variants of Weyl's theorem. Our main tool is a localized version of the single valued extension property. In the last part, we shall consider the property (R) in the framework of polaroid type operators.

We begin with some preliminary definitions and basic properties. Throughout this paper we denote by $L(X)$ the Banach algebra of all bounded linear operators on an infinite-dimensional complex Banach space X. For an

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operator $T \in L(X)$ by $\alpha(T)$ we denote the dimension of the kernel ker T, and by $\beta(T)$ the codimension of the range $T(X)$. Let

$$\Phi_+(X) := \{T \in L(X) : \alpha(T) < \infty \text{ and } T(X) \text{ is closed}\}$$

be the class of all *upper semi-Fredholm* operators, and let

$$\Phi_-(X) := \{T \in L(X) : \beta(T) < \infty\}$$

be the class of all *lower semi-Fredholm* operators. The class of all semi-Fredholm operators is defined by $\Phi_\pm(X) := \Phi_+(X) \cup \Phi_-(X)$, while the class of all Fredholm operators is defined by $\Phi(X) := \Phi_+(X) \cap \Phi_-(X)$. If $T \in \Phi_\pm(X)$, the *index* of T is defined by ind $(T) := \alpha(T) - \beta(T)$. Recall that a bounded operator T is said *bounded below* if it injective and has closed range. The *upper semi-Weyl operators* are defined as

$$W_+(X) := \{T \in \Phi_+(X) : \operatorname{ind} T \leq 0\},$$

the *lower semi-Weyl operators* are defined by

$$W_-(X) := \{T \in \Phi_-(X) : \operatorname{ind} T \geq 0\}.$$

The set of *Weyl operators* is defined by

$$W(X) := W_+(X) \cap W_-(X) = \{T \in \Phi(X) : \operatorname{ind} T = 0\}.$$

Evidently, if T is bounded below, then $T \in W_+(X)$, while if T is onto, then $T \in W_-(X)$. The classes of operators defined above generate the following spectra. The *approximate point spectrum*

$$\sigma_{\mathrm{a}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not bounded below}\},$$

and the *surjectivity spectrum*

$$\sigma_{\mathrm{s}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not surjective}\}.$$

The *Weyl spectrum* is defined by

$$\sigma_{\mathrm{w}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin W(X)\},$$

the *upper semi-Weil spectrum* (also known as the *Weyl essential approximate point spectrum*) is defined by

$$\sigma_{\mathrm{uw}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin W_+(X)\},$$

while the *lower semi-Weil spectrum* (also known as the *Weyl essential surjectivity spectrum*) is defined by

$$\sigma_{\mathrm{lw}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin W_-(X)\}.$$

Obviously, $\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{uw}}(T) \cup \sigma_{\mathrm{lw}}(T)$ and from the classical Fredholm theory we have:

$$\sigma_{\mathrm{uw}}(T) = \sigma_{\mathrm{lw}}(T^*), \quad \sigma_{\mathrm{lw}}(T) = \sigma_{\mathrm{uw}}(T^*).$$

Let $p := p(T)$ be the *ascent* of an operator T; i.e. the smallest non-negative integer p such that ker $T^p =$ ker T^{p+1}. If such integer does not exist we put $p(T) = \infty$. Analogously, let $q := q(T)$ be *descent* of an operator T; i.e. the smallest non-negative integer q such that $T^q(X) = T^{q+1}(X)$, and if such integer does not exist we put $q(T) = \infty$. It is well-known that if $p(T)$

and $q(T)$ are both finite, then $p(T) = q(T)$([22, Proposition 38.3]). Moreover, $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$ precisely when λ is a pole of the resolvent of T, see Proposition 50.2 of Heuser [22]. The class of all *upper semi-Browder operators* is defined

$$B_+(X) := \{T \in \Phi_+(X) : p(T) < \infty\},$$

the class of all *lower semi-Browder operators* is defined

$$B_-(X) := \{T \in \Phi_-(X) : q(T) < \infty\}.$$

The class of all *Browder operators* is defined

$$B(X) := B_+(X) \cap B_-(X).$$

We have

$$B(X) \subseteq W(X), \quad B_+(X) \subseteq W_+(X), \quad B_-(X) \subseteq W_-(X),$$

see [1, Theorem 3.4].

The *Browder spectrum* of $T \in L(X)$ is defined by

$$\sigma_{\mathrm{b}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin B(X)\},$$

the *upper semi-Browder spectrum* is defined by

$$\sigma_{\mathrm{ub}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin B_+(X)\},$$

and analogously the *lower semi-Browder spectrum* is defined by

$$\sigma_{\mathrm{lb}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin B_-(X)\}.$$

Clearly, $\sigma_{\mathrm{b}}(T) = \sigma_{\mathrm{ub}}(T) \cup \sigma_{\mathrm{lb}}(T)$, $\sigma_{\mathrm{w}}(T) \subseteq \sigma_{\mathrm{b}}(T)$ and $\sigma_{\mathrm{uw}}(T) \subseteq \sigma_{\mathrm{ub}}(T)$.

The single valued extension property plays an important role in local spectral theory, see the recent monograph of Laursen and Neumann [24] and Aiena [1]. In this article we shall consider the following local version of this property:

Definition 1.1. *Let X be a complex Banach space and $T \in L(X)$. The operator T is said to have* the single valued extension property *at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at λ_0), if for every open disc $\mathbb{D}$ centered at λ_0, the only analytic function $f : \mathbb{D} \to X$ which satisfies the equation $(\lambda I - T)f(\lambda) = 0$ for all $\lambda \in \mathbb{D}$ is the function $f \equiv 0$.*

An operator $T \in L(X)$ is said to have SVEP if T has SVEP at every point $\lambda \in \mathbb{C}$.

From the identity theorem for analytic function it easily follows that $T \in L(X)$, as well as its dual T^*, has SVEP at every point of the boundary of the spectrum $\sigma(T) = \sigma(T^*)$, so both T and T^* have SVEP at every isolated point of the spectrum. Note that

$$p(\lambda I - T) < \infty \Rightarrow T \text{ has SVEP at } \lambda, \tag{1}$$

and dually

$$q(\lambda I - T) < \infty \Rightarrow T^* \text{ has SVEP at } \lambda, \tag{2}$$

see [1, Theorem 3.8]. Furthermore, from definition of SVEP we have

$$\sigma_{\mathrm{a}}(T) \text{ does not cluster at } \lambda \Rightarrow T \text{ has SVEP at } \lambda, \tag{3}$$

and dually

$$\sigma_{\mathrm{s}}(T) \text{ does not cluster at } \lambda \Rightarrow T^* \text{ has SVEP at } \lambda. \tag{4}$$

An important T- invariant subspace in local spectral theory is the *quasi-nilpotent part* of T, defined as

$$H_0(T) := \{x \in X : \lim_{n\to\infty} \|T^n x\|^{\frac{1}{n}} = 0\}.$$

We also have

$$H_0(\lambda I - T) \text{ closed} \Rightarrow T \text{ has SVEP at } \lambda. \tag{5}$$

Theorem 1.2. ([1, Chapter 3]) *All the implications* (a)–(e) *become equivalences if we assume that* $\lambda I - T \in \Phi_{\pm}(X)$.

2. Weyl's type theorems

Let write iso K for the set of all isolated points of $K \subseteq \mathbb{C}$. If $T \in L(X)$ set

$$p_{00}(T) := \sigma(T) \setminus \sigma_{\mathrm{b}}(T) = \{\lambda \in \sigma(T) : \lambda I - T \in B(X)\}.$$

Note that every $\lambda \in p_{00}(T)$ is a pole of the resolvent and hence an isolated point of $\sigma(T)$, see [22, Proposition 50.2]. Moreover, $p_{00}(T) = p_{00}(T^*)$. Define

$$\pi_{00}(T) := \{\lambda \in \text{iso } \sigma(T) : 0 < \alpha(\lambda I - T) < \infty\}.$$

Obviously,

$$p_{00}(T) \subseteq \pi_{00}(T) \quad \text{for every } T \in L(X). \tag{6}$$

For a bounded operator $T \in L(X)$ let us define

$$\pi^a_{00}(T) := \{\lambda \in \text{iso } \sigma_{\mathrm{a}}(T) : 0 < \alpha(\lambda I - T) < \infty\},$$

and

$$p^a_{00}(T) := \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{ub}}(T) = \{\lambda \in \sigma_{\mathrm{a}}(T) : \lambda I - T \in \mathcal{B}_+(X)\}.$$

Lemma 2.1. *If* $T \in L(X)$, *then*

$$p_{00}(T) \subseteq p^a_{00}(T) \subseteq \pi^a_{00}(T) \quad \textit{and} \quad \pi_{00}(T) \subseteq \pi^a_{00}(T). \tag{7}$$

Proof. Let $\lambda \in p_{00}(T)$. Then $\lambda I - T \in B(X) \subseteq W(X)$, and since $\lambda \in \sigma(T)$ we must have $\alpha(\lambda I - T) > 0$. Hence $\lambda \in \sigma_{\mathrm{a}}(T)$. Obviously, $\lambda I - T \in B_+(X)$, so $\lambda \in p^a_{00}(T)$. This shows the inclusion $p_{00}(T) \subseteq p^a_{00}(T)$. The inclusion $p^a_{00}(T) \subseteq \pi^a_{00}(T)$ follows once observed that if $\lambda \in p^a_{00}(T)$, then $p(\lambda I - T) < \infty$, so T has SVEP at λ and this is equivalent, by Theorem 1.2, to saying that $\lambda \in \text{iso}\, \sigma_{\mathrm{a}}(T)$. Furthermore, $\alpha(\lambda I - T) > 0$, since $\lambda I - T$ has closed range and $\lambda \in \sigma_{\mathrm{a}}(T))$, and $\alpha(\lambda I - T) < \infty$, since $\lambda I - T \in B_+(X)$. The inclusion $\pi_{00}(T) \subseteq \pi^a_{00}(T)$ is obvious. □

Set $\Delta(T) := \sigma(T) \setminus \sigma_{\mathrm{w}}(T)$. Since $\lambda I - T \in W(X)$ implies that $(\lambda I - T)(X)$ is closed, we can write

$$\Delta(T) = \{\lambda \in \mathbb{C} : \lambda I - T \in W(X),\, 0 < \alpha(\lambda I - T)\}.$$

Analogously, if we set $\Delta_{\mathrm{a}}(T) := \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{uw}}(T)$, then

$$\Delta_{\mathrm{a}}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \in W_{+}(X),\, 0 < \alpha(\lambda I - T)\}.$$

Following Coburn [17], we say that *Weyl's theorem holds* for $T \in L(X)$ if $\Delta(T) = \pi_{00}(T)$. There are several other variants of Weyl's theorem. Two variants of Weyl's theorem, introduced by Harte and W. Y. Lee [21], are defined as follows:

(I) $T \in L(X)$ is said to satisfy *Browder's theorem* if $\Delta(T) = p_{00}(T)$, or equivalently $\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{b}}(T)$.

(II) $T \in L(X)$ is said to satisfy *a-Browder's theorem* if $\Delta_{\mathrm{a}}(T) = p_{00}^{a}(T)$, or equivalently $\sigma_{\mathrm{uw}}(T) = \sigma_{\mathrm{ub}}(T)$.

Note that Weyl's theorem for T entails Browder's theorem for T. Moreover, a-Browder's theorem for T entails Browder's theorem holds for T and the converse is not true. It is known that both Browder's theorem and a-Browder's theorem hold if T or T^* has SVEP. Precisely, we have that a-Browder's theorem holds for T if and only if T has SVEP at every $\lambda \notin \sigma_{\mathrm{uw}}(T)$, and, dually, a-Browder's theorem holds for T^* if and only if T has SVEP at every $\lambda \notin \sigma_{\mathrm{lw}}(T)$, see [9, Theorem 2.3].

The following approximate point spectrum variants of Weyl's theorem have been introduced by Rakočević ([27], [26]):

(III) $T \in L(X)$ is said to satisfy *a-Weyl's theorem* if $\Delta_a(T) = \pi_{00}^{a}(T)$.

(IV) $T \in L(X)$ is said to satisfy *property* (w) if $\Delta_a(T) = \pi_{00}(T)$.

The class of operators satisfying a-Weyl's theorem has been studied by several authors (see, for instance, [27], [19], [18], [2]). Property (w) and its perturbation properties has been studied in very recent papers ([11], [6], [10], [3]). The following diagram resume the relationships between Weyl's theorems, a-Browder's theorem and property (w).

$$\begin{array}{ccc} \text{Property } (w) & \Rightarrow & a\text{-Browder's theorem} \\ \Downarrow & & \Uparrow \\ \text{Weyl's theorem} & \Leftarrow & a\text{-Weyl's theorem} \end{array}$$

(see [26] and [11]). Examples of operators satisfying Weyl's theorem but not property (w) may be found in [11]. Property (w) is not intermediate between Weyl's theorem and a-Weyl's theorem, see [11] for examples. The precise relationships between Browder type theorems and Weyl type theorems are described by the following theorem:

Theorem 2.2. *Suppose that $T \in L(X)$. Then we have*

(i) *T satisfies Weyl's theorem if and only if Browder's theorem holds for T and $p_{00}(T) = \pi_{00}(T)$ ([2]).*

(ii) *T satisfies a-Weyl's theorem if and only if a-Browder's theorem holds for T and $p^a_{00}(T) = \pi^a_{00}(T)$ ([2]).*

The equalities $p_{00}(T) = \pi_{00}(T)$ and $p^a_{00}(T) = \pi^a_{00}(T)$ have been characterized in several ways, see [1, Chapter 3]. In this note we study the equality $p^a_{00}(T) = \pi_{00}(T)$.

Definition 2.3. *We say that an operator $T \in L(X)$ satisfies property (R) if the equality $p^a_{00}(T) = \pi_{00}(T)$ holds.*

The next result shows that, roughly speaking, property (R) may be thought as half of the property (w):

Theorem 2.4. *T satisfies property (w) if and only if a-Browder's theorem holds for T and T has property (R) ([11]).*

The first example shows that property (R) is weaker than property (w).

Example 2.5. Let $R \in L(\ell^2(\mathbb{N}))$ denote the classical unilateral right shift, let Q denote a quasi-nilpotent operator. Define $T := R \oplus R' \oplus Q$, R' the Hilbert adjoint of R. It is well-known that R' is an unilateral left shift. Clearly, $\sigma_{\mathrm{a}}(T) = \sigma_{\mathrm{ub}}(T) = D(0,1)$, where $D(0,1)$ denotes the the closed unit disc. Since $\pi_{00}(T) = \emptyset$, then T satisfies property (R), while T does not satisfy property (w), since $\sigma_{\mathrm{uw}}(T) = \Gamma \cup \{0\}$, where Γ denotes the unit circle of $\mathbb{C}$, so $\sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{uw}}(T) \neq \emptyset = \pi_{00}(T)$.

The following example shows that property (R) for an operator T is not transmitted to the dual T^*.

Example 2.6. Let $T \in \ell^2(\mathbb{N}))$ be the weighted right unilateral shift defined by

$$T(x_1, x_2, \dots) := (0, \frac{x_1}{2}, \frac{x_2}{3}, \dots) \quad \text{for all } x = (x_1, x_2, \dots) \in \ell^2(\mathbb{N}).$$

Clearly, T is quasi-nilpotent, $\sigma_{\mathrm{a}}(T) = \sigma_{\mathrm{ub}}(T) = \{0\}$, $p^a_{00}(T) = \emptyset$, so T satisfies property (R). On the other hand T^* does not satisfy property (R).

Definition 2.7. *A bounded operator $T \in L(X)$ is said to be* left Drazin invertible *if $p := p(T) < \infty$ and $T^{p+1}(X)$ is closed. We say that λ is a* left pole *if $\lambda \in \sigma_{\mathrm{a}}(T)$ and $\lambda I - T$ is left Drazin invertible. A left pole λ is said to be of* finite rank *if $\alpha(\lambda I - T) < \infty$. Analogously, $T \in L(X)$ is said to be* right Drazin invertible *if $q := q(T) < \infty$ and $T^q(X) = T^{q+1}(X)$ is closed. We say that λ is a* right pole *if $\lambda \in \sigma_{\mathrm{s}}(T)$ and $\lambda I - T$ is right Drazin invertible. A right pole λ is said to be of* finite rank *if $\beta(\lambda I - T) < \infty$.*

Clearly, if λ is a pole of the resolvent of T, then λ is either a left-pole and a right pole of the resolvent of T. Indeed, if $p := p(\lambda I - T) = q(\lambda I - T)$, then the subspace $(\lambda I - T)^{p+1}(X) = (\lambda I - T)^p(X)$ is closed, since it coincides with the kernel of the spectral projection associated with the set $\{\lambda\}$.

Remark 2.8. If $\alpha(T) < \infty$, then $\alpha(T^n) < \infty$ for all $n \in \mathbb{N}$, and analogously if $\beta(T) < \infty$, then $\beta(T^n) < \infty$ for all $n \in \mathbb{N}$, for a proof see [12, Lemma 2.2].

Clearly, $p_{00}(T)$ is the set of all poles of the resolvent having finite rank. The next result gives a similar characterization of $p_{00}^a(T)$.

Lemma 2.9. *Let $T \in L(X)$. Then we have:*

(i) *$\lambda \in p_{00}^a(T)$ if and only if λ is a left pole of finite rank for T.*
(ii) *$\lambda \in p_{00}^a(T^*)$ if and only if λ is a right pole of finite rank for T.*

Proof. (i) Suppose λ be a left pole of finite rank. We may assume $\lambda = 0$. Then, $0 \in \sigma_{\mathrm{a}}(T)$, T is left Drazin invertible, so $p(T) < \infty$. The condition of left Drazin invertibility is equivalent to saying that T is upper semi B-Browder, i.e. there exists $n \in \mathbb{N}$ such that $T^n(X)$ is closed and the restriction $T_n := T|T^n(X)$ is upper semi-Browder (see [7, Theorem 2.5] for details), in particular upper semi-Fredholm. Since λ is a left pole of finite rank we have $\alpha(T) < \infty$ and hence $\alpha(T^n) < \infty$, so $T^n \in \Phi_+(X)$ and from the classical Fredholm theory this implies that $T \in \Phi_+(X)$. Since $p(T) < \infty$ we then conclude that $T \in B_+(X)$, so $0 \notin \sigma_{\mathrm{ub}}(T)$, and consequently $0 \in \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{ub}}(T) = p_{00}^a(T)$.

Conversely, assume that $0 \in p_{00}^a(T)$. Then $0 \in \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{ub}}(T)$, hence $p := p(T) < \infty$ and $T \in \Phi_+(X)$. From Fredholm theory we know that $T^n \in \Phi_+(X)$ for all $n \in \mathbb{N}$, so $T^{p+1}(X)$ is closed. Thus T is left Drazin invertible. But $0 \in \sigma_{\mathrm{a}}(T)$, thus 0 is a left pole having finite rank, since $\alpha(T) < \infty$.

(ii) Suppose λ be a right pole of finite rank. We may assume $\lambda = 0$. Then $0 \in \sigma_{\mathrm{s}}(T) = \sigma_{\mathrm{a}}(T^*)$, T is right Drazin invertible and $q(T) < \infty$. The condition of right Drazin invertibility is equivalent to saying that T is lower semi B-Browder, i.e. there exists $n \in \mathbb{N}$ such that $T^n(X)$ is closed and the restriction $T_n := T|T^n(X)$ is lower semi-Browder (see [7, Theorem 2.5]), in particular lower semi-Fredholm. Since $\beta(T) < \infty$, then $\beta(T^n) < \infty$, hence $T^n \in \Phi_-(X)$, from which we obtain that $T \in \Phi_-(X)$. Since $q(T) < \infty$ we then conclude that $T \in B_-(X)$, or equivalently $T^* \in B_+(X^*)$, hence $0 \notin \sigma_{\mathrm{ub}}(T^*)$. Therefore, $0 \in \sigma_{\mathrm{a}}(T^*) \setminus \sigma_{\mathrm{ub}}(T^*) = p_{00}^a(T^*)$.

Conversely, assume that $0 \in p_{00}^a(T^*)$. Then $0 \in \sigma_{\mathrm{a}}(T^*) \setminus \sigma_{\mathrm{ub}}(T^*)$ and since, by duality $\sigma_{\mathrm{ub}}(T^*) = \sigma_{\mathrm{lb}}(T)$, it then follows that $0 \in \sigma_{\mathrm{s}}(T) \setminus \sigma_{\mathrm{lb}}(T)$. Therefore, $q := q(T) < \infty$ and $T \in \Phi_-(X)$. From Fredholm theory we know that $T^n \in \Phi_-(X)$ for all $n \in \mathbb{N}$, in particular $T^q(X)$ is closed. Thus T is right Drazin invertible. But $0 \in \sigma_{\mathrm{s}}(T)$, thus 0 is a right pole of T. Finally, since $T \in \Phi_-(X)$ we have $\beta(T) < \infty$ and consequently 0 is a right pole of finite rank for T. □

Theorem 2.10. *If $T \in L(X)$, then we have:*

(i) *T satisfies property (R) if and only if $\pi_{00}(T)$ coincides with the set of left poles of T having finite rank.*
(ii) *T^* satisfies property (R) if and only if $\pi_{00}(T^*)$ coincides with the set of right poles of T having finite rank.*
(iii) *If T satisfies property (R), then $\pi_{00}(T) = p_{00}(T)$. In particular, every left pole of finite rank of T is a pole.*

Proof. The statements (i) and (ii) are clear by Lemma 2.9.

To show (iii) observe first that by (6) the inclusion $p_{00}(T) \subseteq \pi_{00}(T)$ holds for all $T \in L(X)$. To show the opposite inclusion, suppose that T satisfies (R) and let $\lambda \in \pi_{00}(T) = p_{00}^a(T)$. Then $p(\lambda I - T) < \infty$, and since $\lambda \in \operatorname{iso} \sigma(T)$, then T^* has SVEP at λ. By Theorem 1.2, since $\lambda I - T \in B_+(X)$, the SVEP for T^* at λ is equivalent to saying that $q(\lambda I - T) < \infty$. Moreover, $\alpha(\lambda I - T) < \infty$, since $\lambda \in \pi_{00}(T)$. From Theorem 3.4 of [1] it then follows that $\beta(\lambda I - T) < \infty$, so that $\lambda I - T \in B(X)$. Since $\alpha(\lambda I - T) > 0$ we then conclude that $\lambda \in \sigma(T) \setminus \sigma_{\mathrm{b}}(T) = p_{00}(T)$, thus $\pi_{00}(T) = p_{00}(T)$. The last assertion is clear: $p_{00}(T) = p_{00}^a(T)$. □

The following example shows that neither of the two equalities $\pi_{00}(T) = p_{00}(T)$, $\pi_{00}^a(T) = p_{00}^a(T)$ implies $\pi_{00}(T) = p_{00}^a(T)$.

Example 2.11. Let X be the Hilbert space $\ell_2(\mathbb{N})$ provided by the canonical basis $\{e_1, e_2, \dots\}$, and define for $0 < \varepsilon < 1$

$$S(x_1, x_2, \dots) = (\varepsilon x_1, 0, x_2, x_3, \dots) \quad (x_n) \in \ell_2(\mathbb{N}).$$

Clearly, $\sigma(S) = \sigma(S^*) = D(0,1)$. We show that S^* does not satisfies property (R). It is easily seen that

$$\alpha(\lambda I - S) = 1 \quad \text{for all } |\lambda| < 1,\ \lambda \neq \varepsilon,$$

while $\alpha(\varepsilon I - S) = 2$. We claim that $\lambda I - S$ is a Fredholm operator for all $|\lambda| < 1$. In fact, $\ell_2(\mathbb{N})$ is the direct sum of the one dimensional subspace generated by $\{e_1\}$ and its orthogonal complement M, so to the restriction of $\lambda I - T|M$ we can apply the results of [23, Example IV.5.24]. From the index theorem we can deduce that

$$\beta(\lambda I - S) = 0 \quad \text{for } |\lambda| < 1\ \lambda \neq \varepsilon,$$

while $\beta(\varepsilon I - S) = 1$. Now, $\varepsilon \in \sigma_{\mathrm{s}}(S)$ and $\sigma_{\mathrm{s}}(S) \subseteq \Gamma \cup \{\varepsilon\}$, where Γ denotes the unit circle. On the other hand, for every operator the approximate point spectrum of an operator contains always the boundary of the spectrum [1, Theorem 2.42], hence $\sigma_{\mathrm{s}}(S) = \sigma_{\mathrm{a}}(S^*) \supseteq \Gamma$, and consequently

$$\sigma_{\mathrm{s}}(S) = \sigma_{\mathrm{a}}(S^*) = \Gamma \cup \{\varepsilon\}.$$

Now, from above we know that $\lambda I - S \in \Phi_-(X)$ and $\operatorname{ind}(\lambda I - S) > 0$ for all $|\lambda| < 1$, thus $\sigma_{\mathrm{lw}}(S) \subseteq \Gamma$. The following simple argument shows that the opposite inclusion also holds. Suppose that for some $\mu \in \Gamma$ we have $\mu \notin \sigma_{\mathrm{lw}}(S)$, i.e. $\mu I - S \in W_-(X)$. Since both S and S^* have SVEP at μ it then follows from (1) and (2) that $p(\mu I - S) = q(\mu I - S) < \infty$, so μ is a pole of the resolvent, hence an isolated point of the spectrum, a contradiction. Therefore,

$$\sigma_{\mathrm{lw}}(S) = \sigma_{\mathrm{uw}}(S^*) = \Gamma.$$

Now, S^* has SVEP at the points of Γ, since these points belong to the boundary of the spectrum, and S^* has SVEP at ε, since this point is an isolated point of $\sigma_{\mathrm{a}}(S^*)$. Therefore, S^* has SVEP and consequently both Browder's theorem and a-Browder's theorem hold for S^*, i.e.

$$\sigma_{\mathrm{uw}}(S^*) = \sigma_{\mathrm{ub}}(S^*) = \Gamma.$$

Since $\sigma(S^*)$ has no isolated points we have

$$p_{00}(S^*) = \pi_{00}(S^*) = \emptyset,$$

while S^* does not satisfy property (R), since

$$\sigma_{\mathrm{a}}(S^*) \setminus \sigma_{\mathrm{ub}}(S^*) = \{\varepsilon\} \neq \pi_{00}(S^*).$$

Observe that the operator S^* also satisfies the equality $p_{00}^a(S^*) = \pi_{00}^a(S^*)$. Indeed, $\varepsilon I - S^*$ is Fredholm and ε is an isolated point of $\sigma_{\mathrm{a}}(S^*)$, so

$$\pi_{00}^a(S^*) = \{\varepsilon\} = \sigma_{\mathrm{a}}(S^*) \setminus \sigma_{\mathrm{ub}}(S^*) = p_{00}^a(S^*). \qquad \square$$

In the case of Hilbert space operators instead of the dual T^* is more appropriate to consider the Hilbert adjoint T'. It is well-known that the relationship between T' and T^* is determined by the classical Frechet-Riesz representation theorem: If U is the conjugated linear isometry defined as $U : y \in H \to f_y \in H^*$ where $f_y(x) :=< x, y >$ for all $x \in H$, then we have:

$$U(\overline{\lambda} I - T') = (\lambda I - T^*)U. \tag{8}$$

From (8) it then easily follows

$$\lambda I - T^* \in \Phi_+(H^*) \Leftrightarrow \overline{\lambda} I - T' \in \Phi_+(H). \tag{9}$$

The following result is surely known. For sake of completeness we give the proof.

Lemma 2.12. *Let $T \in L(H)$, H a Hilbert space. Then $\sigma_{\mathrm{ub}}(T^*) = \overline{\sigma_{\mathrm{ub}}(T')}$.*

Proof. We prove that $\lambda I - T^* \in B_+(H^*)$ if and only if $\overline{\lambda} I - T' \in B_+(H)$. Suppose that $p := p(\lambda I - T^*) < \infty$ and let $x \in \ker(\overline{\lambda} I - T')^{p+1}$ be arbitrary. Then

$$U(\overline{\lambda} I - T')^{p+1} x = (\lambda I - T^*)^{p+1} x = 0,$$

so $Ux \in \ker(\lambda I - T^*)^{p+1} = \ker(\lambda I - T^*)^p$ from which we obtain $(\lambda I - T^*)^p Ux = U(\overline{\lambda} I - T')^p x = 0$. Since U is injective we then have $(\overline{\lambda} I - T')^p x = 0$, so $\ker(\overline{\lambda} I - T')^{p+1} \subseteq \ker(\overline{\lambda} I - T')^p$. Since the opposite inclusion always holds we then conclude $p(\overline{\lambda} I - T') \leq p$. A similar argument shows that if $p(\overline{\lambda} I - T') < \infty$, then $p(\lambda I - T^*) \leq p(\overline{\lambda} I - T')$. Therefore, $p(\lambda I - T^*) = p(\overline{\lambda} I - T')$. Taking into account (9) we then conclude that $\lambda I - T^* \in B_+(H^*)$ if and only if $\overline{\lambda} I - T' \in B_+(H)$. $\square$

Theorem 2.13. *Let $T \in L(H)$, H a Hilbert space. Then T^* has property (R) if and only if T' has property (R).*

Proof. By Theorem 2.12 we have

$$p_{00}^a(T^*) = \sigma_{\mathrm{a}}(T^*) \setminus \sigma_{\mathrm{ub}}(T^*) = \overline{\sigma_{\mathrm{a}}(T') \setminus \sigma_{\mathrm{ub}}(T')} = \overline{p_{00}^a(T')}$$

and obviously, $\pi_{00}(T^*) = \overline{\pi_{00}(T')}$. $\square$

The condition $\pi_{00}(T) = p_{00}(T)$ may be characterized in several ways, for instance:

$$\pi_{00}(T) = p_{00}(T) \Leftrightarrow \dim H_0(\lambda I - T) < \infty \text{ for all } \lambda \in \pi_{00}(T), \tag{10}$$

see [1, Theorem 3.84].

As noted in Example 2.11 the condition $\pi_{00}(T) = p_{00}(T)$ is strictly weaker than property (R). However, we have:

Theorem 2.14. *$T \in L(X)$ satisfies property (R) if and only if the following two conditions hold:*

(i) *$p^a_{00}(T) \subseteq iso\,\sigma(T)$.*

- (ii)*] dim $H_0(\lambda I - T) < \infty$ for all $\lambda \in \pi_{00}(T)$*

Proof. If T satisfies property (R), then $p^a_{00}(T) = \pi_{00}(T) \subseteq \text{iso}\,\sigma(T)$ and by Theorem 2.10 we have $\pi_{00}(T) = p_{00}(T)$, thus $H_0(\lambda I - T)$ is finite-dimensional for all $\lambda \in \pi_{00}(T)$. Conversely, suppose that both (i) and (ii) hold. If $\lambda \in p^a_{00}(T) = \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{ub}}(T)$, then $\lambda I - T \in B_+(X)$, hence $\lambda I - T$ has closed range. Since $\lambda \in \sigma_{\mathrm{a}}(T)$, then $0 < \alpha(\lambda I - T) < \infty$, from which we conclude that $p^a_{00}(T) \subseteq \pi_{00}(T)$. The condition (ii) is equivalent to saying that $\pi_{00}(T) = p_{00}(T)$, so, by (6), we also have $\pi_{00}(T) \subseteq p^a_{00}(T)$. Therefore $p^a_{00}(T) = \pi_{00}(T)$. $\square$

The following examples show that Weyl's theorem and property (R) are independent.

Example 2.15. Let S be defined as in Example 2.11. As already observed, S^* does not satisfy property (R) while and S^* has SVEP and hence Browder's theorem holds for S^*. By part (i) of Theorem 2.2 then S^* satisfies Weyl's theorem.
To show that property (R) does not entail Weyl's theorem, consider the operator $T := L \oplus R$, where L and R are the left shift and the right shift on $\ell_2(\mathbb{N})$, respectively. We have $\alpha(T) = \beta(T) = 1$ and $p(T) = \infty$. Therefore, $0 \notin \sigma_{\mathrm{w}}(T)$ while $0 \in \sigma_{\mathrm{b}}(T)$, so Browder's theorem (and hence Weyl's theorem) does not hold for T. On the other hand, $\sigma(T) = D(0,1)$, so $\pi_{00}(T) = \emptyset$, and since $\sigma_{\mathrm{a}}(T) = \sigma_{\mathrm{ub}}(T) = \Gamma$ we then have $\pi_{00}(T) = p^a_{00}(T)$, so property (R) holds for T.

It has been already noted that both a-Browder's theorem and property (R) (or equivalently, property (w)) entails Weyl's theorem. We can improve this result as follows:

Theorem 2.16. *If $T \in L(X)$ satisfies both Browder's theorem and property (R), then T satisfies Weyl's theorem. Moreover, $\sigma(T) \setminus \sigma_{\mathrm{w}}(T) = p^a_{00}(T)$.*

Proof. T satisfies Browder's theorem and $p_{00}(T) = \pi_{00}(T)$, by part (iii) of Theorem 2.10. Therefore, Weyl's theorem holds for T by part (i) of Theorem 2.2, i.e. $\sigma(T) \setminus \sigma_{\mathrm{w}}(T) = \pi_{00}(T)$. Property (R) then implies $\sigma(T) \setminus \sigma_{\mathrm{w}}(T) = p^a_{00}(T)$. $\square$

The class of operators T satisfying the equality $\sigma(T) \setminus \sigma_{\mathrm{w}}(T) = p_{00}^a(T)$ has been recently introduced and studied in [16], and an operator T is said to have *property* (ab) exactly when this equality holds. The following examples show that properties (R) and (ab) are independent.

Example 2.17. Let $T \in \ell^2(\mathbb{N})$ be defined as in Example 2.15. Then T satisfies property (R), while property (ab) does not hold for T, since $0 \in \sigma(T) \setminus \sigma_{\mathrm{w}}(T)$ and $p_{00}^a(T) = \emptyset$. This example also shows that without the assumption that T satisfies Browder's theorem the result of Corollary 2.16 does not hold.

An example of operator satisfying property (ab) but not property (R) is the following: let $Q \in \ell^2(\mathbb{N})$ be defined

$$Q(x_1, x_2, \dots) := (0, 0, \frac{x_2}{3}, \frac{x_3}{4}, \dots) \quad \text{for all } (x_n) \in \ell^2(\mathbb{N}).$$

Clearly, Q is quasi-nilpotent and hence

$$\sigma(Q) = \sigma_{\mathrm{a}}(Q) = \sigma_{\mathrm{w}}(Q) = \sigma_{\mathrm{ub}}(Q) = \{0\}.$$

We have $\alpha(Q) = 1$, so that $0 \in \pi_{00}(Q)$, and since $0 \notin p_{00}^a(Q) = \emptyset$ it then follows that Q does not satisfy property (R). On the other hand, Q has property (ab), since $\sigma(Q) \setminus \sigma_{\mathrm{w}}(Q) = \emptyset$.

As observed before, the SVEP of T at every $\lambda \notin \sigma_{\mathrm{uw}}(T)$ is equivalent to saying that T satisfies a-Browder's theorem. Consequently, by Theorem 2.4, if T has SVEP at every $\lambda \notin \sigma_{\mathrm{uw}}(T)$, then property (R) and property (w) are equivalent for T. However, the following example shows that, always under the assumption of SVEP of T at the points $\lambda \notin \sigma_{\mathrm{uw}}(T)$, this equivalence cannot be extended to a-Weyl's theorem.

Example 2.18. Let R denote the right shift on $\ell_2(\mathbb{N})$ and let Q be defined as

$$Q(x_1, x_2, x_3, \dots) = (\frac{x_2}{2}, \frac{x_3}{3}, \cdots) \quad \text{for all } (x_n) \in \ell_2(\mathbb{N}).$$

Define $T := R \oplus Q$ on $H := \ell_2(\mathbb{N}) \oplus \ell_2(\mathbb{N})$. We have $\sigma(T) = D(0,1)$, thus $\pi_{00}(T) = \emptyset$, and $\sigma_{\mathrm{a}}(T) = \sigma_{\mathrm{uw}}(T) = \Gamma \cup \{0\}$. If $\lambda \notin \sigma_{\mathrm{uw}}(T)$, then $\lambda \notin \sigma_{\mathrm{a}}(T)$, so T has SVEP at λ. Therefore a-Browder's theorem holds for T, i.e. $\sigma_{\mathrm{uw}}(T) = \sigma_{\mathrm{ub}}(T)$. From this we obtain

$$p_{00}^a(T) = \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{uw}}(T) = \emptyset,$$

so T satisfies property (R), or equivalently T satisfies property (w) (and hence Weyl's theorem holds for T). On the other hand, $\pi_{00}^a(S) = \{0\}$, so a-Weyl's theorem does not hold for T.

The next result shows that the equivalence of property (R), property (w) and a-Weyl's theorem is true whenever we assume that T^* has SVEP at the points $\lambda \notin \sigma_{\mathrm{uw}}(T)$.

Theorem 2.19. *Suppose that T^* has SVEP at every $\lambda \notin \sigma_{\mathrm{uw}}(T)$. Then the following statements are equivalent:*

(i) $\pi_{00}(T) = p_{00}(T)$;
(ii) $\pi_{00}^a(T) = p_{00}^a(T)$;

(iii) $\pi_{00}(T) = p^a_{00}(T)$.

Consequently, property (R), property (w), Weyl's theorem and a-Weyl's theorem are equivalent for T.

Proof. First we show that

$$\sigma_{\mathrm{a}}(T) = \sigma(T) \quad \text{and} \quad \sigma_{\mathrm{ub}}(T) = \sigma_{\mathrm{b}}(T). \tag{11}$$

To show $\sigma_{\mathrm{a}}(T) = \sigma(T)$, suppose that $\lambda \notin \sigma_{\mathrm{a}}(T)$. Then $p(\lambda I - T) = 0$ and $\lambda I - T \in W_+(X)$, so $\lambda \notin \sigma_{\mathrm{uw}}(T)$ and hence by assumption T^* has SVEP at λ. By Theorem 1.2 it then follows that $q(\lambda I - T) < \infty$ and hence $p(\lambda I - T) = q(\lambda I - T) = 0$, i.e. $\lambda \notin \sigma(T)$. This proves the equality $\sigma_{\mathrm{a}}(T) = \sigma(T)$.

To show the equality $\sigma_{\mathrm{ub}}(T) = \sigma_{\mathrm{b}}(T)$, observe first that $\sigma_{\mathrm{ub}}(T) \subseteq \sigma_{\mathrm{b}}(T)$ holds for every operator. To show the opposite inclusion, let $\lambda \notin \sigma_{\mathrm{ub}}(T)$. Then $\lambda I - T \in B_+(X)$ and hence both the quantities $\alpha(\lambda I - T)$ and $p(\lambda I - T)$ are finite. But $\sigma_{\mathrm{uw}}(T) \subseteq \sigma_{\mathrm{ub}}(T)$ holds for every operator, so $\lambda \notin \sigma_{\mathrm{uw}}(T)$ and the SVEP of T^* at λ implies, by Theorem 1.2, that also $q(\lambda I - T) < \infty$. Therefore, by [1, Theorem 3.4], we have $\beta(\lambda I - T) = \alpha(\lambda I - T) < \infty$, so $\lambda \notin \sigma_{\mathrm{b}}(T)$. Hence $\sigma_{\mathrm{ub}}(T) = \sigma_{\mathrm{b}}(T)$.
From the equalities (11) we deduce that $\pi_{00}(T) = \pi^a_{00}(T)$ and

$$p^a_{00}(T) = \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{ub}}(T) = \sigma(T) \setminus \sigma_{\mathrm{b}}(T) = p_{00}(T),$$

from which the equivalence of (i), (ii) and (iii) easily follows. To show the last statement observe that the SVEP of T^* at the points $\lambda \notin \sigma_{\mathrm{uw}}(T)$ entails that a-Browder's theorem (and hence Browder's theorem) holds for T, see [9, Theorem 2.3]. By Theorem 2.2 and Theorem 2.4 then property (R), property (w), Weyl's theorem and a-Weyl's theorem are equivalent for T. □

Dually, we have

Theorem 2.20. *Suppose that T has SVEP at every $\lambda \notin \sigma_{\mathrm{lw}}(T)$. Then the following statements are equivalent:*

(i) $\pi_{00}(T^*) = p_{00}(T^*)$;
(ii) $\pi^a_{00}(T^*) = p^a_{00}(T^*)$;
(iii) $\pi_{00}(T^*) = p^a_{00}(T^*)$.

Consequently, property (R), property (w), Weyl's theorem and a-Weyl's theorem are equivalent for T^*.

Proof. The proof is dual to that of Theorem 2.19. In fact, by using dual arguments to those of the proof of Theorem 2.19, if T has SVEP at every $\lambda \notin \sigma_{\mathrm{lw}}(T)$, then $\sigma_{\mathrm{s}}(T) = \sigma(T)$ and $\sigma_{\mathrm{lb}}(T) = \sigma_{\mathrm{b}}(T)$. Consequently, $\sigma_{\mathrm{a}}(T^*) = \sigma(T^*)$ and $\sigma_{\mathrm{ub}}(T^*) = \sigma_{\mathrm{b}}(T^*)$, from which we obtain $\pi_{00}(T^*) = \pi^a_{00}(T^*)$ and $p^a_{00}(T^*) = p_{00}(T^*)$. The SVEP of T at every $\lambda \notin \sigma_{\mathrm{lw}}(T)$ ensures that a-Browder's theorem holds for T^* and hence, by Theorem 2.2 and Theorem 2.4, property (R), property (w), Weyl's theorem and a-Weyl's theorem are equivalent for T^*. □

3. Property (R) for polaroid type operators

In this section we consider classes of operators for which the isolated points of the spectrum are poles of the resolvent-

Definition 3.1. *A bounded operator $T \in L(X)$ is said to be* polaroid *if every isolated point of $\sigma(T)$ is a pole of the resolvent of T. $T \in L(X)$ is said to be a-* polaroid *if every isolated point of $\sigma_{\mathrm{a}}(T)$ is a pole of the resolvent of T.*

It is easily seen that

$$T \ a\text{-polaroid} \Rightarrow T \text{ polaroid}, \tag{12}$$

while, in general, the converse does not hold. It is well known that λ is a pole of the resolvent of T if and only if λ is a pole of the resolvent of T^*. Since $\sigma(T) = \sigma(T^*)$ we then have

$$T \text{ is polaroid} \Leftrightarrow T^* \text{ is polaroid}. \tag{13}$$

From the proof of Theorem 2.19 we know that if T^* has SVEP, then $\sigma(T) = \sigma_{\mathrm{a}}(T)$. Therefore, if T^* has SVEP, then

$$T \ a\text{-polaroid} \Leftrightarrow T \text{ polaroid}. \tag{14}$$

If T has SVEP, from the proof of Theorem 2.20 we know that $\sigma(T^*) = \sigma_{\mathrm{a}}(T^*)$. Therefore, if T has SVEP, then

$$T^* \ a\text{-polaroid} \Leftrightarrow T^* \text{ polaroid} \Leftrightarrow T \text{ polaroid}. \tag{15}$$

It should be noted that general a polaroid operator has not SVEP. A trivial example is the left shift T on $\ell^2(\mathbb{N})$. This operator is polaroid, since $\sigma(T)$ is the unit disc of $\mathbb{C}$, so $\operatorname{iso}\sigma(T) = \emptyset$ and it is well known that T fails SVEP at 0.

Theorem 3.2. *Suppose that $T \in L(X)$ is a-polaroid. Then T satisfies property (R).*

Proof. Let $\lambda \in p_{00}^a(T)$. By Lemma 1.2 then $\lambda \in \pi_{00}^a(T)$, so λ is an isolated point of $\sigma_{\mathrm{a}}(T)$. Since T is a-polaroid, then λ is a pole of the resolvent of T and hence an isolated point of the spectrum. Clearly, $0 < \alpha(\lambda I - T) < \infty$, thus $\lambda \in \pi_{00}(T)$ and consequently $p_{00}^a(T) \subseteq \pi_{00}(T)$. To show the opposite inclusion $\pi_{00}(T) \subseteq p_{00}^a(T)$, let $\lambda \in \pi_{00}(T)$ be arbitrary given. Since $0 < \alpha(\lambda I - T)$, then $\lambda \in \operatorname{iso}\sigma_{\mathrm{a}}(T)$ and hence λ a pole of the resolvent of T, or equivalently $\lambda I - T$ has both ascent and descent finite. Since $\alpha(\lambda I - T) < \infty$, then $\beta(\lambda I - T) < \infty$, see [1, Theorem 3.4], hence $\lambda I - T \in B(X)$, in particular $\lambda \notin \sigma_{\mathrm{ub}}(T)$. Therefore $\lambda \in \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{ub}}(T) = p_{00}^a(T)$, as desired. □

The next example shows that under the weaker condition of being T polaroid the result of Theorem 3.2 does not hold.

Example 3.3. Let $R \in \ell^2(\mathbb{N})$ be the unilateral right shift and

$$U(x_1, x_2, \dots) := (0, x_2, x_3, \cdots) \quad \text{for all } (x_n) \in \ell^2(\mathbb{N}).$$

If $T := R \oplus U$, then $\sigma(T) = D(0,1)$, so $\operatorname{iso}\sigma(T) = \pi_{00}(T) = \emptyset$. Therefore, T is polaroid. Moreover, $\sigma_{\mathrm{a}}(T) = \Gamma \cup \{0\}$, where Γ denotes the unit circle, so $\operatorname{iso}\sigma_{\rm a}(T) = \{0\}$. Since R is injective and $p(U) = 1$ it then follows that $p(T) = p(R) + p(U) = 1$. Furthermore, $T \in \Phi_+(X)$ and hence T is upper semi-Browder, so $0 \in \sigma_{\rm a}(T) \setminus \sigma_{\rm ub}(T) = p^a_{00}(T)$, from which we conclude that $p^a_{00}(T) \neq \pi_{00}(T)$.

Let $\mathcal{H}_{nc}(\sigma(T))$ denote the set of all analytic functions, defined on an open neighborhood of $\sigma(T)$, such that f is non constant on each of the components of its domain. Define, by the classical functional calculus, $f(T)$ for every $f \in \mathcal{H}_{nc}(\sigma(T))$.

Theorem 3.4. *Suppose that $T \in L(X)$ is polaroid and $f \in \mathcal{H}_{nc}(\sigma(T))$.*

(i) *If T^* has SVEP, then property (R) holds for $f(T)$, or equivalently property (w), Weyl's theorem and a-Weyl's theorem hold for $f(T)$.*

(ii) *If T has SVEP, then property (R) holds for $f(T^*)$, or equivalently property (w), Weyl's theorem and a-Weyl's theorem hold for $f(T^*)$.*

Proof. (i) By [5, Lemma 3.11] we know that $f(T)$ is polaroid. By [1, Theorem 2.40] $f(T^*)$ has SVEP, hence from the equivalence (14) we conclude that $f(T)$ is a-polaroid. By Theorem 3.2 it then follows that property (R) holds for $f(T)$ and this, by Theorem 2.19), is equivalent to saying that property (w), Weyl's theorem and a-Weyl's theorem hold for $f(T)$.

(ii) From the equivalence (13) we know that T^* is polaroid and hence, again by [5, Lemma 3.11], $f(T^*)$ is polaroid. Moreover, always by [1, Theorem 2.40], $f(T)$ has SVEP, hence from the equivalence (15) we conclude that $f(T^*)$ is a-polaroid. By Theorem 3.2 it then follows that property (R) holds for $f(T^*)$ and this, by Theorem 2.20), is equivalent to saying that property (w), Weyl's theorem and a-Weyl's theorem hold for $f(T^*)$. □

In the case of Hilbert space operators the SVEP for the dual T^* and the SVEP for the Hilbert adjoint T' are equivalent. Consequently, for Hilbert space operators in the statements of Theorem 3.4, T^* may be replaced by T'.

The condition of being polaroid may be characterized by means of the quasi-nilpotent part: T is polaroid if and only if there exists $p := p(\lambda I - T) \in \mathbb{N}$ such that

$$H_0(\lambda I - T) = \ker(\lambda I - T)^p \quad \text{for all } \lambda \in \operatorname{iso}\sigma(T), \tag{16}$$

see [8]. The class of polaroid operators having SVEP is very large. An interesting class of polaroid operators is given by the $H(p)$-operators, where an operator $T \in L(X)$ is said to belong to the class $H(p)$ if there exists a natural $p := p(\lambda)$ such that:

$$H_0(\lambda I - T) = \ker(\lambda I - T)^p \quad \text{for all } \lambda \in \mathbb{C}. \tag{17}$$

From the implication (5) it is obvious that every operator T which belongs to the class $H(p)$ has SVEP. Moreover, from (16) we also see that every $H(p)$ operator T is polaroid. The class $H(p)$ has been introduced by Oudghiri in [25]. Property $H(p)$ is satisfied by every generalized scalar operator (see [24] for definition and properties of this class), and in particular the property $H(p)$ is satisfied by p-hyponormal, log-hyponormal or M-hyponormal operators on Hilbert spaces, see [25]. Multipliers of commutative semi-simple Banach algebras T are $H(1)$, in particular every convolution T_μ operator on $L_1(G)$, G a locally compact Abelian group is $H(1)$, see [13]. Moreover, every convolution operator T_μ on $L_1(G)$ is a-polaroid, since $\sigma_{\mathrm{a}}(T_\mu) = \sigma(T_\mu)$, see [1, Corollary 5.88].

Other examples of polaroid operators having SVEP are given by the *completely hereditarily normaloid* operators on Banach spaces. In particular, all paranormal operators on Hilbert spaces and all (p,k)-*quasihyponormal* operators on Hilbert spaces are polaroid and have SVEP, see for details [20]. Also the *algebraically quasi-class A operators* on a Hilbert space considered in [14] are polaroid and have SVEP.

Definition 3.5. *A bounded operator $T \in L(X)$ is said to be* left polaroid *if every isolated point of $\sigma_{\mathrm{a}}(T)$ is a left pole of the resolvent of T. The operator $T \in L(X)$ is said to be* right polaroid *if every isolated point of $\sigma_{\mathrm{s}}(T)$ is a right pole of the resolvent of T.*

Trivially,

$$T \text{ } a\text{-polaroid} \Rightarrow T \text{ left polaroid.} \tag{18}$$

The following example shows that the implication (18) cannot be reversed.

Example 3.6. Let $T := R \oplus U$ be the operator defined in Example . We have $T \in \Phi_+(X)$ and hence $T^2 \in \Phi_+(X)$, so that $T^2(X)$ is closed. We also have $p(T) = 1$ so that 0 is a left pole of T. Since $\sigma_{\mathrm{a}}(T) = \Gamma \cup \{0\}$ it then follows that T is left polaroid. On the other hand $q(R) = \infty$, so that $q(T) = q(R)+q(U) = \infty$, so T is neither a-polaroid or polaroid. This example also shows that a left polaroid operator in general does not satisfy property (R).

Furthermore,

$$T \text{ left and right polaroid} \Rightarrow T \text{ polaroid,}$$

see [5], and the operator T defined in Example 2.18 provides an example of operator which is polaroid but not left polaroid. In fact, $0 \in \operatorname{iso}\sigma_{\mathrm{a}}(T)$ and $p(T) = \infty$. Moreover, T^* is polaroid but not right polaroid

Theorem 3.7. *Suppose that $T \in L(X)$ and $f \in \mathcal{H}_{nc}(\sigma(T))$. Then the following assertions hold:*

- (i) *If T^* has SVEP and T is left polaroid, then property (R) holds for $f(T)$, or equivalently property (w), Weyl's theorem and a-Weyl's theorem holds for $f(T)$.*
- (ii) *If T has SVEP and T is right polaroid, then property (R) holds for $f(T^*)$, or equivalently property (w), Weyl's theorem and a-Weyl's theorem holds for T^*.*

Proof. (i) Let $\lambda \in \operatorname{iso}\sigma(T)$. Since T^* has SVEP we have $\sigma(T) = \sigma_{\mathrm{a}}(T)$, see the proof of Theorem 2.19, so $\lambda \in \operatorname{iso}\sigma_{\mathrm{a}}(T)$ and hence a left pole for T. In particular, $\lambda I - T$ is left Drazin invertible and hence $p(\lambda I - T) < \infty$. By Theorem 2.5 of [7] we know that $\lambda I - f(T)$ is semi B-Fredholm (i.e. there exists a natural $n \in \mathbb{N}$ such that $(\lambda I - T)^n(X)$ is closed and the restriction $\lambda I - T|(\lambda I - T)^n(X)$ is semi-Fredholm, in particular $\lambda I - T$ is quasi-Fredholm, see [15] for details). By Theorem [4, Theorem 2.3] the SVEP for T^* at λ entails that $q(\lambda I - T) < \infty$. Hence λ is a pole of the resolvent of T. This proves that T is polaroid. By [5, Lemma 3.11] $f(T)$ is polaroid and since, by [1, Theorem 2.40], $f(T^*)$ has SVEP, the assertion follows from part (i) of Theorem 3.4.

(ii) We show that T^* is polaroid, or equivalently that T is polaroid. Suppose that T is right polaroid and has SVEP. The SVEP for T entails that $\sigma(T) = \sigma_{\mathrm{s}}(T)$, see the proof of Theorem 2.19. Let $\lambda \in \operatorname{iso}\sigma(T)$. Then $\lambda \in \operatorname{iso}\sigma_{\mathrm{s}}(T)$ and hence is a right pole of T. Therefore, $q(\lambda I - T) < \infty$. On the other hand, since $\lambda I - T$ is left Drazin invertible, then $\lambda I - T$ is semi B-Fredholm, again by Theorem 2.5 of [7], in particular $\lambda I - T$ is quasi-Fredholm. By Theorem [4, Theorem 2.7] the SVEP for T at λ entails that $p(\lambda I - T) < \infty$. Consequently, λ is a pole of the resolvent of T and hence T is polaroid. By [5, Lemma 3.11] it follows that $f(T^*)$ is polaroid and since $f(T)$ has SVEP, then the assertion follows from part (ii) of Theorem 3.4. □

Remark 3.8. Note that the result of part (ii) of Theorem 3.7 does not hold if we replace the SVEP for T^* by the SVEP for T. For instance, if T is defined as in Example 3.6, we have $\operatorname{iso}\sigma_{\mathrm{a}}(T) = \{0\}$ and 0 is clearly a left pole of T, since $0 \in p_{00}^a(T)$. Therefore T is a left polaroid. Moreover, T has SVEP by (3), since $\sigma_{\mathrm{a}}(T)$ clusters only on the boundary of the spectrum (where every operator has SVEP). As it has been observed in Example 3.6, T does not satisfy property (R).

However, we have:

Theorem 3.9. *Suppose that $T \in L(X)$ is left polaroid and has SVEP. Then we have:*

(i) *T satisfies a-Weyl's theorem.*
(ii) *T satisfies property (R) if and only if $p_{00}^a(T) \subseteq \pi_{00}(T)$.*

Proof. (i) The SVEP for T entails that a-Browder's holds for T. By Theorem 2.2 to show that a-Weyl's theorem holds for T it suffices to prove that $p_{00}^a(T) = \pi_{00}^a(T)$. By Lemma 2.1 the inclusion $p_{00}^a(T) \subseteq \pi_{00}^a(T)$ holds for every operator. Conversely, suppose that $\lambda \in \pi_{00}^a(T)$. Then λ is an isolated point of $\sigma_{\mathrm{a}}(T)$ and hence a left pole of T. Moreover, $\alpha(\lambda I - T) < 0$, so λ has finite rank and hence, by Lemma 2.9, $\lambda \in p_{00}^a(T)$.

(ii) If T has property (R), then, by definition, $\pi_{00}(T) = p_{00}^a(T)$. Conversely, suppose that $p_{00}^a(T) \subseteq \pi_{00}(T)$. Let $\lambda \in \pi_{00}(T)$. Then $\lambda \in \pi_{00}^a(T)$, by Lemma 2.9, and since T has SVEP, a-Browder's theorem holds for T, i.e. $\sigma_{\mathrm{uw}}(T) = \sigma_{\mathrm{ub}}(T)$. By part (i) we then have

$$\lambda \in \pi_{00}^a(T) = \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{uw}}(T) = \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{ub}}(T) = p_{00}^a(T),$$

so the equality $p_{00}^a(T) = \pi_{00}(T)$ holds. □

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